On isolated singularities of Kirchhoff–type Laplacian problems

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Abstract

In this paper, we study isolated singular positive solutions for the following Kirchhoff–type Laplacian problem:

$$-\left(\theta + \int_{\Omega} |\nabla u| dx\right) \Delta u = u^p \quad \text{in} \quad \Omega \setminus \{0\}, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

where $p > 1$, $\theta \in \mathbb{R}$, Ω is a bounded smooth domain containing the origin in $\mathbb{R}^N$ with $N \geq 2$. In the subcritical case: $1 < p < N/(N-2)$ if $N \geq 3$, $1 < p < +\infty$ if $N = 2$, we employ the Schauder fixed-point theorem to derive a sequence of positive isolated singular solutions for the above problem such that $M_\theta(u) > 0$. To estimate $M_\theta(u)$, we make use of the rearrangement argument. Furthermore, we obtain a sequence of isolated singular solutions such that $M_\theta(u) < 0$, by analyzing relationship between the parameter λ and the unique solution u_λ of

$$-\Delta u + \lambda u^p = k\delta_0 \quad \text{in} \quad B_1(0), \qquad u = 0 \quad \text{on} \quad \partial B_1(0).$$

In the supercritical case: $N/(N-2) \leq p < (N+2)/(N-2)$ with $N \geq 3$, we obtain two isolated singular solutions u_i with $i = 1, 2$ such that $M_\theta(u_i) > 0$ under some appropriate assumptions.

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1. Introduction and main results

A model with small variation of tension due to the changes of the length of a string is described by D'Alembert wave equation, it is also well-known as the Kirchhoff equation, see [15], which states as follows

$$m \frac{\partial^2 u}{\partial t^2} - \left[\tau_0 + \frac{\kappa}{2L_0} \int_\alpha^\beta \left|\frac{\partial u}{\partial x}\right|^2 dx\right] \frac{\partial^2 u}{\partial x^2} = 0,$$

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where τ_0 is the tension, $L_0 = \beta - \alpha$ is the length of the string at rest, m is the mass density, κ is the Young's modulus. The Kirchhoff–type problems have been attracted great attentions in the analysis of different nonlinear term due to the gradient term, see [9, 11, 24, 35].

Observe that in the prototype of Kirchhoff model, the tension, for small deformations of the string, takes the linear form as follows:

$$M(u) = a + b \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx, \tag{1.1}$$

where $a > 0$, $b > 0$. When the displacement gradient is small, i.e. $|\nabla u| \ll 1$, $M(u) \backsim a + b|\Omega| + \frac{b}{2} \int_\Omega |\nabla u|^2 dx$. The advantage for this approximation makes the problem have variational structure and the approximating solution could be constructed by variational methods. For example, the stationary analogue and qualitative properties of solutions to the Kirchhoff–type equation

$$-\left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x,u) \quad \text{in} \quad \Omega$$

has been extensively studied in [9, 10, 14, 19, 16, 28] and extended into the fractional setting in [25, 26, 31] and the references therein. In this case, $M(u) = a + b \int_\Omega |\nabla u|^2 dx$ is often called Kirchhoff function. In fact, the Kirchhoff function has been greatly extended for recent years. For example, the case $a = 0, b > 0$, which is called degenerate, has been intensely investigated recently, we refer to [32] for a physical explanation and [8, 38] for related results in this direction.

Our interest of this paper is to study a new Kirchhoff–type problem by taking into account that $|\nabla u|$ is not small in a bounded smooth domain Ω and the tension could be vector in a proper coordinate axis. In this situation, the Kirchhoff function (1.1) may be taken as

$$M_\theta(u) = \theta + \int_\Omega |\nabla u| dx, \tag{1.2}$$

where θ is assumed to be real number. Given a sequence of extra pressures $\{\sigma_m\}$ with the support in $B_{\frac{1}{m}}(0)$ and the total force $F = \int_\Omega \sigma_m dx = 1$ keeps invariant. The limit of $\{\sigma_m\}$ as $m \to +\infty$ in the distributional sense is Dirac mass. As we know that the corresponding solutions may blow up at the origin or blow up in the whole domain. Our aim is to clarify this limit phenomena of the solutions to some elliptic problems involving the Kirchhoff–type function (1.2).

More precisely, in this article we are interested in nonnegative singular solutions of the following Kirchhoff–type equation:

$$\begin{cases} -M_\theta(u)\Delta u = u^p & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where $p > 1$, M_θ is defined by (1.2) with $\theta \in \mathbb{R}$ and Ω is a bounded, smooth domain containing the origin in $\mathbb{R}^N$ with $N \geq 2$. The following parameter plays an important role in obtaining the solutions of (1.3):

$$a_p = \sup_{x \in \Omega} \frac{w_1}{w_0}, \tag{1.4}$$

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where $w_0 = \mathbb{G}_\Omega[\delta_0]$ and $w_1 = \mathbb{G}_\Omega[w_0^p]$, $\mathbb{G}_\Omega$ is Green operator defined as

$$\mathbb{G}_\Omega[u](x) = \int_\Omega G_\Omega(x,y)u(y)dy,$$

here G_Ω is the Green kernel of $-\Delta$ in $\Omega \times \Omega$ with zero Dirichlet boundary condition. Note that a_p is well-defined when p is subcritical, that is, $p < p^*$, where

$$p^* = \begin{cases} \dfrac{N}{N-2} & \text{if} \quad N \geq 3, \\ +\infty & \text{if} \quad N = 2. \end{cases} \tag{1.5}$$

Our first existence result about isolated singular solutions with $M_\theta(u) > 0$ is stated as follows.

