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CURVES AND SURFACES WITH CONSTANT NONLOCAL MEAN CURVATURE: MEETING ALEXANDROV AND DELAUNAY

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Abstract. We are concerned with hypersurfaces of $\mathbb{R}^N$ with constant nonlocal (or fractional) mean curvature. This is the equation associated to critical points of the fractional perimeter under a volume constraint. Our results are twofold. First we prove the nonlocal analogue of the Alexandrov result characterizing spheres as the only closed embedded hypersurfaces in $\mathbb{R}^N$ with constant mean curvature. Here we use the moving planes method. Our second result establishes the existence of periodic bands or "cylinders" in $\mathbb{R}^2$ with constant nonlocal mean curvature and bifurcating from a straight band. These are Delaunay type bands in the nonlocal setting. Here we use a Lyapunov-Schmidt procedure for a quasilinear type fractional elliptic equation.

1. Introduction and main results

Let $\alpha \in (0,1)$, and let E be an open set in $\mathbb{R}^N$ (not necessarily connected, neither bounded) with C^2-boundary. Then for every $x \in \partial E$, the nonlocal or fractional mean curvature of ∂E at x (that we call NMC for short) is given by

$$H_E(x) = -PV \int_{\mathbb{R}^N} \tau_E(y)|x-y|^{-(N+\alpha)}\,dy := -\lim_{\varepsilon\to 0}\int_{|y-x|\geq\varepsilon} \tau_E(y)|x-y|^{-(N+\alpha)}\,dy \quad (1.1)$$

and is well defined. Here and in the following, we use the notation

$$\tau_E(y) := 1_E(y) - 1_{E^c}(y),$$

where E^c is the complement of E in $\mathbb{R}^N$ and 1_A denotes the characteristic function of A. In the first integral PV denotes the principal value sense, and sometimes will be omitted. The minus sign in front of the integrals makes that balls have constant positive NMC. For the asymptotics α tending to 0 or 1, H_E should be renormalized with a positive constant factor $C_{N,\alpha}$. Since constant factors are not relevant for the results of this paper, we use the simpler expression in (1.1) without the constant $C_{N,\alpha}$.

The following is a more geometric expression for the NMC. We will not use it in this paper. We have that

$$H_E(x) = -\frac{2}{\alpha} PV \int_{\partial E} |x-y|^{-(N+\alpha)}(x-y)\cdot\nu(y)\,dy, \quad (1.2)$$

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where $\nu(y)$ denotes the outer unit normal to ∂E. This follows easily from (1.1) after an integration by parts using that $\nabla_y \cdot \left\{(x-y)|x-y|^{-(N+\alpha)}\right\} = \alpha|x-y|^{-(N+\alpha)}$. On the other hand, [1] introduces the notion of nonlocal directional curvatures —neither used in this paper.

The nonlocal mean curvature is the Euler-Lagrange equation for the fractional perimeter functional, as first discovered in [7]; see also [20]. Nonlocal minimal surfaces are hypersurfaces with zero NMC and were introduced in 2009 by Caffarelli, Roquejoffre, and Savin [7]. They are the limiting configurations of some fractional diffusions —for instance, of the fractional Allen-Cahn equation; see the exposition in [26] and its references. The seminal paper [7] established the first existence and regularity theorems. Within these years, there have been important efforts and results concerning nonlocal minimal surfaces, the following being the main ones. Here we refer to minimizing nonlocal minimal surfaces —a smaller class than that of surfaces with zero NMC. When α is close to 1, Caffarelli and Valdinoci [8, 9] proved their $C^{1,\gamma}$ regularity up to dimension $N \leq 7$. This, together with the subsequent results of Barrios, Figalli, and Valdinoci [4], leads, for α close to 1, to their C^∞ regularity up to dimension $N \leq 7$. For $N = 2$ and any α, Savin and Valdinoci [22] established that they are C^∞. Figalli and Valdinoci [19] have proved that if they are Lipschitz in $\mathbb{R}^N$, then they are C^∞. Finally, Dávila, del Pino, and Wei [14] initiate the important study of nonlocal minimal cones in any dimension, characterizing the stability or instability of α-Lawson cones. Besides, they also construct surfaces of revolution with zero NMC, for instance the fractional catenoid. Still, apart from dimension $N = 2$, there is a lot to be understood, mainly for the classification of all stable nonlocal minimal cones.

Instead, to our knowledge, there are no works on CNMC hypersurfaces, that is, hypersurfaces with constant nonlocal mean curvature. The purpose of this article is twofold. We establish results both of Alexandrov and of Delaunay type for the nonlocal mean curvature.

Our first result is the nonlocal or fractional counterpart of the classical result by Alexandrov [2] on the characterization of spheres as the only closed embedded CMC-hypersurfaces. The precise statement is the following.

Theorem 1.1. *Suppose that E is a nonempty bounded open set with $C^{2,\beta}$-boundary for some $\beta > \alpha$ and with the property that H_E is constant on ∂E. Then E is a ball.*

After completing our proof, we have learnt that this result has also been established, at the same time and independently of ours, by Ciraolo, Figalli, Maggi, and Novaga [10] —a paper in which they also prove stability results with respect to this rigidity theorem.

Our proof of the result relies on the moving planes method introduced by Alexandrov in [2]. We also use a formula for the tangential derivatives of the nonlocal mean curvature, Proposition 2.1 below. In contrast with the classical case, there is no local comparison principle related to the fractional mean curvature. However, nonlocal elliptic operators enjoy (by their definition) a very strong global comparison principle. One of its consequences is the following. Connectedness of E is obviously a necessary assumption in classification results for CMC-hypersurfaces, whereas the assumptions of

Theorem 1.1 allow for disconnected sets E with finitely many connected components. Related to this result is the one in [18] where fractional overdetermined problems are studied on smooth bounded open sets using moving plane techniques and comparison principles. See also [12].

The second purpose of our paper is to establish a nonlocal analogue in the plane of the classical result of Delaunay [16] on periodic cylinders with constant mean curvature. We study sets $E \subset \mathbb{R}^2$ with constant nonlocal mean curvature which have the form of bands or "cylinders" in the plane

$$E = \{(s_1, s_2) \in \mathbb{R}^2 : -u(s_1) < s_2 < u(s_1)\},$$

where $u : \mathbb{R} \to (0, \infty)$ is a positive function. In contrast with classical mean curvature, we note that a straight band $\{-R < s_2 < R\}$ has *positive* constant nonlocal mean curvature. This can be easily seen either from (1.1) or from (1.2).

We establish the existence of a continuous family of bands which are periodic in the first variable s_1, have all the same constant nonlocal mean curvature, and converge to the straight band. The width $2R$ of the straight band will be chosen so that the periods of the new bands converge to 2π as the bands tend to $\{-R < s_2 < R\}$. Our result is of perturbative nature and thus we find periodic bands which are all very close to the straight one.

Therefore we show that, in the nonlocal setting, these objects already exist in dimension 2 —while they only exist in dimensions 3 and higher in the classical CMC setting. Our precise result is the following.

Theorem 1.2. *For every $\alpha \in (0, 1)$ there exist $R > 0$ and a small $\nu > 0$, both depending only on α, and a continuous family of periodic functions $u_a : \mathbb{R} \to \mathbb{R}$ parameterized by $a \in (-\nu, \nu)$, for which the sets*

$$E_a = \{(s_1, s_2) \in \mathbb{R}^2 : -u_a(s_1) < s_2 < u_a(s_1)\}$$

have all the same nonlocal mean curvature, equal to a constant $h_R > 0$ depending only on α, and converge to the straight band $\{-R < s_2 < R\}$ as $a \to 0$. Moreover, $E_a \neq E_{a'}$ for $a \neq a'$, u_a has minimal period $2\pi/\lambda(a)$ if $a \neq 0$, and the periods

$$2\pi/\lambda(a) \to 2\pi \qquad \text{as } a \to 0.$$

In addition, u_a is of class $C^{1,\beta}(\mathbb{R})$ for some $\beta \in (\alpha, 1)$, even with respect to $s_1 = 0$, and of the form

$$u_a(s) = R + \frac{a}{\lambda(a)} \{\cos(\lambda(a)s) + v_a(\lambda(a)s)\} \tag{1.3}$$

with $v_a \to 0$ as $a \to 0$ in the norm of $C^{1,\beta}(\mathbb{R})$, and with $v_a = v_a(\sigma)$ satisfying $\int_0^{2\pi} v_a(\sigma) \cos(\sigma)\, d\sigma = 0$.

The smoothness of these curves, and in general of graphs in $\mathbb{R}^N$ with constant NMC, follows from the methods and results of Barrios, Figalli, and Valdinoci [4] on nonlocal minimal graphs.

In a forthcoming work, [5], we treat this Delaunay type result in $\mathbb{R}^N$. In [5] we will also show further properties of the CNMC curves $s_2 = u_a(s_1)$; for instance, for $a > 0$, they are decreasing in $(0, \pi/\lambda(a))$ (the first half of their period).

For a related nonlocal equation, but different than nonlocal mean curvature, the recent paper [15] by Dávila, del Pino, Dipierro, and Valdinoci establishes variationally the existence of periodic and cylindrical symmetric hypersurfaces. These surfaces minimize a certain fractional perimeter under a volume constraint.

Regarding nonlocal minimal curves in the plane, let us mention here that Savin and Valdinoci [22] have proved that the only minimizing nonlocal minimal curves in the plane are the straight lines. A minimizing nonlocal minimal curve is the boundary of a set that minimizes fractional perimeter in compact sets (and, in particular, has zero NMC). Another important rigidity theorem is that of Figalli and Valdinoci [19] establishing that, in $\mathbb{R}^3$, nonlocal minimal graphs of functions $u : \mathbb{R}^2 \to \mathbb{R}$ are necessarily planes.

To prove Theorem 1.2 we use the Lyapunov-Schmidt procedure. We were inspired by the results on periodic solutions in Ambrosetti and Prodi [3], as well as in our work in a simpler setting (the semilinear one) that we are carrying out in [6]. The first step in the proof, as in the case of nonlocal minimal cones in [14] and nonlocal minimal graphs in [4], is to write the NMC operator acting on graphs of functions —the functions u_a above. The result is a nonlinear fractional operator of quasilinear type acting on functions $u = u(s)$ —see the operators (4.1), (4.7), (4.14), and (4.16) below. Its linearization gives rise to the fractional Laplacian $(-\Delta)^{(1+\alpha)/2}$ and some convolution operators. We use both Fourier series and Hölder spaces to treat all these operators. Let us mention that the implicit function theorem in weighted Hölder spaces has already been used in the important study of nonlocal minimal cones by Dávila, del Pino, and Wei [14].

Related to our work, the nice papers by Sicbaldi [24] and by Schlenk and Sicbaldi [23] establish the existence of periodic and cylindrical symmetric domains in $\mathbb{R}^N$ whose first Dirichlet eigenfunction has constant Neumann data on the boundary. This is therefore a nonlinear nonlocal operator "based" on the half-Laplacian or Dirichlet to Neumann map. Their papers also use the Lyapunov-Schmidt procedure.

The articles [13,17] also use the Lyapunov-Schmidt method, here for some semilinear fractional elliptic equations.

Our paper is organized as follows. The following two sections concern the Alexandrov type result. Section 2 treats the moving planes method, while section 3 establishes a formula for the tangential derivatives of the NMC. Sections 4 to 6 are dedicated to the Delaunay type result. The first one sets up the nonlinear nonlocal operator to be studied, as well as the Lyapunov-Schmidt procedure and functional spaces to be used. Section 5 is devoted to the study of the linearization of our nonlinear problem, while section 6 establishes the C^1 character of our nonlinear operators.

2. The moving plane argument

This section is devoted to the proof of Theorem 1.1. We need the following proposition which might also be useful for other problems. It gives a formula for the tangential derivatives along ∂E of the NMC.

Proposition 2.1. *If $E \subset \mathbb{R}^N$ is bounded and ∂E is of class $C^{2,\beta}$ for some $\beta > \alpha$, then H_E is of class C^1 on ∂E, and we have*

$$\partial_v H_E(x) = (N+\alpha)\, PV \int_{\mathbb{R}^N} \tau_E(y)|x-y|^{-(N+2+\alpha)}(x-y)\cdot v\, dy$$

for $x \in \partial E$ and $v \in T_x\partial E$.

Here, $T_x\partial E$ denotes the tangent space to ∂E at x. Since the proof of this proposition is somewhat technical, we postpone it to section 3. With the help of this proposition, we may now complete the

Proof of Theorem 1.1. Since the fractional mean curvature is invariant under reflections, translations and rotations, it suffices to show that every set $E \subset \mathbb{R}^N$ satisfying the assumptions of Theorem 1.1 is convex and symmetric in x_1 after a translation in the x_1-direction. To prove this, we introduce some notation. As usual, we let ν denote the unit normal vector field on ∂E pointing outside E.