Theorem 1.1. *Assume that $N \geq 2$, M_θ is defined by (1.2) with $\theta \in \mathbb{R}$, a_p is given by (1.4), p^* is given by (1.5) and Ω is a bounded smooth domain containing the origin such that*

$$B_1(0) \subset \Omega \quad \text{and} \quad |B_{r_0}(0)| = |\Omega|$$

where $1 \leq r_0 < +\infty$.
   Let $k > r_0\theta_-$ with $\theta_- := \min\{0,\theta\}$ be such that

$$\frac{k^{p-1}}{\theta + r_0^{-1}k} \leq \frac{1}{a_p p}\left(\frac{p-1}{p}\right)^{p-1}. \tag{1.6}$$

Then for $p \in (1, p^*)$, problem (1.3) has a nonnegative solution u_k satisfying that

$$M_\theta(u_k) \geq \theta + r_0^{-1}k > 0 \tag{1.7}$$

and u_k has following asymptotic behaviors at the origin

$$\lim_{|x|\to 0^+} u_k(x)\Phi^{-1}(x) = c_N k, \tag{1.8}$$

where $c_N > 0$ is the normalized constant and

$$\Phi(x) = \begin{cases} |x|^{2-N} & \text{if} \quad N \geq 3, \\ -\ln|x| & \text{if} \quad N = 2. \end{cases}$$

Furthermore, u_k is a distributional solution of

$$\begin{cases} -\Delta u = \dfrac{u^p}{M_\theta(u)} + k\delta_0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.9}$$

where δ_0 is Dirac mass concentrated at the origin.

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Remark 1.1. Note that a_p depends on p and Ω and the value $p = 2$ is critical for assumption (1.6) for $N = 2, 3$. Indeed, $p^* > 2$ occurs only for $N = 2$ and $N = 3$. Due to the parameter θ, (1.6) gives a rich structure of isolated singular solutions for problem (1.3). Moreover, a discussion is put in Proposition 2.2 in Section 2.

Involving Kirhchoff function $M_\theta(u)$, the classical method of Lions' iteration argument in [17] does not work due to the lack of monotonicity of nonlinearity $M_\theta^{-1}(u)u^p$, and also the variational method in [27] fails, since (1.3) has no variational structure. Furthermore, it is difficult to calculate precise value for $\int_\Omega |\nabla u_k|$ to express $M_\theta(u_k)$, especially, when Ω is a general bounded domain. To overcome these difficulties, we make use of the rearrangement argument to estimate the value of $M_\theta(u)$ and employ the Schauder fixed-point theorem to obtain the existence of isolated singular solutions in the class set of $M_\theta(u) > 0$.

When $\theta < 0$, we can derive a branch of singular solutions such that $M_\theta(u) < 0$.

Theorem 1.2. (i) Let $N \geq 2$, $p \in (1, p^*)$, $\theta < 0$ and $\Omega = B_1(0)$. For $k \in (0, -\theta)$, problem (1.3) has a nonnegative solution u_k, which is a distributional solution of (1.9) with $\Omega = B_1(0)$, satisfying that
$$\theta < M_\theta(u_k) < k + \theta < 0$$
and u_k has the asymptotic behavior (1.8).

(ii) Let $N \geq 2$, $p \in (\frac{N+1}{N-1}, p^*)$, $\theta < 0$ and Ω is a bounded smooth domain containing the origin. Then problem (1.3) has a nonnegative solution u_p, which is not a distributional solution of (1.9), satisfying that
$$M_\theta(u_p) < 0$$
and u_p has the asymptotic behavior
$$\lim_{|x|\to 0^+} u_p(x)|x|^{\frac{2}{p-1}} = c_p(-M_\theta(u_p))^{\frac{1}{p-1}},$$
where $c_p = \left[\frac{2}{p-1}\left(\frac{2}{p-1} + 2 - N\right)\right]^{\frac{1}{p-1}}$.

For $M_\theta(u) < 0$, problem (1.3) could be written as
$$-\Delta u + \lambda u^p = 0 \quad \text{in} \quad B_1(0) \setminus \{0\}, \quad u = 0 \quad \text{on} \quad \partial B_1(0), \tag{1.10}$$

where $\lambda = -M_\theta^{-1}(u) > 0$. For $\lambda = 1$, the nonlinearity in problem (1.10) is an absorption and Lions showed in [17] that it is always studied by considering the very weak solutions of
$$-\Delta u + \lambda u^p = k\delta_0 \quad \text{in} \quad B_1(0). \tag{1.11}$$

Véron in [37] gave a survey on the isolated singularities of (1.10), in which $B_1(0)$ is replaced by general bounded domain containing the origin. With a general Radon measure and a more general absorption nonlinearity $g : \mathbb{R} \to \mathbb{R}$ satisfies the subcritical assumption:
$$\int_1^{+\infty} (g(s) - g(-s))s^{-1-p^*}ds < +\infty,$$

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problem (1.11) has been studied by Benilan-Brézis [1], Brézis [3], by approximating the measure by a sequence of regular functions, and find classical solutions which converges to a weak solution. For this approach to work, uniform bounds for the sequence of classical solutions are necessary to be established. The uniqueness is then derived by Kato's inequality. Such a method has been applied to solve equations with boundary measure data in [13, 20, 21, 22] and other extensions in [2, 5].