For $\lambda \in \mathbb{R}$, we let $Q_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ denote the reflection with respect to the hyperplane $\{x_1 = \lambda\}$. Moreover, we put $E^\lambda := Q_\lambda(E)$ and

$$E^\lambda_- := \{x \in E^\lambda \,:\, x_1 < \lambda\}.$$

Note that $E^\lambda_- = \varnothing$ if $\lambda > 0$ is sufficiently large, whereas $E^\lambda_- = E^\lambda \not\subset E$ if $-\lambda$ is sufficiently large. We define

$$\lambda^* := \inf\{\lambda \in \mathbb{R} \,:\, E^\mu_- \subset E \text{ for } \mu \geq \lambda, \text{ and } \nu_1(x) > 0 \text{ for all } x \in \partial E \text{ with } x_1 > \lambda\}.$$

We claim that

$$E = E^{\lambda^*}. \tag{2.1}$$

This will complete the proof of the theorem. To prove the claim, we first note that

$$E^{\lambda^*}_- \subset E, \tag{2.2}$$

since ∂E is of class C^2. Moreover, by standard arguments detailed e.g. in [21, Section 5.2] (see also a sketch of the argument in Remark 2.2 below), there exists a point $x \in \partial E$ having one of the following properties:

Case 1 (interior touching): $x_1 < \lambda^*$ and $x \in \partial E^{\lambda^*}$.

Case 2 (non-transversal intersection): $x_1 = \lambda^*$ and $e_1 := (1, 0, \ldots, 0) \in T_x E$.

We treat these cases separately, and we adjust the notation first. For simplicity, we will write λ instead of λ^* and Q instead of $Q_\lambda = Q_{\lambda^*}$ in the following. Obviously, we have that

$$\tau_E \equiv \tau_{E^\lambda} \qquad \text{in } E \cap E^\lambda \text{ and in } \mathbb{R}^N \setminus (E \cup E^\lambda), \tag{2.3}$$

whereas

$$\tau_E \equiv -\tau_{E^\lambda} \equiv 1 \text{ in } E \setminus E^\lambda \qquad \text{and} \qquad \tau_{E^\lambda} \equiv -\tau_E \equiv 1 \text{ in } E^\lambda \setminus E. \tag{2.4}$$

We now let $\mathcal{H}$ denote the half space $\{x_1 < \lambda\}$, and we note that, by (2.2) —recall that $\lambda = \lambda^*$—,

$$E \setminus E^\lambda \subset \mathcal{H} \qquad \text{and} \qquad E^\lambda \setminus E = Q(E \setminus E^\lambda) \subset Q(\mathcal{H}).$$

Finally, for $\varepsilon > 0$ and $x \in \mathbb{R}^N$ the point in either Case 1 or Case 2, we set

$$A^1_\varepsilon := \{y \in E \setminus E^\lambda : |y-x| \geq \varepsilon\} \subset \mathcal{H} \quad \text{and} \quad A^2_\varepsilon := \{y \in E^\lambda \setminus E : |y-x| \geq \varepsilon\} \subset Q(\mathcal{H}).$$

We first consider **Case 1:** From (2.3) and (2.4), it then follows that

$$\begin{aligned}
0 &= \frac{1}{2}\Big(H_E(Q(x)) - H_E(x)\Big) = \frac{1}{2}\Big(H_{E^\lambda}(x) - H_E(x)\Big) \\
&= \frac{1}{2} PV \int_{\mathbb{R}^N} \big(\tau_E(y) - \tau_{E^\lambda}(y)\big)|x-y|^{-(N+\alpha)}\, dy \\
&= \lim_{\varepsilon \to 0}\Big(\int_{A^1_\varepsilon} |x-y|^{-(N+\alpha)}\, dy - \int_{A^2_\varepsilon} |x-y|^{-(N+\alpha)}\, dy\Big) \\
&= \lim_{\varepsilon \to 0} \int_{A^1_\varepsilon} |x-y|^{-(N+\alpha)}\, dy - \int_{E^\lambda \setminus E} |x-y|^{-(N+\alpha)}\, dy.
\end{aligned}$$

Here we have used that ∂E and ∂E^λ are sets of measure zero since ∂E is C^2, and the fact that the second integral in the last line exists since $x \in \mathcal{H}$ and $E^\lambda \setminus E \subset Q(\mathcal{H})$. Thus, by monotone convergence we have

$$\int_{E^\lambda \setminus E} |x-y|^{-(N+\alpha)}\, dy = \lim_{\varepsilon \to 0} \int_{A^1_\varepsilon} |x-y|^{-(N+\alpha)}\, dy = \int_{E \setminus E^\lambda} |x-y|^{-(N+\alpha)}\, dy$$

and therefore

$$\begin{aligned}
0 &= \int_{E \setminus E^\lambda} |x-y|^{-(N+\alpha)}\, dy - \int_{E^\lambda \setminus E} |x-y|^{-(N+\alpha)}\, dy \\
&= \int_{E \setminus E^\lambda} \Big(|x-y|^{-(N+\alpha)} - |x-Q(y)|^{-(N+\alpha)}\Big)\, dy.
\end{aligned}$$

Since $E \setminus E^\lambda \subset \mathcal{H}$ and $|x-y| < |x-Q(y)|$ for $y \in \mathcal{H}$, we deduce that $|E \setminus E^\lambda| = 0$ and thus E equals E^λ up to a set of measure zero. Since both E and E^λ are bounded sets with C^2 boundary, we conclude that $E = E^\lambda$, as claimed in (2.1).

We now consider **Case 2:** Since $e_1 := (1, 0, \dots, 0) \in T_x E \cap T_x E^\lambda$ and H_E (respectively, H_{E^λ}) are constant functions on ∂E (respectively, ∂E^λ), we have $\partial_{e_1} H_E(x) =$

$\partial_{e_1} H_{E^\lambda}(x) = 0$. Applying Proposition 2.1 and recalling (2.3) and (2.4), we find that

$$\begin{aligned}
0 &= \frac{1}{2}\big(\partial_{e_1} H_E(x) - \partial_{e_1} H_{E^\lambda}(x)\big) \\
&= \frac{1}{2}(N+\alpha) PV \int_{\mathbb{R}^N} (x_1 - y_1)\big(\tau_E(y) - \tau_{E^\lambda}(y)\big)|x-y|^{-(N+2+\alpha)}\, dy \\
&= (N+\alpha) \lim_{\varepsilon\to 0}\Big(\int_{A^1_\varepsilon} (\lambda - y_1)|x-y|^{-(N+2+\alpha)}\, dy - \int_{A^2_\varepsilon} (\lambda - y_1)|x-y|^{-(N+2+\alpha)}\, dy\Big) \\
&= (N+\alpha) \lim_{\varepsilon\to 0}\Big(\int_{A^1_\varepsilon} |\lambda - y_1||x-y|^{-(N+2+\alpha)}\, dy + \int_{A^2_\varepsilon} |\lambda - y_1||x-y|^{-(N+2+\alpha)}\, dy\Big).
\end{aligned}$$

Hence it follows that

$$\lim_{\varepsilon\to 0} \int_{A^1_\varepsilon} |\lambda - y_1||x-y|^{-(N+2+\alpha)}\, dy = 0.$$

Since $A^1_{\varepsilon'} \supset A^1_\varepsilon$ for $0 < \varepsilon' \le \varepsilon$ and the integrand is a continuous positive function in $A^1_\varepsilon \subset \mathcal{H}$, this implies that $|A^1_\varepsilon| = 0$ for every $\varepsilon > 0$ and thus $|E \setminus E^\lambda| = 0$. As in Case 1, we conclude that $E = E^\lambda$, as claimed in (2.1). □

Remark 2.2. Here we sketch the argument showing that either Case 1 or 2 in the previous proof must happen when $\lambda = \lambda^*$.

Let us denote points by $x = (x_1, x')$ and, for every λ, $T^\lambda := \{x' \in \mathbb{R}^{N-1} : (\lambda, x') \in E\} \subset \mathbb{R}^{N-1}$. By the definition of λ^* it is clear that, for every $x' \in T^{\lambda^*}$, the set $\{x_1 \in \mathbb{R} : x_1 > \lambda^* \text{ and } (x_1, x') \in E\}$ is an open interval. Therefore, since in addition $\nu_1(x) > 0$ for all $x \in \partial E$ with $x_1 > \lambda^*$ (by definition of λ^*), we have that

$$\partial E \cap \{x_1 > \lambda^*\} = \{(\varphi^{\lambda^*}(x'), x') : x' \in T^{\lambda^*}\} \tag{2.5}$$

for a C^1 function φ^{λ^*} on the open set $T^{\lambda^*} \subset \mathbb{R}^{N-1}$. The previous statement follows from the inverse function theorem.

Suppose now that Case 2 did not happen. Then, we would have $\nu_1(x) > 0$ for all $x \in \partial E$ with $x_1 \ge \lambda^*$. Thus, by the inverse function theorem, φ^{λ^*} would be C^1 up to the boundary ∂T^{λ^*}. The implicit function theorem also gives that (2.5) would also hold with λ^* replaced by μ if $\mu \in (\lambda^* - \delta, \lambda^*)$ for some small $\delta > 0$ —with φ^{λ^*} replaced by a new C^1 function φ^μ.

Hence, for a smaller $\eta > 0$, the reflection of $\{E \cap \{\mu < x_1 < \lambda^* + \eta\}\}$ with respect to $\{x_1 = \mu\}$ is still contained in E, if $\mu \in (\lambda^* - \eta, \lambda^*)$. Now, the reflection of $K := \overline{E} \cap \{x_1 \ge \lambda^* + \eta\}$ with respect to $\{x_1 = \lambda^*\}$ is a compact set contained in $\overline{E}$. If we assume that Case 1 neither happens, then $K \cap \partial E = \varnothing$ and thus K is at a positive distance from ∂E. By continuity, it follows that all reflections of K with respect to $\{x_1 = \mu\}$ are also contained in E for $\mu \in (\lambda^* - \eta', \lambda^*)$ (for a perhaps smaller $\eta' > 0$). This is a contradiction, since then $E^\mu_- \subset E$ for $\mu \in (\lambda^* - \eta', \lambda^*)$, contradicting the definition of λ^*.

3. Tangential derivatives of the NMC. Proof of Proposition 2.1

In this section we prove Proposition 2.1. Let $E \subset \mathbb{R}^N$ be bounded and such that ∂E is of class $C^{2,\beta}$ for some $\beta > \alpha$. As before, we let ν denote the unit normal vector field on ∂E pointing outside E.

Lemma 3.1. *For $\varepsilon > 0$ sufficiently small, the function*

$$H_\varepsilon : \mathbb{R}^N \to \mathbb{R}, \qquad H_\varepsilon(x) = -\int_{|z|\geq\varepsilon} \tau_E(x+z)|z|^{-(N+\alpha)}\, dz$$

is of class C^1 with

$$\frac{\partial H_\varepsilon}{\partial x_i}(x) = -(N+\alpha)\int_{|z|\geq\varepsilon} \tau_E(x+z)|z|^{-(N+2+\alpha)} z_i\, dz + \varepsilon^{-(N+1+\alpha)}\int_{|z|=\varepsilon} \tau_E(x+z) z_i\, d\sigma(z) \tag{3.1}$$

for $i = 1, \dots, N$.

Note that the boundary integral in (3.1) vanishes if $\partial B_\varepsilon(x) \subset E$ or $\partial B_\varepsilon(x) \subset E^c$.

Proof. Since $E \subset \mathbb{R}^N$ is bounded and ∂E is of class $C^{2,\beta}$, we have, for $\varepsilon > 0$ sufficiently small,

$$\mathcal{H}^{N-1}(\partial B_\varepsilon(x) \cap \partial E) = 0 \qquad \text{for all } x \in \mathbb{R}^N \tag{3.2}$$

—that we will assume from now on— and also that the boundary integral in (3.1) is continuous in x.