In the case $\lambda = -M_\theta^{-1}(u)$, depending on the unknown function u, a different approach has to be taken into account to study problem (1.11). A branch of solutions such that $M_\theta(u) < 0$ are derived from the observations that the function $F(\lambda) = -M_\theta^{-1}(u) - \lambda$ is continuous and it has a zero, because we will find two values λ_1, λ_2 such that $F(\lambda_1)F(\lambda_2) < 0$, where v_λ is the unique solution of problem (1.11). This zero indicates a solution of problem (1.3).

For the singularity as $|x|^{-2/(p-1)}$, the diffusion and the nonlinear terms play the predominant roles in (1.3), so we just consider λu_p, where u_p is the solution of $-\Delta u + u^p = 0$ in $\Omega \setminus \{0\}$. By scaling λ to meet the Kirchhoff function and then a solution with this type singularity is derived in Theorem 1.2. This scaling technique could be extended to obtain solutions in the supercritical case in Theorem 5.1 in Section 5.

It is worth pointing out that the method of searching solutions with the weak singularities as Φ in Theorem 1.1 could be extended into dealing with general nonlinearity $f(u)$ when $0 \leq f(u) \leq c|u|^p$ with $p \in (1, p^*)$. This method to prove Theorem 1.2 is based on the homogeneous property of nonlinearity and when the nonlinearity is not a power function, it is open but challenging to obtain solutions with such isolated singularity.

The rest of this paper is organized as follows. In Section 2, we introduce the very weak solution of equation (1.3) involving Dirac mass and give a discussion of (1.6). Section 3 is devoted to show the existence of a solution to (1.3) with $M_\theta(u) > 0$ in Theorem 1.1. In Section 4, we search the solutions of (1.3) with $M_\theta(u) < 0$ in Theorem 1.2. The supercritical case: $N/(N-2) \leq p < (N+2)/(N-2)$ with $N \geq 3$, is considered in Section 5, and we obtain there multiple isolated singular solutions of (1.3) such that $M_\theta(u_i) > 0$.

2. Preliminary

2.1. Kirchhoff-type problem with Dirac mass

In order to drive solutions of (1.3) with singularity (1.8), it is always transformed into finding solutions of (1.9). A function u is said to be a super (resp. sub) distributional solution of (1.9), if $u \in L^1(\Omega)$, $|\nabla u| \in L(\Omega)$, $u^p \in L^1(\Omega, \rho dx)$ and

$$\int_\Omega \left[u(-\Delta)\xi - \frac{u^p}{M_\theta(u)}\xi \right] d\mu \geq \ (\text{resp. } \leq) \ k\xi(0), \quad \forall \xi \in C_0^{1.1}(\Omega), \ \xi \geq 0, \qquad (2.1)$$

where $\rho(x) = \text{dist}(x, \partial\Omega)$. A function u is a distributional solution of (1.9) if u is both super and sub distributional solutions of of (1.9).

Next we build the connection between the singular solutions of (1.3) and the distributional solutions of (1.9).

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Theorem 2.1. *Assume that $N \geq 2$, $p > 1$ and $u \in L^1(\Omega)$ is a nonnegative classical solution of problem (1.3) satisfying that $M_\theta(u) \neq 0$ and $u^p \in L^1(\Omega, \rho dx)$. Then u is a very weak solution of problem (1.9) for some $k \geq 0$. Furthermore,*

Case 1: $M_\theta(u) < 0$.
(i) For $N \geq 3$, $p \geq p^*$, problem (1.3) only has zero solution and $\theta < 0$.
(ii) For $N \geq 2$, $1 < p < p^*$, we have that $k > 0$ and

$$\lim_{|x| \to 0^*} u(x) \Phi^{-1}(x) = c_N k. \tag{2.2}$$

Case 2: $M_\theta(u) > 0$.
(i) For $N \geq 3$, $p \geq p^*$, we have that $k = 0$ and

$$\lim_{|x| \to 0} u(x) |x|^{N-2} = 0;$$

(ii) Assume more that $1 < p < p^*$. If $k = 0$, then u is removable at the origin, and if $k > 0$, then u satisfies (1.8).

In order to prove Theorem 2.1, we need the following lemmas.

Lemma 2.1. *Let $\tau \in (0, N)$, then for $x \in B_{1/2}(0) \setminus \{0\}$,*

$$\mathbb{G}_\Omega[|\cdot|^{-\tau}](x) \leq \begin{cases} c_2 |x|^{-\tau+2} & \text{if} \quad \tau > 2, \\ -c_2 \log(|x|) & \text{if} \quad \tau = 2, \\ c_2 & \text{if} \quad \tau < 2. \end{cases} \tag{2.3}$$

For $N \geq 3$, $p \in (1, p^*)$, there holds

$$\mathbb{G}_\Omega[\mathbb{G}_\Omega^p[\delta_0]] \leq \begin{cases} c_2 |x|^{p(2-N)+2} & \text{if} \quad p \in (\frac{2}{N-2}, p^*), \\ -c_2 \log(|x|) & \text{if} \quad p = \frac{2}{N-2}, \\ c_2 & \text{if} \quad p < \frac{2}{N-2} \end{cases} \tag{2.4}$$

Proof. We follow the idea of Lemma 2.3 in [6]. In fact, from [2, Propsition 2.1] it follows that the Green kernel verifies that

$$G_\Omega(x, y) \leq c_N \Phi(x - y),$$

By direct computation, we get (2.3). Since $\lim_{|x| \to 0^+} \mathbb{G}_\Omega[\delta_0](x) \Phi^{-1}(x) \to c_N$, (2.3) with $\tau = (2 - N)p$ implies (2.4). $\qquad \square$ $\qquad \qquad \square$

Proposition 2.1. *([34] or [7, Propostion 5.1]) Let $h \in L^s(\Omega)$ with $s \geq 1$, then there exists $c_3 > 0$ such that*

(i)
$$\|\mathbb{G}_\Omega[h]\|_{L^\infty(\Omega)} \le c_3 \|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s} < \frac{2}{N}; \tag{2.5}$$

(ii)
$$\|\mathbb{G}_\Omega[h]\|_{L^r(\Omega)} \le c_3 \|h\|_{L^s(\Omega)} \quad \text{if} \quad \frac{1}{s} \le \frac{1}{r} + \frac{2}{N} \quad \text{and} \quad s > 1; \tag{2.6}$$

(iii)
$$\|\mathbb{G}_\Omega[h]\|_{L^r(\Omega)} \le c_3 \|h\|_{L^1(\Omega)} \quad \text{if} \quad 1 < \frac{1}{r} + \frac{2}{N}. \tag{2.7}$$

Proof of Theorem 2.1. For $M_\theta(u) \ne 0$, we rewrite (1.3) as

$$\begin{cases} -\Delta u = \dfrac{u^p}{M_\theta(u)} & \text{in} \quad \Omega \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.8}$$

Since $u^p \in L^1(\Omega, \rho dx)$ and $u \in L^1(\Omega)$, we may define the operator L by the following

$$L(\xi) := \int_\Omega \left[u(-\Delta)\xi - \frac{u^p}{M_\theta(u)}\xi \right] dx, \quad \forall \xi \in C_c^\infty(\mathbb{R}^N). \tag{2.9}$$

First we claim that for any $\xi \in C_c^\infty(\Omega)$ with the support in $\Omega \setminus \{0\}$,

$$L(\xi) = 0.$$

In fact, since $\xi \in C_c^\infty(\Omega)$ has the support in $\Omega \setminus \{0\}$, then there exists $r \in (0,1)$ such that $\xi = 0$ in $B_r(0)$ and then

$$L(\xi) = \int_{\Omega \setminus B_r(0)} \left[u(-\Delta)\xi - \frac{u^p}{M_\theta(u)}\xi \right] dx = \int_{\Omega \setminus B_r(0)} \left(-\Delta u - \frac{u^p}{M_\theta(u)} \right) \xi\, dx = 0.$$

From Theorem 1.1 in [4], it implies that

$$L = k\delta_0 \quad \text{for some} \quad k \ge 0, \tag{2.10}$$

that is,

$$L(\xi) = \int_\Omega \left[u(-\Delta)\xi - \frac{u^p}{M_\theta(u)}\xi \right] dx = k\xi(0), \quad \forall \xi \in C_c^\infty(\mathbb{R}^N). \tag{2.11}$$

Then u is a weak solution of (1.9) for some $k \ge 0$.

Case 1: $M_\theta(u) < 0$. We observe that

$$u = k\mathbb{G}_\Omega[\delta_0] - \frac{1}{-M_\theta(u)}\mathbb{G}_\Omega[u^q] \le k\mathbb{G}_\Omega[\delta_0],$$

then

$$k\mathbb{G}_\Omega[\delta_0] - \frac{k^p}{-M_\theta(u)}\mathbb{G}_\Omega[\mathbb{G}_\Omega[\delta_0]^p] \le u \le k\mathbb{G}_\Omega[\delta_0] \quad \text{in} \quad \Omega \setminus \{0\}.$$

So if $k = 0$, we obtain that $u \equiv 0$, which implies $M_\theta(u) = \theta < 0$; and if $k > 0$

$$\lim_{|x| \to 0^+} u(x) \Phi^{-1}(x) = c_N k.$$

We prove that $k = 0$ if $p \geq p^*$ with $N \geq 3$. By contradiction, if $k > 0$, then

$$u \geq (k/2)\Phi \quad \text{in} \quad B_{r_0}(0) \setminus \{0\},$$

which implies that

$$u^p(x) \geq (k/2)^p |x|^{(2-N)p}, \quad \forall x \in B_{r_0}(0) \setminus \{0\},$$

where $(2 - N)p \leq -N$ and $r_0 > 0$ is such that $B_{2r_0}(0) \subset \Omega$. A contradiction is obtained that $u^p \notin L^1(\Omega)$. Therefore, when $p \geq p^*$, there is no nontrivial nonnegative solution (1.3) such that $M_\theta(u) < 0$.

Case 2: $M_\theta(u) > 0$. We refer to [17] for the proof. For the reader's convenience, we give the details. When $p \in (1, N/(N-2))$ and $k = 0$, then

$$u = \frac{1}{M_\theta(u)} \mathbb{G}_\Omega[u^p].$$

We infer from $u^p \in L^{t_0}(\Omega)$ with $t_0 = \frac{1}{2}(1 + \frac{1}{p}\frac{N}{N-2}) > 1$ and Proposition 2.1 that $u \in L^{t_1 p}(\Omega)$ and $u^p \in L^{t_1}(\Omega)$ with

$$t_1 = \frac{1}{p}\frac{N}{N - 2t_0} t_0 > t_0.$$

If $t_1 > Np/2$, by Proposition 2.1, $u \in L^\infty(\Omega)$ and then it could be improved that u is a classical solution of

$$-\Delta u = \frac{1}{M_\theta(u)} u^p \quad \text{in} \quad \Omega. \tag{2.12}$$