Let $\tau_n \in C^1_c(\mathbb{R}^N)$, $n \in \mathbb{N}$, be chosen such that

$$|\tau_n| \leq 1 \quad \text{in } \mathbb{R}^N \text{ for all } n \in \mathbb{N} \tag{3.3}$$

and such that

$$\tau_n \to \tau_E \qquad \text{uniformly on compact subsets of } \mathbb{R}^N \setminus \partial E. \tag{3.4}$$

Moreover, for $n \in \mathbb{N}$, we consider

$$h^n : \mathbb{R}^N \to \mathbb{R}, \quad h^n(x) = \int_{|z|\geq\varepsilon} \tau_n(x+z)|z|^{-(N+\alpha)}\, dz.$$

By a standard application of Lebesgue's theorem, h^n is of class C^1, and by integration by parts we have

$$\begin{aligned}\frac{\partial h^n}{\partial x_i}(x) &= \int_{|z|\geq\varepsilon} \frac{\partial \tau_n}{\partial x_i}(x+z)\, |z|^{-(N+\alpha)}\, dz \\ &= (N+\alpha)\int_{|z|\geq\varepsilon} \tau_n(x+z)|z|^{-(N+2+\alpha)}\, z_i\, dz - \varepsilon^{-(N+1+\alpha)}\int_{|z|=\varepsilon} \tau_n(x+z) z_i\, d\sigma(z).\end{aligned}$$

Let $R > 0$. From (3.3) and (3.4), it easily follows that

$$\sup_{|x|\leq R}\int_{\varepsilon\leq|z|\leq\rho} |\tau_n(x+z) - \tau_E(x+z)| dz \to 0 \qquad \text{as } n \to \infty \text{ for every } \rho > \varepsilon,$$

and by using (3.3) again this implies that

$$\sup_{|x|\le R}\Big|\int_{|z|\ge\varepsilon}\big(\tau_n(x+z)-\tau_E(x+z)\big)|z|^{-(N+2+\alpha)}\,z_i\,dz\Big|\to 0\qquad\text{as } n\to\infty \tag{3.5}$$

for $i=1,\dots,N$. A similar argument, using (3.2), (3.3) and (3.4), shows that

$$\sup_{|x|\le R}\Big|\int_{|z|=\varepsilon}\big(\tau_n(x+z)-\tau_E(x+z)\big)z_i\,dz\Big|\to 0\qquad\text{as } n\to\infty \tag{3.6}$$

for $i=1,\dots,N$. Since also $h^n\to -H_\varepsilon$ as $n\to\infty$ uniformly in $B_R(0)$, it follows that H_ε is of class C^1 in $B_R(0)$ with partial derivatives given by (3.1). Since $R>0$ was arbitrary, the claim follows. □

In the following, we set

$$A(\varepsilon',\varepsilon):=\{z\in\mathbb{R}^N\,:\,\varepsilon'<|z|<\varepsilon\}\qquad\text{for } 0<\varepsilon'\le\varepsilon.$$

Lemma 3.2. *There exist $\varepsilon_0>0$ and $C>0$ with the following property. For every $x\in\partial E$, $v\in T_x\partial E$ and $0<\varepsilon'<\varepsilon<\varepsilon_0$ we have*

$$\Big|\int_{A(\varepsilon',\varepsilon)}\tau_E(x+z)|z|^{-(N+2+\alpha)}\,z\cdot v\,dz\Big|\le C|v|\varepsilon^{\beta-\alpha} \tag{3.7}$$

and

$$\Big|\int_{|z|=\varepsilon}\tau_E(x+z)z\cdot v\,d\sigma(z)\Big|\le C|v|\varepsilon^{N+1+\beta}. \tag{3.8}$$

Proof. Without loss of generality, we may assume that $x=0$, $v=e_1$ and $\nu(0)=e_N$. For $\varepsilon>0$, we consider the sets

$$B_\varepsilon^{N-1}:=\{y\in\mathbb{R}^{N-1}\,:\,|y|<\varepsilon\}\qquad\text{and}\qquad C_\varepsilon:=B_\varepsilon^{N-1}\times(-\varepsilon,\varepsilon)\subset\mathbb{R}^N,$$

so that we have the inclusions

$$A(\varepsilon',\varepsilon)\subset B_\varepsilon(0)\subset C_\varepsilon\qquad\text{for } 0<\varepsilon'\le\varepsilon.$$

Since ∂E is of class $C^{2,\beta}$, there exists $\varepsilon_0\in(0,1]$ and a $C^{2,\beta}$-function $h:B_{\varepsilon_0}^{N-1}\to\mathbb{R}$ with $h(0)=0$, $\nabla h(0)=0$ and such that

$$\begin{aligned}
&C_{\varepsilon_0}\cap\partial E=\{(y,h(y))\,:\,y\in B_{\varepsilon_0}^{N-1}\};\\
&C_{\varepsilon_0}\cap E=\{(y,t)\,:\,y\in B_{\varepsilon_0}^{N-1},\ t<h(y)\};\\
&C_{\varepsilon_0}\cap E^c=\{(y,t)\,:\,y\in B_{\varepsilon_0}^{N-1},\ t\ge h(y)\}.
\end{aligned}$$

Moreover, making ε_0 smaller if necessary, we find $c>0$ such that

$$\Big|h(y)-q(y)\Big|\le c|y|^{2+\beta}\qquad\text{for } y\in B_{\varepsilon_0}^{N-1}\quad\text{with } q(y):=\sum_{i,j=1}^N\partial_{ij}h(0)y_iy_j.$$

Next, for $0 < \varepsilon' \le \varepsilon \le \varepsilon_0$, we split $A(\varepsilon', \varepsilon)$ into the subsets

$$\begin{aligned} A^+(\varepsilon',\varepsilon) &:= \{(y,t) \in A(\varepsilon',\varepsilon) \,:\, t > q(y) + c|y|^{2+\beta}\} \;\subset\; E^c, \\ A^-(\varepsilon',\varepsilon) &:= \{(y,t) \in A(\varepsilon',\varepsilon) \,:\, t < q(y) - c|y|^{2+\beta}\} \;\subset\; E, \\ A^0(\varepsilon',\varepsilon) &:= \{(y,t) \in A(\varepsilon',\varepsilon) \,:\, q(y) - c|y|^{2+\beta} \le t \le q(y) + c|y|^{2+\beta}\}. \end{aligned}$$

Since the sets $A^\pm(\varepsilon',\varepsilon)$ are invariant under the reflection $(y,t) \mapsto (-y,t)$ and $\mp\tau_E \equiv 1$ in $A^\pm(\varepsilon',\varepsilon)$, we find that

$$\int_{A^\pm(\varepsilon',\varepsilon)} \tau_E(z)|z|^{-(N+2+\alpha)} z_1 \, dz = \mp \int_{A^\pm(\varepsilon',\varepsilon)} |z|^{-(N+2+\alpha)} z_1 \, dz = 0.$$

We thus have that

$$\begin{aligned} \Big| \int_{A(\varepsilon',\varepsilon)} \tau_E(z)|z|^{-(N+2+\alpha)} z_1 \, dz \Big| &\le \int_{A^0(\varepsilon',\varepsilon)} |z|^{-(N+1+\alpha)} \, dz \\ &\le \int_{B_\varepsilon^{N-1}} \int_{q(y)-c|y|^{2+\beta}}^{q(y)+c|y|^{2+\beta}} |(y,t)|^{-(N+1+\alpha)} \, dt \, dy \le \int_{B_\varepsilon^{N-1}} |y|^{-(N+1+\alpha)} \int_{q(y)-c|y|^{2+\beta}}^{q(y)+c|y|^{2+\beta}} dt \, dy \\ &= 2c \int_{B_\varepsilon^{N-1}} |y|^{-N+1+\beta-\alpha} dy = 2c\, \omega_{N-2} \int_0^\varepsilon r^{\beta-\alpha-1} dr = \frac{2c\,\omega_{N-2}}{\beta-\alpha} \varepsilon^{\beta-\alpha}, \end{aligned}$$

where ω_{N-2} denotes the surface area of the unit sphere in $\mathbb{R}^{N-1}$.

To see (3.8), we split $\partial B_\varepsilon(0) \subset \mathbb{R}^N$ into the subsets

$$\begin{aligned} S^+(\varepsilon) &:= \{(y,t) \in \partial B_\varepsilon(0) \,:\, t > q(y) + c|y|^{2+\beta}\} \;\subset\; E^c, \\ S^-(\varepsilon) &:= \{(y,t) \in \partial B_\varepsilon(0) \,:\, t < q(y) - c|y|^{2+\beta}\} \;\subset\; E, \\ S^0(\varepsilon) &:= \{(y,t) \in \partial B_\varepsilon(0) \,:\, q(y) - c|y|^{2+\beta} \le t \le q(y) + c|y|^{2+\beta}\}. \end{aligned}$$

Since the sets $S^\pm(\varepsilon)$ are invariant under the reflection $(y,t) \mapsto (-y,t)$ and $\mp\tau_E \equiv 1$ on $S^\pm(\varepsilon)$, we find that

$$\int_{S^\pm(\varepsilon)} \tau_E(z) z_1 \, d\sigma(z) = \mp \int_{S^\pm(\varepsilon)} z_1 \, d\sigma(z) = 0$$

and therefore

$$\Big| \int_{\partial B_\varepsilon(0)} \tau_E(z) z_1 \, d\sigma(z) \Big| \le \int_{S^0(\varepsilon)} |z| \, d\sigma(z) = \varepsilon \mathcal{H}^{N-1}(S^0(\varepsilon)) = \varepsilon^N \mathcal{H}^{N-1}(S^1(\varepsilon)) \tag{3.9}$$

with

$$S^1(\varepsilon) := \{(y,t) \in \partial B_1(0) \,:\, \varepsilon q(y) - c\varepsilon^{1+\beta}|y|^{2+\beta} \le t \le \varepsilon q(y) + c\varepsilon^{1+\beta}|y|^{2+\beta}\}.$$

To estimate $\mathcal{H}^{N-1}(S^1(\varepsilon))$, we fix $\bar q \ge \max_{\partial B_1(0)} |q|$ and recall that $\varepsilon \le \varepsilon_0 \le 1$. Thus for $(y,t) \in S^1(\varepsilon)$ we have $|t| \le \varepsilon(\bar q + c)$, and hence

$$\big|q(\frac{y}{|y|}) - q(y)\big| = \big|q(\frac{y}{|y|})\big|(1-|y|^2) = \big|q(\frac{y}{|y|})\big| t^2 \le \varepsilon^2 \bar q (\bar q + c)^2 = c_1 \varepsilon^2$$

with $c_1 = \bar{q}(\bar{q}+c)^2$ and therefore, with $c_2 := c_1 + c$,

$$\varepsilon q(\frac{y}{|y|}) - c_2\varepsilon^{1+\beta} \le \varepsilon q(\frac{y}{|y|}) - c_1\varepsilon^3 - c\varepsilon^{1+\beta} \le \varepsilon q(y) - c\varepsilon^{1+\beta} \le t \le \dots$$
$$\le \varepsilon q(\frac{y}{|y|}) + c_2\varepsilon^{1+\beta}.$$

Denoting by S^{N-2} the unit sphere in $\mathbb{R}^{N-1}$, we may thus estimate

$$\mathcal{H}^{N-1}(S^1(\varepsilon)) \le \int_{S^{N-2}} \int_{\varepsilon q(\rho)-c_2\varepsilon^{1+\beta}}^{\varepsilon q(\rho)+c_2\varepsilon^{1+\beta}} (1-t^2)^{\frac{N-3}{2}}\, dt\, d\rho.$$

If $N \ge 3$, this implies that

$$\mathcal{H}^{N-1}(S^1(\varepsilon)) \le \int_{S^{N-2}} 2c_2\varepsilon^{1+\beta}\, d\rho = 2c_2\, \omega_{N-2}\varepsilon^{1+\beta},$$

whereas in the case $N = 2$ we have

$$(1-t^2)^{\frac{N-3}{2}} \le (1-\varepsilon^2(\bar{q}+c)^2)^{-\frac{1}{2}} \le 2 \qquad \text{for } (y,t) \in S^1(\varepsilon) \text{ if } \varepsilon \le \frac{\sqrt{3}}{2(\bar{q}+c)}.$$

and therefore

$$\mathcal{H}^{N-1}(S^1(\varepsilon)) \le \int_{S^{N-2}} 4c_2\varepsilon^{1+\beta}\, d\rho = 4c_2\, \omega_{N-2}\varepsilon^{1+\beta} \qquad \text{if } \varepsilon \le \frac{\sqrt{3}}{2(\bar{q}+c)}.$$

Combining this with (3.9) and assuming without loss that $\varepsilon_0 \le \frac{\sqrt{3}}{2(\bar{q}+c)}$, we find in both cases that

$$\Big|\int_{\partial B_\varepsilon(0)} \tau_E(z) z_1\, d\sigma(z)\Big| \le 4c_2\, \omega_{N-2}\, \varepsilon^{N+1+\beta} \qquad \text{for } 0 < \varepsilon \le \varepsilon_0.$$