If $t_1 < Np/2$, we proceed as above. By Proposition 2.1, $u \in L^{t_2 p}(\Omega)$, where

$$t_2 = \frac{1}{p}\frac{Nt_1}{N - 2t_1} > \frac{1}{p}\frac{N}{N - 2t_0} t_1 = \left(\frac{1}{p}\frac{N}{N - 2t_0}\right)^2 t_0.$$

Inductively, let us define

$$t_m = \frac{1}{p}\frac{Nt_{m-1}}{N - 2t_{m-1}} > \left(\frac{1}{p}\frac{N}{N - 2t_0}\right)^m t_0 \to +\infty \quad \text{as} \quad m \to +\infty.$$

Then there exists $m_0 \in \mathbb{N}$ such that

$$t_{m_0} > \frac{1}{2}Np$$

and by part (i) in Proposition 2.1,

$$u \in L^\infty(\Omega).$$

It then follows that u is a classical solution of (2.12).

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When $p \in (1, N/(N-2))$ and $k \neq 0$, we observe that

$$\lim_{x \to 0} \mathbb{G}_\Omega[\delta_0](x)|x|^{N-2} = c_{N,\alpha}$$

and

$$u = \frac{1}{M_\theta(u)} \mathbb{G}_\Omega[u^p] + k\mathbb{G}_\Omega[\delta_0]. \tag{2.13}$$

We let

$$u_1 = \frac{1}{M_\theta(u)} \mathbb{G}_\Omega[u^p] \quad \text{and} \quad \Gamma_0 = k\mathbb{G}_\Omega[\delta_0].$$

Then by Young's inequality,

$$u^p \leq 2^p \left(u_1^p + \Gamma_0^p \right). \tag{2.14}$$

By the definition of u_1 and (2.14), we obtain

$$u_1 \leq 2^p \mathbb{G}_\Omega[u_1^p] + \Gamma_1, \tag{2.15}$$

where $u_1 \in L^s(\Omega)$ for any $s \in (1, N/(N-2))$ and

$$\Gamma_1 = 2^p \mathbb{G}_\Omega[\Gamma_0^p].$$

Denoting $\mu_1 = 2 + (2-N)p$, then for $0 < |x| < 1/2$,

$$\Gamma_1(x) \leq \begin{cases} c_1|x|^{\mu_1} & \text{if} \quad \mu_1 < 0, \\ -c_1 \log|x| & \text{if} \quad \mu_1 = 0, \\ c_1 & \text{if} \quad \mu_1 > 0. \end{cases}$$

If $\mu_1 \leq 0$, letting

$$u_2 = 2^p \mathbb{G}_\Omega[u_1^p],$$

then $u_2 \in L^s(\Omega)$ with $s \in [1, \frac{N}{N-2})$, $u_1 \leq u_2 + \Gamma_1$ and

$$u_2 \leq 2^p \left(\mathbb{G}_\Omega[u_2^p] + \mathbb{G}_\Omega[\Gamma_1^p] \right).$$

Let $\mu_2 = \mu_1 p + 2$, then $\mu_2 > \mu_1$ and for $0 < |x| < \frac{1}{2}$,

$$\Gamma_2(x) := 2^p \mathbb{G}_\Omega[\Gamma_1^p](x) \leq \begin{cases} c_2|x|^{\mu_2} & \text{if} \quad \mu_2 < 0, \\ -c_2 \log|x| & \text{if} \quad \mu_2 = 0, \\ c_2 & \text{if} \quad \mu_2 > 0. \end{cases}$$

Inductively, we assume that

$$u_{n-1} \leq 2^p \mathbb{G}_\Omega[u_{n-1}^p] + 2^p \mathbb{G}_\Omega[\Gamma_{n-2}^p],$$

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where $u_{n-1} \in L^s(\Omega)$ for $s \in [1, N/(N-2))$, $\Gamma_{n-2}(x) \le |x|^{\mu_{n-2}}$ for $\mu_{n-2} < 0$.

Let
$$u_n = 2^p \mathbb{G}_\Omega[u_{n-1}^p], \qquad \Gamma_{n-1} = 2^p \mathbb{G}_\Omega[\Gamma_{n-2}^p],$$

and
$$\mu_{n-1} = \mu_{n-2} p + 2.$$

Then $u_n \in L^s(\Omega)$ for $s \in [1, N/(N-2))$ and for $0 < |x| < 1/2$,

$$\Gamma_{n-1}(x) := \mathbb{G}_\Omega[\Gamma_{n-2}^p](x) \le \begin{cases} c_n |x|^{\mu_{n-1}} & \text{if} \quad \mu_{n-1} < 0, \\[2mm] -c_n \log |x| & \text{if} \quad \mu_{n-1} = 0, \\[2mm] c_n & \text{if} \quad \mu_{n-1} > 0. \end{cases}$$

We observe that

$$\mu_{n-1} - \mu_{n-2} = p(\mu_{n-2} - \mu_{n-3}) = p^{n-3}(\mu_2 - \mu_1)$$
$$\to +\infty \quad \text{as} \quad n \to +\infty.$$

Then there exists $n_2 \ge 1$ such that

$$\mu_{n_2-1} > 0 \quad \text{and} \quad \mu_{n_2-2} \le 0$$

and
$$u \le u_{n_2} + \sum_{i=1}^{n_2-1} \Gamma_i + \Gamma_0, \tag{2.16}$$

where $\Gamma_i \le c|x|^{\mu_i}$ and
$$u_{n_2} \le 2^p(\mathbb{G}_\Omega[u_{n_2}^p] + 1).$$