Since ∂E is compact and of class $C^{2+\beta}$, the constants $\varepsilon_0, c, \bar{q}$ can be chosen independently of $x \in \partial E$, and then also $c_1, c_2 > 0$ do not depend on $x \in \partial E$. Hence both (3.7) and (3.8) hold with $C := \omega_{N-2} \max\{\frac{2c}{\beta-\alpha}, 4c_2\}$. □

Proof of Proposition 2.1 (completed). Let $(\varepsilon_k)_k$ be a decreasing sequence of positive numbers with $\varepsilon_k \to 0$. For $k \in \mathbb{N}$, we consider the functions

$$H_E^k : \partial E \to \mathbb{R}, \qquad H_E^k(x) = -\int_{|z|\ge\varepsilon_k} \tau_E(x+z)|z|^{-(N+\alpha)}\, dy.$$

We then have that

$$\lim_{k\to\infty} H_E^k(x) = -\lim_{\varepsilon_k\to 0} \int_{|x-y|\ge\varepsilon_k} \tau_E(y)|x-y|^{-(N+\alpha)}\, dy = H_E(x) \qquad \text{for } x \in \partial E.$$

Moreover, we may pass to a subsequence such that all functions H_E^k are of class C^1 on ∂E by Lemma 3.1. Moreover, for a given C^1-vector field v on ∂E and $k \in \mathbb{N}$, we have $\partial_{v(x)} H_E^k(x) = \rho^k(x) + \sigma^k(x)$ for $x \in \partial E$ with $\rho^k, \sigma^k : \partial E \to \mathbb{R}$ given by

$$\rho^k(x) := -(N+\alpha) \int_{|z|\ge\varepsilon_k} \tau_E(x+z)|z|^{-(N+2+\alpha)} z \cdot v(x)\, dz$$

and

$$\sigma^k(x) := \varepsilon_k^{-(N+1+\alpha)} \int_{|z|=\varepsilon_k} \tau_E(x+z) z \cdot v(x) d\sigma(x).$$

Since $\beta > \alpha$, Lemma 3.2 implies that $(\rho^k)_k$, $k \in \mathbb{N}$ is a Cauchy sequence in $C(\partial E)$, whereas $\sigma^k \to 0$ uniformly on ∂E as $k \to \infty$. Moreover, a standard sequence mixing argument shows that, independently of the choice of the sequence $(\varepsilon_k)_k$, we have

$$\lim_{k\to\infty} \rho^k(x) = -(N+\alpha) \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} \tau_E(x+z)|z|^{-(N+2+\alpha)} z \cdot v(x)\, dz \qquad \text{for } x \in \partial E.$$

Since the vector field v was chosen arbitrarily, it thus follows that H_E is of class C^1 on ∂E with

$$\begin{aligned}\partial_{v(x)} H_E(x) &= \lim_{k\to\infty} \rho^k(x) = -(N+\alpha) \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} \tau_E(x+z)|z|^{-(N+2+\alpha)} z \cdot v(x)\, dz \\ &= (N+\alpha) \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} \tau_E(y)|x-y|^{-(N+2+\alpha)} (x-y) \cdot v(x)\, dz,\end{aligned}$$

as claimed. □

4. Periodic CNMC curves in $\mathbb{R}^2$

In this section we set up the Lyapunov-Schmidt procedure to prove Theorem 1.2 on Delaunay-type curves with CNMC in the plane, that is, curves with constant nonlocal mean curvature. The full proof of the theorem will be completed in this and the following two sections.

The following easy lemma gives a formula for the nonlocal mean curvature of a set given by $\{-u(s_1) < s_2 < u(s_1)\}$ in terms of the positive function $u = u(s)$. The same computation already appears in [4, 14].

Lemma 4.1. *If $E = \{(s_1, s_2) \in \mathbb{R}^2 : -u(s_1) < s_2 < u(s_1)\}$, where $u : \mathbb{R} \to (0, \infty)$ is a function of class $C^{1,\beta}$ for some $\beta \in (\alpha, 1)$ such that $0 < m_1 \leq u \leq m_2$ for two positive constants m_i, its nonlocal mean curvature H_E —that we will denote by $H(u)$— at a point $(s, u(s))$ is given by*

$$\begin{aligned}\frac{1}{2} H(u)(s) = &\int_{\mathbb{R}} F\left(\frac{u(s)-u(s-t)}{|t|}\right) d\mu(t) \\ &- \int_{\mathbb{R}} \left\{ F\left(\frac{u(s)+u(s-t)}{|t|}\right) - F(+\infty) \right\} d\mu(t),\end{aligned} \tag{4.1}$$

where the integrals are to be understood in the principal value sense,

$$d\mu(t) = \frac{dt}{|t|^{1+\alpha}}, \tag{4.2}$$

and

$$F(q) = \int_0^q \frac{d\tau}{(1+\tau^2)^{\frac{2+\alpha}{2}}}. \tag{4.3}$$

In the fractional perimeter functional (whose Euler-Lagrange equation is the NMC operator), the first integral in (4.1) corresponds to the interactions of points in the upper curve $\{s_2 = u(s_1)\}$ of $\{-u(s_1) < s_2 < u(s_1)\}$ with points in the same upper curve. The second integral corresponds to interactions of points in the upper curve with points in the lower disjoint curve $\{s_2 = -u(s_1)\}$.

Proof of Lemma 4.1. We have that

$$-H(u)(s) = \int_{\mathbb{R}^2} \frac{1_E((s_1,s_2)) - 1_{E^c}((s_1,s_2))}{|(s,u(s)) - (s_1,s_2)|^{2+\alpha}}\, ds_1\, ds_2 = \int_{\mathbb{R}} I(s,s_1)\, ds_1,$$

where 1_A denotes the characteristic function of A, and

$$\begin{aligned} I(s,s_1) &= \left\{ \int_{-u(s_1)}^{u(s_1)} ds_2 - \int_{-\infty}^{-u(s_1)} ds_2 - \int_{u(s_1)}^{+\infty} ds_2 \right\} \left((s-s_1)^2 + (s_2 - u(s))^2 \right)^{-\frac{2+\alpha}{2}} \\ &= \left\{ 2\int_{-u(s_1)}^{u(s_1)} ds_2 - \int_{\mathbb{R}} ds_2 \right\} |s-s_1|^{-(2+\alpha)} \left(1 + \left(\frac{s_2 - u(s)}{|s-s_1|} \right)^2 \right)^{-\frac{2+\alpha}{2}}. \end{aligned}$$

Then, with the new variable $\tau = (s_2 - u(s))/|s-s_1|$, one gets

$$I(s,s_1) = |s-s_1|^{-(1+\alpha)} \left(2\int_{\frac{-u(s_1)-u(s)}{|s-s_1|}}^{\frac{u(s_1)-u(s)}{|s-s_1|}} d\tau - \int_{\mathbb{R}} d\tau \right) (1+\tau^2)^{-\frac{2+\alpha}{2}}$$

$$= |s-s_1|^{-(1+\alpha)} \left\{ 2\left(F\left(\frac{u(s_1)-u(s)}{|s-s_1|} \right) - F\left(\frac{-u(s_1)-u(s)}{|s-s_1|} \right) \right) - F(+\infty) + F(-\infty) \right\}$$

$$= -2|s-s_1|^{-(1+\alpha)} \left\{ F\left(\frac{u(s)-u(s_1)}{|s-s_1|} \right) - F\left(\frac{u(s)+u(s_1)}{|s-s_1|} \right) + F(+\infty) \right\}.$$

Changing the variable $s_1 = s - t$, we arrive at the expression of the lemma. □

With the use of formula (4.1) we can compute the NMC —that we denote by h_R— of the straight band of width $u_R \equiv 2R$:

$$H(u_R) = -2\int_{\mathbb{R}} \left\{ F\left(\frac{2R}{|t|} \right) - F(+\infty) \right\} d\mu(t) =: h_R > 0. \tag{4.4}$$

Note that the functions $u_a = u_a(s)$ in (1.3),

$$u_a(s) = R + \frac{a}{\lambda}\{\cos(\lambda s) + v_a(\lambda s)\}, \qquad \text{with } \lambda = \lambda(a),$$

have a period that may change with a. It will be very convenient, in order to apply the implicit function theorem, to work with functions all with the same period 2π. Thus, we rescale the variables $s = s_1$ and $u(s) = s_2$ and see how the NMC changes after

rescaling. Since the NMC is a geometric quantity, the following comes as no surprise. If, for $\lambda > 0$, $w^\lambda(s) := \lambda w(s/\lambda)$, then with the change of variables $\tau = t/\lambda$ we see that

$$\begin{aligned} H(w^\lambda)(s) &= 2\int_{\mathbb{R}} F\left(\frac{\lambda w(s/\lambda) - \lambda w((s-t)/\lambda)}{|t|}\right)\, d\mu(t) \\ &- \ 2\int_{\mathbb{R}} \left\{F\left(\frac{\lambda w(s/\lambda) + \lambda w((s-t)/\lambda)}{|t|}\right) - F(+\infty)\right\}\, d\mu(t) = \lambda^{-\alpha} H(w)(s/\lambda). \end{aligned}$$

Thus, we must look for functions $u = (u_a)^\lambda$ of the form

$$u(s) = \lambda R + a\{\cos(s) + v_a(s)\}, \tag{4.5}$$

with v_a even, 2π-periodic, and orthogonal to $\cos(\cdot)$ in $[0,\pi]$. In addition, since we want the NMC for the band given by u_a to be h_R, we need that $H(u) = \lambda^{-\alpha} h_R$. Thus, from (4.4) and making below the change of variables $t = \overline{t}/\lambda$, we see that u must satisfy

$$\begin{aligned} \frac{1}{2}H(u) &= \frac{1}{2}\lambda^{-\alpha} h_R = -\lambda^{-\alpha}\int_{\mathbb{R}} \{F(2R/|t|) - F(+\infty)\}\, d\mu(t) \\ &= \lambda^{-\alpha}\int_{\mathbb{R}}\int_{2R/|t|}^{+\infty} (1+\tau^2)^{-\frac{2+\alpha}{2}}\, d\tau\, \frac{dt}{|t|^{1+\alpha}} = \int_{\mathbb{R}}\int_{2\lambda R/|\overline{t}|}^{+\infty} (1+\tau^2)^{-\frac{2+\alpha}{2}}\, d\tau\, \frac{d\overline{t}}{|\overline{t}|^{1+\alpha}} \\ &= \int_{\mathbb{R}} d\mu(\overline{t}) \int_{2\lambda R/|\overline{t}|}^{+\infty} (1+\tau^2)^{-\frac{2+\alpha}{2}}\, d\tau = \int_{\mathbb{R}} d\mu(t) \int_{2\lambda R/|t|}^{+\infty} (1+\tau^2)^{-\frac{2+\alpha}{2}}\, d\tau \\ &= -\int_{\mathbb{R}} \{F(2\lambda R/|t|) - F(+\infty)\}\, d\mu(t). \end{aligned}$$

Finally, recalling the expression (4.1) for $H(u)$, we conclude that the function u given by (4.5) must satisfy the equation

$$\int_{\mathbb{R}} \left\{F\left(\frac{u(s) - u(s-t)}{|t|}\right) - \left\{F\left(\frac{u(s) + u(s-t)}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right)\right\}\right\} d\mu(t) = 0. \tag{4.6}$$

Recall that we look for solutions of this equation of the form

$$u(s) = \lambda R + a\{\cos(s) + v_a(s)\} = \lambda R + a\varphi(s).$$

In our proof, after a Lyapunov-Schmidt reduction, we will apply the implicit function theorem at $a = 0$, $\lambda = 1$, and $v_a = 0$ —taking v_a orthogonal to $\cos(\cdot)$ in $L^2(0,\pi)$. This will give us λ and v_a as functions of a. To have the linearized operator to be invertible and be able to use directly the implicit function theorem, it is necessary to divide equation (4.6) by a (as in [6] and as in the classical paper by Crandall-Rabinowitz [11]) and work with the new operator

$$\begin{aligned} \Phi(a,\lambda,\varphi)(s) := &\int_{\mathbb{R}} \frac{1}{a} F\left(a\, \frac{\varphi(s) - \varphi(s-t)}{|t|}\right) d\mu(t) \\ &- \int_{\mathbb{R}} \frac{1}{a}\left\{F\left(\frac{2\lambda R + a(\varphi(s) + \varphi(s-t))}{|t|}\right) - F\left(\frac{2\lambda R}{|t|}\right)\right\} d\mu(t). \end{aligned} \tag{4.7}$$

We need to solve the nonlinear equation

$$\Phi(a,\lambda,\varphi)=0.$$

Let us introduce the functional spaces in which we work. We take β such that

$$0<\alpha<\beta<\min\{1,2\alpha+1/2\}. \tag{4.8}$$

The condition $\beta<2\alpha+1/2$ is technical (to simplify a proof on regularity) and could be avoided. Consider the spaces

$$X:=C^{1,\beta}_{p,e}=\{\varphi:\mathbb{R}\to\mathbb{R}\ :\ \varphi\in C^{1,\beta}(\mathbb{R}) \text{ is } 2\pi\text{-periodic and even}\} \tag{4.9}$$

and

$$Y:=C^{0,\beta-\alpha}_{p,e}=\{\tilde\varphi:\mathbb{R}\to\mathbb{R}\ :\ \tilde\varphi\in C^{0,\beta-\alpha}(\mathbb{R}) \text{ is } 2\pi\text{-periodic and even}\}. \tag{4.10}$$

Note that if $\varphi\in X$, then

$$\varphi'(0)=\varphi'(\pi)=0.$$

Since the functions in these spaces are even and 2π-periodic, we can take as norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, the $C^{1,\beta}([0,\pi])$ and $C^{0,\beta-\alpha}([0,\pi])$ norms. That is,

$$\|\varphi\|_X:=\|\varphi\|_{L^\infty(0,\pi)}+\|\varphi'\|_{L^\infty(0,\pi)}+\sup_{0\le s<\overline{s}\le\pi}\frac{|\varphi'(s)-\varphi'(\overline{s})|}{|s-\overline{s}|^\beta} \tag{4.11}$$

and

$$\|\tilde\varphi\|_Y:=\|\tilde\varphi\|_{L^\infty(0,\pi)}+\sup_{0\le s<\overline{s}\le\pi}\frac{|\tilde\varphi(s)-\tilde\varphi(\overline{s})|}{|s-\overline{s}|^{\beta-\alpha}}.$$

We must study the operator

$$\Phi(a,\lambda,\varphi)=\Phi_1(a,\varphi)-\Phi_2(a,\lambda,\varphi),$$

where

$$\Phi_1(a,\varphi):=\int_{\mathbb{R}}\frac{1}{a}F(a\delta_-\varphi)\ d\mu(t),$$

$$\Phi_2(a,\lambda,\varphi):=\int_{\mathbb{R}}\frac{1}{a}\left\{F\left(\frac{2\lambda R}{|t|}+a\frac{\delta_0\varphi}{|t|}\right)-F\left(\frac{2\lambda R}{|t|}\right)\right\}d\mu(t), \tag{4.12}$$

and we define

$$\delta_-\varphi(s,t):=\frac{\varphi(s)-\varphi(s-t)}{|t|},$$

$$\delta_+\varphi(s,t):=\frac{\varphi(s)-\varphi(s+t)}{|t|} \tag{4.13}$$

(δ_+ does not appear above, but it will be used later), and

$$\delta_0\varphi(s,t):=\varphi(s)+\varphi(s-t).$$

The previous operators —and their equivalent expressions below— are seen to be well defined (some in the principal value sense) in section 6, where we will establish much more: their differentiable character between the spaces X and Y above.

To express the first term Φ_1 of the operator in a nicer manner, observe that

$$F_1(a,q) := \frac{1}{a}F(aq) = \frac{1}{a}\int_0^{aq} \frac{d\rho}{(1+\rho^2)^{\frac{2+\alpha}{2}}} = \int_0^q \frac{d\tau}{(1+a^2\tau^2)^{\frac{2+\alpha}{2}}}$$

is a smooth function both of a and q, and also that $F_1(0,q) = q$. Thus, we have that

$$\Phi_1(a,\varphi) := \int_{\mathbb{R}} F_1(a, \delta_-\varphi)\, d\mu(t), \tag{4.14}$$

where

$$F_1(a,q) := \int_0^q \frac{d\tau}{(1+a^2\tau^2)^{\frac{2+\alpha}{2}}}. \tag{4.15}$$

Next, we deal with the second term Φ_2 in the operator Φ. Recall that it is given by expression (4.12). Using the change of variables $\tau = (2\lambda R + a\delta_0\varphi\,\overline{\tau})/|t|$, we see that

$$\begin{aligned}
\frac{1}{a}\left\{F\left((2\lambda R + a\delta_0\varphi)/|t|\right) - F\left(2\lambda R/|t|\right)\right\} &= \frac{1}{a}\int_{2\lambda R/|t|}^{(2\lambda R + a\delta_0\varphi)/|t|} \frac{d\tau}{(1+\tau^2)^{\frac{2+\alpha}{2}}} \\
&= \frac{\delta_0\varphi}{|t|}\int_0^1 \left\{1 + ((2\lambda R + a\delta_0\varphi\,\overline{\tau})/|t|)^2\right\}^{-\frac{2+\alpha}{2}} d\overline{\tau} \\
&= |t|^{1+\alpha}\delta_0\varphi \int_0^1 \frac{d\overline{\tau}}{\{t^2 + (2\lambda R + a\delta_0\varphi\,\overline{\tau})^2\}^{\frac{2+\alpha}{2}}}.
\end{aligned}$$

Therefore,

$$\Phi_2(a,\lambda,\varphi) = \int_{\mathbb{R}} \delta_0\varphi\, F_2(t,a,\lambda,\delta_0\varphi)\, dt, \tag{4.16}$$

where

$$F_2(t,a,\lambda,r) = \int_0^1 \frac{d\overline{\tau}}{\{t^2 + (2\lambda R + ar\overline{\tau})^2\}^{\frac{2+\alpha}{2}}}. \tag{4.17}$$

Note that the measure in (4.16) is dt and not $d\mu(t)$ as above for Φ_1, and that δ_0 is not an increment but a sum which is not divided by $|t|$.

Let us write our operator Φ when $a = 0$. The following expression can be computed from (4.7) differentiating with respect to a at $a = 0$, or can be obtained from (4.14) and (4.16) —taking into account that $(t^2 + (2\lambda R)^2)^{-\frac{2+\alpha}{2}} = F'(2\lambda R/|t|)/|t|^{2+\alpha}$. We have

$$\Phi(0,\lambda,\varphi)(s) = \int_{\mathbb{R}} (\varphi(s) - \varphi(s-t))\,\frac{dt}{|t|^{2+\alpha}} - \int_{\mathbb{R}} (\varphi(s) + \varphi(s-t))\, F'\left(\frac{2\lambda R}{|t|}\right)\frac{dt}{|t|^{2+\alpha}}, \tag{4.18}$$

where, as always, the integrals are to be understood in the principal value sense.

We choose now $R > 0$ such that $\Phi(a = 0, \lambda = 1, \varphi = \cos(\cdot)) = 0$ holds, i.e., such that

$$\begin{aligned}
0 = \Phi(0,1,\cos(\cdot))(s) = &\int_{\mathbb{R}} (\cos(s) - \cos(s-t))\,\frac{dt}{|t|^{2+\alpha}} \\
&- \int_{\mathbb{R}} (\cos(s) + \cos(s-t))\, F'\left(\frac{2\lambda R}{|t|}\right)\frac{dt}{|t|^{2+\alpha}}
\end{aligned} \tag{4.19}$$

for all $s \in \mathbb{R}$. We will see in Lemma 5.3 below that there exists a unique $R > 0$, depending only on α, such that (4.19) holds for all $s \in \mathbb{R}$.

The next task is to study the linearized operator of Φ at $(a = 0, \lambda = 1, \varphi = \cos(\cdot))$. This will be done in all detail in next section. Let us say here that we will see that the function

$$\cos(\cdot) \text{ belongs to the kernel of } D_\varphi \Phi(0, 1, \cos(\cdot)),$$

and this is why we must use a Lyapunov-Schmidt reduction corresponding to the following subspaces of X and Y.

In $L^2(0, \pi)$ we consider the orthogonal basis given by $\{1, \cos(\cdot), \cos(2\cdot), \cos(3\cdot), \ldots\}$ and the subspaces

$$V_1 = \langle \cos(\cdot) \rangle \qquad \text{and} \qquad V_2 = \langle \cos(\cdot) \rangle^\perp = \langle 1, \cos(2\cdot), \cos(3\cdot), \ldots \rangle. \tag{4.20}$$

We also consider

$$X_1 = X \cap V_1, \qquad X_2 = X \cap V_2, \qquad Y_1 = Y \cap V_1, \qquad Y_2 = Y \cap V_2,$$

meaning for the first intersection, for instance, the functions defined in $\mathbb{R}$ belonging to X such that, when restricted to $(0, \pi)$, belong to V_1. Denote the standard projections by

$$\Pi_1 : Y \to Y_1 \qquad \text{and} \qquad \Pi_2 : Y \to Y_2.$$

Retaking our notation above, we look for functions φ of the form

$$\varphi(s) = \cos(s) + v(s), \qquad \text{with } v \in X_2.$$

We then write our operator acting on the function $v \in X_2$ as

$$\overline{\Phi} : A \subset \mathbb{R} \times \mathbb{R} \times X_2 \to Y, \qquad \overline{\Phi}(a, \lambda, v) := \Phi(a, \lambda, \cos(\cdot) + v),$$

where A will be an appropriate open set containing $(0, 1, 0)$.

The key result to apply the implicit function theorem is the following proposition. It will be proved in the next two sections. The differentiability properties stated in the proposition are proved in section 6 and ensure, in particular, that all operators and integrals above are well defined. Recall that, by (4.19), we have that $\overline{\Phi}(0, 1, 0) = 0$.

Proposition 4.2. *There exists $\nu > 0$ small enough (depending only on α) for which the operator $\overline{\Phi} = \overline{\Phi}(a, \lambda, v) : (-\nu, \nu) \times (1/2, 3/2) \times B_1(0) \subset \mathbb{R} \times \mathbb{R} \times X_2 \to Y$ is of class C^1. Moreover, the linear operator*

$$(D_\lambda \overline{\Phi}, D_v \overline{\Phi})(0, 1, 0) : \mathbb{R} \times X_2 \to Y$$

is continuous and invertible.

From this result and the implicit function theorem, Theorem 1.2 follows immediately.

Proof of Theorem 1.2. As mentioned above, we have $\overline{\Phi}(0, 1, 0) = 0$. Proposition 4.2 allows to apply the implicit function theorem and obtain the family of functions u_a in the statement of the theorem. All properties stated in the theorem follow immediately from the setting described in this section, with the exception of the following two statements that need to be justified.

It remains to prove that the minimal period of u_a is $2\pi/\lambda(a)$ if $a \neq 0$, and that $u_a \not\equiv u_{a'}$ if $a \neq a'$. Let us start from the first statement. Clearly, it is equivalent to prove that, after the rescaling, the function

$$u(s) = \lambda R + a\{\cos(s) + v_a(s)\},$$

with $a \neq 0$ and v_a orthogonal to $\cos(\cdot)$ in $L^2(0,\pi)$, has minimal period 2π. This is an easy task, by expressing $v_a(s)$ as a Fourier series $a_0 + \sum_{k=2}^{\infty} a_k \cos(ks)$. Now, if T is the minimal period of u, we must have

$$\begin{aligned} &\cos(s) + v_a(s) = \cos(s+T) + v_a(s+T) \\ &\quad = \cos(s)\cos(T) - \sin(s)\sin(T) + a_0 + \sum_{k=2}^{\infty} a_k\{\cos(ks)\cos(kT) - \sin(ks)\sin(kT)\}. \end{aligned}$$

Multiplying the first and last terms in the above expression by $\cos(s)$ and integrating in $(0, 2\pi)$, we deduce that $\cos(T) = 1$. Hence the minimal period is $T = 2\pi$.

Finally, from this we can easily deduce that $u_a \not\equiv u_{a'}$ if $a \neq a'$. Indeed, if $u_a \equiv u_{a'}$ then their minimal periods would agree, and thus $\lambda(a) = \lambda(a')$. Now, this leads to $a = a'$, since $u_a(s) = R + \frac{a}{\lambda(a)}\{\cos(\lambda(a)s) + v_a(\lambda(a)s)\}$ and $v_a(\sigma)$ is orthogonal to $\cos(\sigma)$ in $L^2(0,\pi)$. □

5. The linearized operator acting on even periodic functions

In this section we study the linearized operator of Φ at $(0, 1, \cos(\cdot))$ and we establish the last statement on invertibility in Proposition 4.2. For this, the main results that we prove are collected in the following result.