Next, we claim that $u_{n_2} \in L^\infty(\Omega)$. Since $u_{n_2} \in L^s(\Omega)$ for $s \in [1, N/(N-2))$, letting

$$t_0 = \frac{1}{2}\left(1 + \frac{1}{p}\frac{N}{N-2}\right) \in \left(1, \frac{N}{N-2}\right),$$

then $\frac{1}{p}\frac{N}{N-2t_0} > 1$ and by Proposition 2.1, we have that $u_{n_2} \in L^{t_1}(\Omega)$ with

$$t_1 = \frac{1}{p}\frac{Nt_0}{N-2t_0}.$$

Inductively, it implies by $u_{n_2} \in L^{t_{n-1}}(\Omega)$ that $u_{n_2} \in L^{t_n}(\Omega)$ with

$$t_n = \frac{1}{p}\frac{Nt_{n-1}}{N-2t_{n-1}} > \left(\frac{1}{p}\frac{N}{N-2t_0}\right)^n t_0 \to +\infty \quad \text{as} \quad n \to \infty.$$

Then there exists $n_3 \in \mathbb{N}$ such that
$$s_{n_3} > \frac{Np}{2}$$

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and by part (i) in Proposition 2.1, it infers that

$$u_{n_2} \in L^\infty(\Omega).$$

Therefore, it implies by $u \geq \Gamma_0$ and (2.16) that

$$\lim_{x \to 0} u(x)|x|^{N-2} = c_{N,\alpha} k.$$

This ends the proof. □

2.2. Discussion on (1.6)

The following two functions plays an important role in searching distributional solutions of problem (1.9)

$$w_0 = \mathbb{G}_\Omega[\delta_0], \qquad w_1 = \mathbb{G}_\Omega[w_0^p], \tag{2.17}$$

which are the solutions respectively of

$$\begin{cases} -\Delta u = \delta_0 & \text{in} \quad \Omega, \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.18}$$

and

$$\begin{cases} -\Delta u = w_0^p & \text{in} \quad \Omega, \\ \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.19}$$

Observe that $a_p > 0$ defined in (1.4) is the smallest constant with $p \in (1, \frac{N}{N-2})$ such that

$$w_1 \leq a_p w_0 \quad \text{in} \quad \Omega \setminus \{0\}. \tag{2.20}$$

Obviously, a_p depends on the domain Ω.

Proposition 2.2. *Let $\Omega = B_1(0)$.*
(i) If $\theta > 0$ and $1 < p < \min\{2, p^*\}$, there exists $a_p^* > 0$ depending θ such that when $0 < a_p \leq a_p^*$, (1.6) holds for any $k > 0$; and when $a_p > a_p^*$, (1.6) holds for $0 < k \leq k_1$ and $k_2 \leq k < +\infty$, where $0 < k_1 < k_2 < +\infty$.
If $\theta > 0$, $p^* > 2$ and $2 < p < p^*$, there exists $k_3 > 0$ such that for $0 < k \leq k_3$, (1.6) holds.
If $\theta > 0$, $p^* > 2$ and $p = 2$, then when $a_2 > \frac{1}{4}$, (1.6) holds for $0 < k < \frac{\theta}{4a_2 - 1}$; and when $a_2 \leq \frac{1}{4}$, (1.6) holds for any $k > 0$.
(ii) If $\theta = 0$ and $1 < p < \min\{2, p^*\}$, then (1.6) is equivalent to

$$k \geq \left(\frac{(p-1)^{p-1}}{p^p a_p} \right)^{-\frac{1}{2-p}};$$

If $\theta = 0$, $p^* > 2$ and $2 < p < p^*$, then (1.6) is equivalent to

$$0 < k \leq \left(\frac{(p-1)^{p-1}}{p^p a_p} \right)^{\frac{1}{p-2}}.$$

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If $\theta = 0$, $p^* > 2$ and $p = 2$, then when $a_2 > \frac{1}{4}$, there is no $k > 0$ such that (1.6) holds; and when $a_2 \leq \frac{1}{4}$, (1.6) holds for any $k > 0$.

(iii) If $\theta < 0$ and $1 < p < \min\{2, p^*\}$, then (1.6) holds for $k \geq k_4$, where

$$k_4 > \left(\frac{(p-1)^{p-1}}{p^p a_p} \right)^{-\frac{1}{2-p}} ;$$

If $\theta < 0$, $p^* > 2$ and $2 < p < p^*$, then $a_p^{**} = (-\theta)^{2-p} p^{-p}(p-1)(p-2)^{p-3}$ such that when $0 < a_p \leq a_p^{**}$, (1.6) holds for $k_5 \leq k \leq k_6$, where $0 < k_5 \leq \frac{p-1}{2-p}\theta \leq k_6 < +\infty$; and when $a_p > a_p^*$, there is no $k > 0$ such that (1.6) holds.

If $\theta < 0$, $p^* > 2$ and $p = 2$, then when $a_2 < \frac{1}{4}$, (1.6) holds for $0 < k < \frac{\theta}{4a_2 - 1}$; and when $a_2 \geq \frac{1}{4}$, (1.6) holds for any $k > 0$.

Proof. When $\Omega = B_1(0)$, we have that $r_0 = 1$. Let

$$h(k) = \frac{k^{p-1}}{\theta + k} - \frac{1}{a_p p} \left(\frac{p-1}{p} \right)^{p-1}, \quad k \in (\theta_-, +\infty).$$

Note that

$$h'(k) = \frac{(p-1)k^{p-2}(\theta + k) - k^{p-1}}{(\theta + k)^2}.$$

When $p \neq 2$, $h'(k_0) = 0$ implies that

$$k_0 = \frac{p-1}{2-p}\theta.$$

When $p = 2$,

$$h(k) = \frac{k}{\theta + k} - \frac{1}{4a_2}, \quad k \in (0, +\infty).$$

The rest of the proof is simple and hence we omit it. $\quad\square$ $\qquad\qquad \blacksquare$

When $p = 2$, note that $\frac{1}{4}$ is a critical value for (1.6) and we show that $a_2 < \frac{1}{4}$ when Ω is a ball.

Lemma 2.2. *Assume that $\Omega = B_1(0)$, $N = 2$ or 3, $p = 2$ and a_2 is given by (2.20). Then*

$$a_2 < \frac{1}{4}.$$

Proof. When $\Omega = B_1(0)$, take $\xi(x) = 1 - |x|$ as a test function, we derive

$$\int_{B_1(0)} |\nabla w_0| \, dx = \int_{B_1(0)} \nabla w_0 \cdot \nabla (1 - |x|) \, dx = 1. \qquad (2.21)$$

Since w_1 is radial symmetric and decreasing, then

$$-(r^{N-1} w_1'(r))' = r^{N-1} w_0^2.$$

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So for $N = 3$,
$$w_1'(r) = \frac{1}{16\pi^2}r^{-2}\int_0^r (1-t)^2 dt$$

and
$$w_1(r) = \frac{1}{48\pi^2}\int_r^1 s^{-2}[(1-s)^3 - 1]ds = \frac{1}{48\pi^2}\left[3(r-1) - 3\ln r - \frac{r^2-1}{2}\right].$$

Then
$$\frac{w_1(r)}{w_0(r)} = \frac{1}{12\pi}\left[3r - 3\frac{r\ln r}{1-r} + \frac{r+r^2}{2}\right],$$

then $r \mapsto \frac{w_1(r)}{w_0(r)}$ is increasing, so

$$a_2 = \lim_{r\to 1}\frac{w_1(r)}{w_0(r)} = \frac{w_1'(1)}{w_0'(1)}.$$

So for $N = 2$,
$$w_1'(r) = \frac{1}{4\pi^2}r^{-1}\int_0^r (\ln t)^2 t\, dt$$

and
$$w_1(r) = \frac{1}{8\pi^2}\int_r^1 \left[s(\ln s)^2 - s\ln s - \frac{s}{2}\right]ds.$$

Then
$$\frac{w_1(r)}{w_0(r)} = \frac{-\frac{r^2(\ln r)^2}{2} + r^2\ln r + \frac{1-r^2}{4}}{-\frac{1}{2\pi}\ln r},$$

then $r \mapsto \frac{w_1(r)}{w_0(r)}$ is increasing, so

$$a_2 = \lim_{r\to 1}\frac{w_1(r}{w_0(r)} = \frac{w_1'(1)}{w_0'(1)}.$$

We see that

$$-w_1'(1) = \begin{cases} \dfrac{1}{48\pi^2} & \text{if} \quad N = 3, \\ \dfrac{1}{16\pi^2} & \text{if} \quad N = 2 \end{cases}$$

and

$$-w_0'(1) = \begin{cases} \dfrac{1}{4\pi} & \text{if} \quad N = 3, \\ \dfrac{1}{2\pi} & \text{if} \quad N = 2, \end{cases}$$

so

$$\alpha_2 = \begin{cases} \dfrac{1}{12\pi} & \text{if} \quad N = 3, \\ \dfrac{1}{8\pi} & \text{if} \quad N = 2. \end{cases}$$

Therefore, we have that $\alpha_2 < 1/4$. The proof is thus complete. $\qquad\square$

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Corollary 2.1. *Assume that $N = 2$ or 3, $p = 2$ M_θ is defined by (1.2) with $\theta \geq 0$, a_2 is given by (1.4), $\Omega = B_1(0)$. Then for any $k > 0$, problem (1.3) has a nonnegative solution u_k satisfying (1.7) and (1.8).*

3. Solutions with $M_\theta(u) > 0$

In order to do estimates on $M_\theta(u)$, we introduce the following lemma.

Lemma 3.1. *Let u, v be a radially symmetric, decreasing and nonnegative functions in $C^1(B_1(0) \setminus \{0\}) \cap W_0^{1,1}(B_1(0))$ such that*

$$\|u\|_{L^1(B_1(0))} \geq \|v\|_{L^1(B_1(0))} \quad \text{and} \quad \liminf_{|x| \to 0^+} [u(x) - v(x)]|x|^{N-1} \geq 0. \tag{3.1}$$

Then

$$\int_{B_1(0)} |\nabla u| dx \geq \int_{B_1(0)} |\nabla v| dx.$$

Proof. For radially symmetric decreasing function $f \in C^1(B_1(0) \setminus \{0\}) \cap W_0^{1,1}(B_1(0))$, we have that

$$\omega_N f(r) r^{N-1} + (N-1)\omega_N \int_r^1 f(s) s^{N-2} ds = -\omega_N \int_r^1 f'(s) s^{N-1} ds,$$

then we have that

$$\omega_N \lim_{|x| \to 0^+} (u - v)(x)|x|^{N-1} + (N-1) \int_{B_1(0)} [u(x) - v(x)] dx = \int_{B_1(0)} |\nabla u| dx - \int_{B_1(0)} |\nabla v| dx.$$

From (3.1), we have that

$$\int_{B_1(0)} |\nabla u| dx \geq \int_{B_1(0)} |\nabla v| dx.$$

This finishes the proof. □ □

Proof of Theorem 1.1. We search for distributional solutions of

$$-\Delta u = \frac{1}{M_\theta(u)} u^p + k\delta_0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{3.2}$$

by using the Schauder fixed-point theorem. Let w_0, w_1 be the solutions of (2.17) and denote

$$w_t = tk^p w_1 + kw_0, \tag{3.3}$$

where the parameter $t > 0$.