Proposition 5.1. *We have that*

$$D_\lambda \Phi(0, 1, \cos(\cdot)) = \gamma \cos(\cdot) \tag{5.1}$$

for some constant $\gamma > 0$. On the other hand, for all $\psi \in X$,

$$L\psi := D_\varphi \Phi(0, 1, \cos(\cdot))\, \psi = C_{(1+\alpha)/2}^{-1} (-\Delta)^{\frac{1+\alpha}{2}} \psi - \left(\textstyle\int_{\mathbb{R}} P_R\right) \psi - P_R * \psi, \tag{5.2}$$

where $C_{(1+\alpha)/2}$ is the usual constant factor in the definition of $(-\Delta)^{\frac{1+\alpha}{2}}$ and

$$P_R(t) = \frac{1}{|t|^{2+\alpha}} F'\left(\frac{2R}{|t|}\right) = \frac{1}{\{(2R)^2 + |t|^2\}^{\frac{2+\alpha}{2}}}.$$

The functions $e_k(s) = \cos(ks)$, $k = 0, 1, 2, 3 \ldots$, are all eigenfunctions of L, with eigenvalues satisfying

$$Le_k = \lambda_k e_k, \qquad \lambda_0 < 0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots \qquad \textit{and} \tag{5.3}$$

$$\frac{\lambda_k}{k^{1+\alpha}} \to \mu_\infty > 0 \qquad \textit{as } k \to \infty. \tag{5.4}$$

As a consequence, we will have that the linear operator

$$(D_\lambda \overline{\Phi}, D_v \overline{\Phi})(0, 1, 0) : \mathbb{R} \times X_2 \to Y$$

is continuous and invertible.

The rest of this section is dedicated to prove the above proposition and, in particular, the last statement on invertibility on Hölder spaces in Proposition 4.2.

From (4.18), we compute

$$\begin{aligned} D_\lambda \overline{\Phi}(0,1,0)(s) &= -\int_{\mathbb{R}} (\cos(s)+\cos(s-t))\, F''(2R/|t|)\, 2R\, \frac{dt}{|t|^{3+\alpha}} \\ &= -\cos(s) \int_{\mathbb{R}} (1+\cos(t))\, F''(2R/|t|)\, 2R\, \frac{dt}{|t|^{3+\alpha}}. \end{aligned} \tag{5.5}$$

The last simplification comes from the fact that $\cos(s-t)=\cos(s)\cos(t)+\sin(s)\sin(t)$, $\sin(t)$ is an odd function of t, and $F''(2R/|t|)|t|^{-3-\alpha}$ is an even function of t. Note also that the last integral converges at $t=0$, since $F''(2R/|t|) \sim |t|^{3+\alpha}$ near $t=0$. Observe also that this integral is a strictly negative number, since $1+\cos(t)\geq 0$ and $F''(2R/|t|)<0$.

From (5.5) we deduce that

$$D_\lambda(\Pi_1 \overline{\Phi})(0,1,0)(s) = -\cos(s) \int_{\mathbb{R}} (1+\cos(t))\, F''(2R/|t|)\, 2R\, \frac{dt}{|t|^{3+\alpha}}$$

and that this integral is negative, and also that

$$D_\lambda(\Pi_2 \overline{\Phi})(0,1,0) \equiv 0.$$

As a consequence of these two statements on $D_\lambda(\Pi_i \overline{\Phi})(0,1,0)$, to establish the last statement in Proposition 5.1 it only remains to prove that $D_v(\Pi_2 \overline{\Phi})(0,1,0)$ is an isomorphism between X_2 and Y_2. But, from (4.18), and with P_R as in the statement of Proposition 5.1, we have

$$\begin{aligned} Lw(s) :&= D_v \overline{\Phi}(0,1,0)\, w(s) \\ &= \int_{\mathbb{R}} (w(s)-w(s-t)) \frac{dt}{|t|^{2+\alpha}} - \int_{\mathbb{R}} (w(s)+w(s-t)) F'\left(\frac{2R}{|t|}\right) \frac{dt}{|t|^{2+\alpha}} \\ &= \int_{\mathbb{R}} (w(s)-w(s-t)) \frac{dt}{|t|^{2+\alpha}} - (\textstyle\int_{\mathbb{R}} P_R)\, w(s) - (P_R * w)(s) \\ &= \left\{ C_{(1+\alpha)/2}^{-1} (-\Delta)^{\frac{1+\alpha}{2}} w - (\textstyle\int_{\mathbb{R}} P_R) w - P_R * w \right\}(s). \end{aligned}$$

We emphasize that $P_R \in (C^\infty \cap L^1 \cap L^\infty)(\mathbb{R})$ and that $P_R(t)$ is an even function of t.

The following simple result states that the functions $\cos(k\cdot)$ are always eigenfunctions of any convolution operator with an even kernel.

Lemma 5.2. i) *If $P \in L^1(\mathbb{R})$ is an even function and $e_k(s)=\cos(ks)$, $k=0,1,2,3\ldots$, then*

$$(P * e_k)(s) = (\textstyle\int_{\mathbb{R}} \cos(kt) P(t)\, dt)\ e_k(s)$$

for all $s \in \mathbb{R}$. In particular, e_k is an eigenfunction of the operator $P * \cdot$.

ii) *In a similar way, the functions $e_k(s)=\cos(ks)$ are eigenfunctions of $(-\Delta)^{\frac{1+\alpha}{2}}$.*

Proof. i) This is a simple calculation:

$$\begin{aligned}(P * e_k)(s) &= \int_{\mathbb{R}} \cos(k(s-t))P(t)\ dt = \int_{\mathbb{R}} \{\cos(ks)\cos(kt) + \sin(ks)\sin(kt)\}\, P(t)\, dt \\ &= \left(\int_{\mathbb{R}} \cos(kt)P(t)\ dt\right)\cos(ks),\end{aligned}$$

since P is even.

ii) For the fractional Laplacian, in the principal value sense, we have

$$\int_{\mathbb{R}} \frac{\cos(ks) - \cos(ks-kt)}{|t|^{2+\alpha}}\ dt = \int_{\mathbb{R}} \frac{\cos(ks) - \cos(ks)\cos(kt) - \sin(ks)\sin(kt)}{|t|^{2+\alpha}}\ dt,$$

and this last integral, in the principal value sense, is equal to

$$\left(\int_{\mathbb{R}} \frac{1-\cos(kt)}{|t|^{2+\alpha}}\ dt\right)\cos(ks),$$

as desired. □

Thus,

$$Le_k = \left\{\int_{\mathbb{R}} \frac{1-\cos(kt)}{|t|^{2+\alpha}}\ dt - \int_{\mathbb{R}} \frac{1+\cos(kt)}{|t|^{2+\alpha}} F'\left(\frac{2R}{|t|}\right) dt\right\} e_k =: \lambda_k e_k,$$

for $k = 0, 1, 2, \ldots$. Note that $\lambda_0 = -\int_{\mathbb{R}} 2|t|^{-2-\alpha} F'(2R/|t|)\, dt < 0$. Now,

$$\lambda_1 = \int_{\mathbb{R}} \frac{1-\cos(t)}{|t|^{2+\alpha}}\ dt - \int_{\mathbb{R}} \frac{1+\cos(t)}{|t|^{2+\alpha}} F'\left(\frac{2R}{|t|}\right)\ dt = 0,$$

since this value is the same as the factor multiplying $\cos(s)$ in expression (4.19), defining R, and that we next take to be equal to zero. Thus, $L\cos(\cdot) = Le_1 = \lambda_1 e_1 \equiv 0$.

Let us see now that there exists a unique $R > 0$ for which $\lambda_1 = 0$.

Lemma 5.3. *There exists a unique $R > 0$ —which depends only on α— such that*

$$\lambda_1(R) = \int_{\mathbb{R}} \frac{1-\cos(t)}{|t|^{2+\alpha}}\ dt - \int_{\mathbb{R}} \frac{1+\cos(t)}{|t|^{2+\alpha}} F'\left(\frac{2R}{|t|}\right)\ dt = 0. \tag{5.6}$$

Proof. First,

$$\lambda_1'(R) = -\int_{\mathbb{R}} \frac{1+\cos(t)}{|t|^{2+\alpha}}\ \frac{2}{|t|}\ F''\left(\frac{2R}{|t|}\right)\ dt > 0,$$

and therefore λ_1 is increasing in R. Second, applying the monotone convergence theorem for integrals to

$$\lambda_1(R) = \int_{\mathbb{R}} \left\{\frac{1-\cos(t)}{|t|^{2+\alpha}} - \frac{1+\cos(t)}{|t|^{2+\alpha}}\left(1 + \frac{(2R)^2}{|t|^2}\right)^{-\frac{2+\alpha}{2}}\right\} dt,$$

we deduce that $\lambda_1(R) \nearrow \int_{\mathbb{R}} \frac{1-\cos(t)}{|t|^{2+\alpha}}\ dt > 0$ as $R \to +\infty$, and also that $\lambda_1(R) \searrow \int_{\mathbb{R}} -\frac{2\cos(t)}{|t|^{2+\alpha}}\ dt = -\infty$ as $R \to 0$. Thus, the result is proved. □

Let us see now that the sequence λ_k is increasing in k, and that therefore we have $\lambda_0 < 0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Indeed, using the change of variables $\overline{t} = kt$,

$$\begin{aligned}\lambda_k &= \int_{\mathbb{R}} \{(1-\cos(kt)) - (1+\cos(kt))F'(2R/|t|)\} \frac{dt}{|t|^{2+\alpha}} \\ &= k^{1+\alpha} \int_{\mathbb{R}} \left\{ (1-\cos(\overline{t})) - (1+\cos(\overline{t})) \left(1 + \frac{4R^2}{|\overline{t}/k|^2}\right)^{-\frac{2+\alpha}{2}} \right\} \frac{d\overline{t}}{|\overline{t}|^{2+\alpha}} = k^{1+\alpha}\mu_k,\end{aligned}$$

where we have defined

$$\mu_k = \int_{\mathbb{R}} \left\{ (1-\cos(\overline{t})) - (1+\cos(\overline{t})) \left(1 + \frac{4R^2k^2}{|\overline{t}|^2}\right)^{-\frac{2+\alpha}{2}} \right\} \frac{d\overline{t}}{|\overline{t}|^{2+\alpha}}.$$

But both $k^{1+\alpha}$ and μ_k are increasing in k. And, by the monotone convergence theorem,

$$-\int_{\mathbb{R}} (1+\cos(\overline{t})) \left(1 + \frac{4R^2k^2}{|\overline{t}|^2}\right)^{-\frac{2+\alpha}{2}} \frac{d\overline{t}}{|\overline{t}|^{2+\alpha}} \nearrow 0$$

as $k \to +\infty$. Hence $\mu_k \to \int_{\mathbb{R}} \frac{1-\cos(\overline{t})}{|\overline{t}|^{2+\alpha}}\, d\overline{t} =: \mu_\infty > 0$. Thus, and this will be crucial in the next argument,

$$\frac{\lambda_k}{k^{1+\alpha}} \to \mu_\infty > 0 \qquad \text{as } k \to +\infty.$$

From (5.3), $L\cos(\cdot) = 0$, the asymptotics (5.4), and the definition of fractional Sobolev spaces in terms of Fourier coefficients, we deduce that

$$L_{|V_2} = \Pi_2 L_{|V_2} : H^{1+\alpha}_{p,e} \cap V_2 \to L^2_{p,e} \cap V_2 \quad \text{ is continuous and invertible,}$$

where $H^{1+\alpha}_{p,e}$ denotes the space of functions which are even and 2π periodic in $\mathbb{R}$ and belong to the Sobolev space $H^{1+\alpha}$ in bounded sets of $\mathbb{R}$. The same definition applies to $L^2_{p,e}$.

To complete the proof of Proposition 5.1, we need to show that L is also continuous and invertible from X_2 onto Y_2. The fact that L sends X_2 to Y continuously (and thus into Y_2), follows from $L_{|X_2} = D_v\overline{\Phi}(0,1,0)$ and the first part of Proposition 4.2 (to be proved in next section) stating that $\overline{\Phi}$ is C^1 from its domain in $\mathbb{R}\times\mathbb{R}\times X_2$ into Y.

It remains to establish that L is invertible in these spaces. For this, given $f \in Y_2$, since then $f \in L^2_{p,e} \cap V_2$, we know that we can find a unique $w \in H^{1+\alpha}_{p,e} \cap V_2$ (using Fourier series and the eigenvalues λ_k above) such that $Lw = f$. Recall that L is given by

$$Lw = c_1(-\Delta)^{\frac{1+\alpha}{2}} w - c_2 w - P_R * w$$

for some positive constants c_i. Hence, $Lw = f$ can be written as

$$c_1(-\Delta)^{\frac{1+\alpha}{2}} w = c_2 w + P_R * w + f \quad \text{ in } \mathbb{R}. \tag{5.7}$$

Since $w \in H^{1+\alpha}_{p,e}$ and P_R is smooth and integrable, we have that $P_R * w \in H^{1+\alpha}_{p,e}$. Use now that $H^{1+\alpha}_{p,e} \subset Y = C^{0,\beta-\alpha}_{p,e}$ by Morrey's embedding, since $1+\alpha-1/2 = 1/2+\alpha > \beta-\alpha$ —recall (4.8).

Therefore, the right hand side of (5.7) belongs to $Y = C^{0,\beta-\alpha}_{p,e}$. By standard Hölder regularity for the fractional Laplacian —see Proposition 2.8 of [25]—, it follows that $w \in X = C^{1+\alpha,\beta}_{p,e}$.

6. Differentiability properties of the nonlinear operator acting on even periodic functions

To prove the differentiability properties of the operator $\overline{\Phi} = \overline{\Phi}(a, \lambda, v)$ stated in Proposition 4.2, it is sufficient to establish that

$$\Phi = \Phi(a, \lambda, \varphi) : (-\nu, \nu) \times (1/2, 3/2) \times B_{10}(0) \subset \mathbb{R} \times \mathbb{R} \times X \to Y$$

is of class C^1. Here one should recall that $\varphi = \cos(\cdot) + v$ and note that, by (4.11), $\|\cos(\cdot)\|_X \leq 1 + 1 + \pi < 9$.

We start studying the first term, Φ_1, of the operator —which is the most delicate. It turns out to be a nonlinear version of the fractional Laplacian $(-\Delta)^{\frac{1+\alpha}{2}}$. More precisely, by expression (6.2) below, it is a quasilinear version of the fractional Laplacian — the second order increments in (6.2) are multiplied by a nonlinear "coefficient" F_3 depending only on the first order increments. Expression (6.2) will be most useful to deduce all properties of Φ_1.

Instead, Φ_2, that we study later on in this section, is a nonlinear and nonlocal zero order operator —recall that is given by (4.16). For instance, we will see that Φ_2 sends the space of Lipschitz functions into itself.

We start studying Φ_1, given by (4.14)-(4.15),

$$\Phi_1(a, \varphi) := \int_{\mathbb{R}} F_1(a, \delta_-\varphi)\, d\mu(t), \tag{6.1}$$

where

$$F_1(a, q) := \int_0^q \frac{d\tau}{(1 + a^2\tau^2)^{\frac{2+\alpha}{2}}}.$$

Lemma 6.1. *The operator $\Phi_1 = \Phi_1(a, \varphi) : \mathbb{R} \times X \to Y$ is of class C^1.*

Proof. Changing t by $-t$ in (6.1) and using that $\delta_-\varphi(s, -t) = \delta_+\varphi(s, t)$ (recall (4.13) for the definition of δ_+), we have

$$\Phi_1(a, \varphi) = \frac{1}{2} \int_{\mathbb{R}} \{F_1(a, \delta_-\varphi) + F_1(a, \delta_+\varphi)\}\, d\mu(t).$$

Denoting

$$q = \delta_-\varphi \quad \text{and} \quad p = \delta_+\varphi,$$

we have

$$\begin{aligned} F_1(a, q) + F_1(a, p) &= \left\{\int_0^q + \int_0^p\right\} \frac{d\tau}{(1 + a^2\tau^2)^{\frac{2+\alpha}{2}}} = \left\{\int_0^q + \int_{-p}^0\right\} \frac{d\tau}{(1 + a^2\tau^2)^{\frac{2+\alpha}{2}}} \\ &= (q + p)\, \frac{1}{q + p} \int_{-p}^q \frac{d\tau}{(1 + a^2\tau^2)^{\frac{2+\alpha}{2}}}. \end{aligned}$$

Making the change of variables $\tau = -p + (q+p)\overline{\tau}$, we have that $F_1(a,q) + F_1(a,p) = (q+p)F_3(a,q,p)$, where

$$F_3(a,q,p) := \int_0^1 \frac{d\overline{\tau}}{\left\{1 + a^2\left(-p + (q+p)\overline{\tau}\right)^2\right\}^{\frac{2+\alpha}{2}}}.$$

Therefore, we conclude that

$$\Phi_1(a,\varphi) := \frac{1}{2}\int_{\mathbb{R}} (\delta_-\varphi + \delta_+\varphi)\ F_3(a, \delta_-\varphi, \delta_+\varphi)\ d\mu(t). \tag{6.2}$$

This is the expression that will be useful to establish the lemma.

We first collect several important inequalities for functions in X. For all $\varphi \in X$ and real numbers $s, \overline{s}$, and t, we have

$$|\delta_-\varphi(s,t)| + |\delta_+\varphi(s,t)| \le 2\|\varphi\|_X, \tag{6.3}$$

$$|(\delta_-\varphi + \delta_+\varphi)(s,t)| \le 2\|\varphi\|_X\, |t|^\beta, \tag{6.4}$$

and

$$|\delta_\mp\varphi(s,t) - \delta_\mp\varphi(\overline{s},t)| \le \|\varphi\|_X\, |s-\overline{s}|\, |t|^{\beta-1}. \tag{6.5}$$

Inequalities (6.4) and (6.5) are the crucial point to prove that Φ_1 and its derivatives send functions in X to functions in Y. This will be accomplished with the inequalities following (6.7) —an argument in Hölder spaces already used in [25] for the pure fractional Laplacian.

While (6.3) is obvious, (6.4) is easily proved as follows:

$$\begin{aligned}
|\varphi(s) - \varphi(s-t) + \varphi(s) - \varphi(s+t)| &= \left| -\int_0^1 \frac{d}{d\rho}\varphi(s-\rho t)\, d\rho - \int_0^1 \frac{d}{d\rho}\varphi(s+\rho t) d\rho \right| \\
&= \left| \int_0^1 \left\{\varphi'(s-\rho t) - \varphi'(s+\rho t)\right\} t\, d\rho \right| \\
&\le \int_0^1 \|\varphi\|_X\, |2\rho t|^\beta |t|\, d\rho \le 2\|\varphi\|_X\, |t|^{\beta+1}.
\end{aligned}$$

Similarly, we can establish (6.5), as follows:

$$\begin{aligned}
&\left| \frac{\varphi(s) - \varphi(s-t)}{|t|} - \frac{\varphi(\overline{s}) - \varphi(\overline{s}-t)}{|t|} \right| \\
&\quad = \left| \int_0^1 \frac{d}{d\rho}\, \frac{\varphi\left(\rho s + (1-\rho)\overline{s}\right) - \varphi\left(\rho s + (1-\rho)\overline{s} - t\right)}{|t|}\, d\rho \right| \\
&\quad = \left| (s-\overline{s}) \int_0^1 \frac{\varphi'\left(\rho s + (1-\rho)\overline{s}\right) - \varphi'\left(\rho s + (1-\rho)\overline{s} - t\right)}{|t|}\, d\rho \right| \\
&\quad \le \|\varphi\|_X\, |s-\overline{s}|\, |t|^{\beta-1}.
\end{aligned}$$

The same bound for $\delta_+\varphi$ follows from $\delta_+\varphi(s,t) = \delta_-\varphi(s,-t)$.

We can now check that the integrand in (6.2) is an integrable function, and thus $\Phi_1(a,\varphi)$ is well defined by (6.2). That the function is integrable in $\{|t| \ge 1\}$ follows

from (6.3) and $0 \le F_3 \le 1$ —recall that $d\mu(t) = |t|^{-1-\alpha}\,dt$. Instead for the integral in $(-1,1)$, we use (6.4), $0 \le F_3 \le 1$, and that $\int_{-1}^{1} |t|^{\beta-1-\alpha}\,dt < \infty$.

We now verify that $\Phi_1(a,\varphi) \in Y$ for all $\varphi \in X$. We add and subtract a term to have

$$2\left\{\Phi_1(a,\varphi)(s) - \Phi_1(a,\varphi)(\overline{s})\right\} = \int_{\mathbb{R}} i_1(s,t)\,d\mu(t) + \int_{\mathbb{R}} i_2(s,t)\,d\mu(t),$$

where

$$i_1(s,t) = \left\{(\delta_-\varphi + \delta_+\varphi)(s,t) - (\delta_-\varphi + \delta_+\varphi)(\overline{s},t)\right\} F_3\left(a, \delta_-\varphi(s,t), \delta_+\varphi(s,t)\right)$$

and

$$i_2(s,t) = (\delta_-\varphi + \delta_+\varphi)(\overline{s},t)\left\{F_3\left(a, \delta_-\varphi(s,t), \delta_+\varphi(s,t)\right) - F_3\left(a, \delta_-\varphi(\overline{s},t), \delta_+\varphi(\overline{s},t)\right)\right\}.$$

We claim that

$$(|i_1| + |i_2|)(s,t) \le C \min(|s-\overline{s}|\,|t|^{\beta-1}, |t|^{\beta}), \tag{6.6}$$

where C is a constant depending only on an upper bound for $|a| + \|\varphi\|_X$. Indeed, for i_1, this follows from (6.4), (6.5), and $0 \le F_3 \le 1$. For i_2, the bound by $|t|^\beta$ follows from (6.4) and $0 \le F_3 \le 1$. The first bound for $|i_2|$ follows from (6.3) and the use of the intermediate value formula to express the difference of values for F_3 in the second factor in i_2. Here we also use that $(|\partial_q F_3| + |\partial_p F_3|)(a,q,p) \le Ca^2(|q|+|p|) \le C$ —the last inequality by (6.3).

Using (6.6), we can prove that $\Phi_1(a,\varphi) \in Y$. Indeed, an L^∞ bound for $\Phi_1(a,\varphi)$ has already been discussed above. Now

$$\begin{aligned} |\Phi_1(a,\varphi)(s) - \Phi_1(a,\varphi)(\overline{s})| &\le C\int_{\mathbb{R}} (|i_1| + |i_2|)\,d\mu(t) \qquad (6.7)\\ &\le C\int_{-|s-\overline{s}|}^{|s-\overline{s}|} |t|^\beta \frac{dt}{|t|^{1+\alpha}} + C\int_{\{|t|\ge|s-\overline{s}|\}} |s-\overline{s}||t|^{\beta-1}\frac{dt}{|t|^{1+\alpha}}\\ &= C|s-\overline{s}|^{\beta-\alpha} + C|s-\overline{s}|\,|s-\overline{s}|^{\beta-\alpha-1} = C|s-\overline{s}|^{\beta-\alpha}. \end{aligned}$$

Next we check that Φ_1 is differentiable with respect to a and that $D_a\Phi_1$ is continuous from $\mathbb{R}\times X$ to Y. From (6.2) we have that

$$D_a\Phi_1(a,\varphi) = \frac{1}{2}\int_{\mathbb{R}} (\delta_-\varphi + \delta_+\varphi)\,\partial_a F_3(a, \delta_-\varphi, \delta_+\varphi)\,d\mu(t). \tag{6.8}$$

This expression has exactly the same form as the one for $\Phi_1(a,\varphi)$, but with F_3 replaced by $\partial_a F_3$. As for F_3, note that $\partial_a F_3$ is a smooth and bounded function, since $|q|+|p| = |\delta_-\varphi| + |\delta_+\varphi| \le C$ by (6.3). Thus, as before, $D_a\Phi_1$ is well defined and belongs to Y.

To prove that $D_a\Phi_1$ is continuous from $\mathbb{R}\times X$ to Y, we take a sequence (a_k,φ_k) converging in $\mathbb{R}\times X$ to (a,φ). We need to bound

$$(D_a\Phi_1(a_k,\varphi_k))(s) - (D_a\Phi_1(a,\varphi))(s) - (D_a\Phi_1(a_k,\varphi_k))(\overline{s}) + (D_a\Phi_1(a,\varphi))(\overline{s}).$$

To do this, we need to add and subtract a good number of terms, and thus it is convenient to write the above expression in a more compact (or "symbolic") way.

We write the integrand in (6.8) as the expression $G\phi(s)H\phi(s)$, where $\phi = (a, \varphi)$, $\phi_k = (a_k, \varphi_k)$, t is given,

$$G\phi(s) := (\delta_-\varphi + \delta_+\varphi)(s,t),$$

and

$$H\phi(s) := \partial_a F_3(a, \delta_-\varphi, \delta_+\varphi)(s,t).$$

We must bound

$$G\phi(s)H\phi(s) - G\phi_k(s)H\phi_k(s) - G\phi(\overline{s})H\phi(\overline{s}) + G\phi_k(\overline{s})H\phi_k(\overline{s}).$$

We write this expression as

$$\begin{aligned}
&G\phi(s)(H\phi - H\phi_k)(s)\\
&+(G\phi - G\phi_k)(s)H\phi_k(s)\\
&-G\phi(\overline{s})(H\phi - H\phi_k)(\overline{s})\\
&-(G\phi - G\phi_k)(\overline{s})H\phi_k(\overline{s}).
\end{aligned}$$

We add and subtract one term in the first and third lines (and another one in the second and fourth lines) to have this equal to

$$\begin{aligned}
&(G\phi(s) - G\phi(\overline{s}))(H\phi - H\phi_k)(s)\\
&+(G\phi - G\phi_k)(s)(H\phi_k(s) - H\phi_k(\overline{s}))\\
&+G\phi(\overline{s})((H\phi - H\phi_k)(s) - (H\phi - H\phi_k)(\overline{s}))\\
&+((G\phi - G\phi_k)(s) - (G\phi - G\phi_k)(\overline{s}))H\phi_k(\overline{s}).
\end{aligned}$$

Now, every of these four lines can be controlled in absolute value by the bound

$$c(k)\min(|s-\overline{s}|\,|t|^{\beta-1}, |t|^{\beta}),$$

where the constant $c(k) \to 0$ as $k \to \infty$. This is done in the way explained before, right after (6.6), taking care now to control the smallness of terms like

$$|(H\phi - H\phi_k)(s)| = |\partial_a F_3(a, \delta_-\varphi, \delta_+\varphi)(s,t) - \partial_a F_3(a_k, \delta_-\varphi_k, \delta_+\varphi_k)(s,t)|$$

with the intermediate value theorem. Finally, we can integrate in t and proceed as in (6.7) (and the inequalities following it), to deduce that $D_a\Phi_1(a_k, \varphi_k)$ converges to $D_a\Phi_1(a, \varphi)$ in Y.

Finally, we need to prove the C^1 character of Φ_1 with respect to the φ variable. We have that

$$\begin{aligned}
2D_\varphi\Phi_1(a,\varphi)\,\psi = &\int_{\mathbb{R}} (\delta_-\psi + \delta_+\psi)\, F_3(a, \delta_-\varphi, \delta_+\varphi)\, d\mu(t)\\
&+ \int_{\mathbb{R}} (\delta_-\varphi + \delta_+\varphi)\, \{\partial_q F_3(a, \delta_-\varphi, \delta_+\varphi)\delta_-\psi + \partial_p F_3(a, \delta_-\varphi, \delta_+\varphi)\delta_+\psi\}\, d\mu(t).
\end{aligned} \tag{6.9}$$

Recall that $q = \delta_-\varphi$, $p = \delta_+\varphi$, $\delta_-\psi$, and $\delta_+\psi$ are all bounded, by (6.3). Note that $\partial_q F_3$, $\partial_p F_3$, $\partial_{qq} F_3$, $\partial_{qp} F_3$, and $\partial_{pp} F_3$ are all smooth and bounded functions. Now, to control $|(D_\varphi\Phi_1(a,\varphi)\,\psi)(s) - (D_\varphi\Phi_1(a,\varphi)\,\psi)(\overline{s})|$, we proceed as we did above for

$|\Phi_1(a,\varphi)(s) - \Phi_1(a,\varphi)(\overline{s})|$ adding and subtracting one auxiliary term. In this way we see that $D_\varphi \Phi_1(a,\varphi)\,\psi \in Y$.

The continuity of $D_\varphi \Phi_1$ as a function of (a,φ) with values in the space of bounded linear operators from X to Y is proved in a similar way to the one above for the continuity of $D_a\Phi_1$. Here we must look at

$$(D_\varphi \Phi_1(a_k,\varphi_k)\,\psi)(s) - (D_\varphi \Phi_1(a,\varphi)\,\psi)(s) - (D_\varphi \Phi_1(a_k,\varphi_k)\,\psi)(\overline{s}) + (D_\varphi \Phi_1(a,\varphi)\,\psi)(\overline{s}),$$

use expression (6.9), and add and subtract terms as above for $D_a\Phi_1$. The first integral (6.9) is easier to deal with, while the second has the same structure as in $D_a\Phi_1$. □

Next, we deal with the second term Φ_2 in the operator Φ. Recall that it is given by expression (4.16)-(4.17),

$$\Phi_2(a,\lambda,\varphi) = \int_{\mathbb{R}} \delta_0\varphi\; F_2(t,a,\lambda,\delta_0\varphi)\,dt, \tag{6.10}$$

where

$$F_2(t,a,\lambda,r) = \int_0^1 \frac{d\overline{\tau}}{\{t^2 + (2\lambda R + ar\overline{\tau})^2\}^{\frac{2+\alpha}{2}}}. \tag{6.11}$$

With this expression at hand, we prove our last lemma.

Lemma 6.2. *There exists $\nu > 0$ small enough (depending only on α) for which the operator $\Phi_2 = \Phi_2(a,\lambda,\varphi) : (-\nu,\nu) \times (1/2,3/2) \times B_{10}(0) \subset \mathbb{R}\times\mathbb{R}\times X \to Y$ is of class C^1.*

Proof. Recall that $R > 0$ has been chosen to depend only on α. We take $|a| \le \nu$, with ν small depending on how large is $R = R(\alpha)$. Since $|r| = |\delta_0\varphi| \le 2\|\varphi\|_\infty \le 20$, $\lambda \in (1/2,3/2)$, and $\overline{\tau} \in [0,1]$, in the definition (6.11) of F_2 we have

$$R/2 \le R - 20a \le 2\lambda R + ar\overline{\tau} \le 3R + 20a \le 4R.$$

It follows that F_2 is a smooth, positive, and bounded function of $(t,a,\lambda,r) \in \mathbb{R}\times(-\nu,\nu)\times(1/2,3/2)\times(-20,20)$. In addition, we have that

$$\lim_{|t|\to\infty} \frac{F_2(t,a,\lambda,r)}{|t|^{-2-\alpha}} = 1 \tag{6.12}$$

uniformly as (a,λ,r) belong to the above sets.

It follows that the integrand in (6.10) is integrable, and hence Φ_2 is well defined.

Next, we see that $\Phi_2(a,\lambda,\varphi) \in Y$ and has Y-norm controlled by an upper bound on $|a| + \lambda + \|\varphi\|_X$. Indeed, we can see even more: that $\Phi_2(a,\lambda,\varphi)$ is Lipschitz if φ is Lipschitz, and $\Phi_2(a,\lambda,\varphi)$ has Lipschitz norm controlled by an upper bound on $|a| + \lambda + \|\varphi\|_{\mathrm{Lip}}$. To verify this, the L^∞ norm of $\Phi_2(a,\lambda,\varphi)$ has already been treated above. Next, we look at $\Phi_2(a,\lambda,\varphi)(s) - \Phi_2(a,\lambda,\varphi)(\overline{s})$, add and subtract the term $\int_{\mathbb{R}} \delta_0\varphi(\overline{s},t)\,F_2(t,a,\lambda,\delta_0\varphi(s,t))\,dt$, use that F_2 and $\partial_r F_2$ are bounded, use the intermediate value formula for the last two terms, and that

$$|\delta_0\varphi(s,t) - \delta_0\varphi(\overline{s},t)| \le 2\|\varphi\|_{\mathrm{Lip}}|s-\overline{s}|$$

to conclude the desired bound. For the convergence of the integrals, use (6.12) and that $|\partial_r F_2| \leq C|t|^{-3-\alpha}$ for $|t|$ large.

Finally, we must prove that Φ_2 is C^1 with respect to (a, λ, φ) with values in Y. For this, we compute $D_a\Phi_2$, $D_\lambda\Phi_2$, and $D_\varphi\Phi_2$ at (a, λ, φ) as we did for Φ_1 in the proof of Lemma 6.1. The same argument as in the previous paragraph gives that these functions are Lipschitz and have their Lipschitz norm controlled by an upper bound on $|a| + \lambda + \|\varphi\|_{\mathrm{Lip}}$. From this, it follows the continuity of $D_a\Phi_2$, $D_\lambda\Phi_2$, and $D_\varphi\Phi_2$ as functions of (a, λ, φ) with values in Y. Indeed, if $(a_k, \lambda_k, \varphi_k) \to (a, \lambda, \varphi)$ in $\mathbb{R}\times\mathbb{R}\times X$ then, by the previous bounds, we have that the sequence $D_{(a,\lambda,\varphi)}\Phi_2(a_k, \lambda_k, \varphi_k)$ is uniformly bounded in the Lipschitz norm. By Arzelà-Ascoli theorem, a subsequence converges in the weaker Hölder norm of Y to a function in Y. This function must be $D_{(a,\lambda,\varphi)}\Phi_2(a, \lambda, \varphi)$, since the integrals in the expressions for $D_{(a,\lambda,\varphi)}\Phi_2(a_k, \lambda_k, \varphi_k)$ convergence to the corresponding integrals for (a, λ, φ). This follows from the dominated convergence theorem. As a consequence, the full sequence converges in Y to $D_{(a,\lambda,\varphi)}\Phi_2(a, \lambda, \varphi)$. □

References

[1] N. Abatangelo, E. Valdinoci, A notion of nonlocal curvature, *Numer. Funct. Anal. Optim.* **35**, (2014), 793–815.

[2] A. D. Alexandrov, A characteristic property of spheres, *Ann. Mat. Pura Appl.*, **58** (1962), 303–315.

[3] A. Ambrosetti, G. Prodi, A primer of nonlinear analysis, Cambridge Studies in Advanced Mathematics, 34, Cambridge University Press, Cambridge, 1995.

[4] B. Barrios, A. Figalli, E. Valdinoci, Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces, arXiv:1202.4606.

[5] X. Cabré, M. Fall, J. Solà-Morales, T. Weth, Delaunay hypersurfaces with constant nonlocal mean curvature, forthcoming.

[6] X. Cabré, J. Solà-Morales, Periodic solutions of fractional semilinear elliptic problems, forthcoming.

[7] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, *Comm. Pure Appl. Math.* **63** (2010), 1111–1144.

[8] L. Caffarelli, E. Valdinoci, Uniform estimates and limiting arguments for nonlocal minimal surfaces, *Calc. Var. Partial Differential Equations* **41** (2011), 203–240.

[9] L. Caffarelli, E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments, arXiv:1105.115.

[10] G. Ciraolo, A. Figalli, F. Maggi, M. Novaga, Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature, Preprint 2015.

[11] M. G. Crandall, P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Functional Analysis* **8** (1971), 321–340.

[12] A.-L. Dalibard, D. Gerard-Varet, On shape optimization problems involving the fractional laplacian, *ESAIM. Control, Optimisation and Calculus of Variations* **19** (2013), 976–1013.

[13] J. Dávila, M. del Pino, J. Wei, Concentrating standing waves for the fractional nonlinear Schrödinger equation, *J. Differential Equations* **256** (2014), 858–892.

[14] J. Dávila, M. del Pino, J. Wei, Nonlocal s-minimal surfaces and Lawson cones, arXiv:1402.4173.

[15] J. Dávila, M. del Pino, S. Dipierro, E. Valdinoci, Nonlocal Delaunay surfaces, arXiv:1501.07459.

[16] Ch. Delaunay, Sur la surface de révolution dont la courbure moyenne est constante, *J. Math. Pures Appl. 1re. série* **6** (1841), 309–315.

[17] S. Dipierro, M. Medina, I. Peral, E. Valdinoci, Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^n$, arXiv:1410.3076.

[18] M. M. Fall, S. Jarohs, Overdetermined problems with fractional Laplacian, arXiv:1311.7549, to appear in *ESAIM. Control, Optimisation and Calculus of Variations*.

[19] A. Figalli, E. Valdinoci, Regularity and Bernstein-type results for nonlocal minimal surfaces, arXiv:1307.0234.

[20] A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini, Isoperimetry and stability properties of balls with respect to nonlocal energies, *Communications in Mathematical Physics*, December 2014.

[21] L. E. Fraenkel, An introduction to maximum principles and symmetry in elliptic problems, Cambridge Tracts in Mathematics 128, Cambridge University Press, Cambridge, 2000.

[22] O. Savin, E. Valdinoci, Regularity of nonlocal minimal cones in dimension 2, *Calc. Var. and PDEs* **48** (2012), 33–39.

[23] F. Schlenk, P. Sicbaldi, Bifurcating extremal domains for the first eigenvalue of the Laplacian, *Adv. Math.* **229** (2012), 602–632.

[24] P. Sicbaldi, New extremal domains for the first eigenvalue of the Laplacian in flat tori, *Calc. Var. and PDEs* **37** (2010), 329–344.

[25] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), 67–112.

[26] E. Valdinoci, A fractional framework for perimeters and phase transitions, *Milan J. Math.* **81** (2013), 1–23.

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