We claim that there exists $k_p > 0$ independent of θ such that for $k \in (0, k_p]$, if $\theta + r_0^{-1} k > 0$ there exists $t_p > 0$ such that

$$t_p k^p w_1 \geq \frac{\mathbb{G}_\Omega[w_{t_p}^p]}{\theta + r_0^{-1} k}. \tag{3.4}$$

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We observe that if

$$\frac{(a_p t k^p + k)^p}{\theta + r_0^{-1} k} \leq t k^p, \tag{3.5}$$

then w_t verifies (3.4), since

$$
\begin{aligned}
\frac{\mathbb{G}_\Omega[w_{t_p}^p]}{\theta + r_0^{-1} k} &\leq \frac{(a_p t k^p + k)^p \mathbb{G}_\Omega[w_0^p]}{\theta + r_0^{-1} k} \\
&= \frac{(a_p t k^p + k)^p}{\theta + r_0^{-1} k} w_1 \\
&\geq t k^p w_1.
\end{aligned}
$$

Now we discuss what condition on k guarantee that (3.5) holds for some $t > 0$. In fact, (3.5) is equivalent to

$$(a_p t k^{p-1} + 1)^p \leq t(\theta + r_0^{-1} k) \tag{3.6}$$

or in the form

$$s = t(\theta + r_0^{-1} k) \quad \text{and} \quad \left(\frac{a_p k^{p-1}}{\theta + r_0^{-1} k} s + 1 \right)^p \leq s.$$

For $p > 1$, since the function $f(s) = (\frac{1}{p}(\frac{p-1}{p})^{p-1} s + 1)^p$ intersects the line $g(s) = s$ at the unique point $s_p = \left(\frac{p}{p-1} \right)^p$, so k may be chosen such that

$$\frac{a_p k^{p-1}}{\theta + r_0^{-1} k} \leq \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1}. \tag{3.7}$$

In fact, (1.6) implies (3.7). Therefore, for $k > r_0 \theta_-$ satisfying (1.6) and taking $t_p = (\theta + k)^{-1} \left(\frac{p}{p-1} \right)^p$, function w_{t_p} verifies (3.4).

Let

$$\mathcal{D}_k = \left\{ u \in W_0^{1.1}(\Omega) : 0 \leq u \leq t_p k^p w_1 \right\}.$$

Denote

$$\mathcal{T} u = \frac{1}{M_\theta(u + k w_0)} \mathbb{G}_\Omega[(u + k w_0)^p], \quad \forall u \in \mathcal{D}_k.$$

We claim that

$$M_\theta(u + k w_0) \geq \theta + r_0^{-1} k > 0 \quad \text{for} \quad u \in \mathcal{D}_k. \tag{3.8}$$

For $u \in \mathcal{D}_k$, we may let $v_n \in C_0^1(\Omega)$ be a sequence of nonnegative functions converging to u in $W_0^{1,1}(\Omega)$. Let $u_n = v_n + k w_0$, and by the fact that $w_0 \in C^1(\Omega \setminus \{0\}) \cap W_0^{1,1}(\Omega)$, then $u_n \in C^1(\Omega \setminus \{0\}) \cap W_0^{1,1}(\Omega)$, $u_n \geq k w_0$ in $\Omega \setminus \{0\}$ and u_n converge to $u + k w_0$ in $W^{1,1}(\Omega)$. By the symmetric decreasing arrangement, we may denote u_n^*, the symmetric decreasing rearranged function of u_n in $B_{r_0}(0)$, where $r_0 \geq 1$ such that $|B_{r_0}(0)| = |\Omega|$. Observe that

$$\liminf_{|x| \to 0^+} u_n^*(x) |x|^{N-1} \geq 0 = k \lim_{|x| \to 0^+} w_0(x) |x|^{N-1}$$

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and
$$\int_\Omega u_n dx \geq k \int_\Omega w_0 dx.$$

By Pólya-Szegő inequality, we have that

$$\|\nabla u_n\|_{L^1(\Omega)} \geq \|\nabla u_n^*\|_{L^1(B_{r_0}(0))} = r_0^{-1}\|\nabla w_n^*\|_{L^1(B_1(0))}.$$

where $w_n^*(x) = r_0^{-N} u_n^*(r_0 x)$ for $x \in B_1(0)$.

Let $w_{B_1(0)} = k\mathbb{G}_{B_1(0)}[\delta_0]$, since $B_1(0) \subset \Omega$, Kato's inequality implies that

$$\int_\Omega w_0 dx \geq \int_{B_1(0)} w_{B_1(0)} dx.$$

Thus,

$$\int_{B_1(0)} w_n^* dx = \int_{B_{r_0}(0)} u_n^* dx \geq \int_{B_1(0)} w_{B_1(0)} dx. \tag{3.9}$$

Thus, by Lemma 3.1, (3.9) and (2.21), we have

$$\|\nabla w_n^*\|_{L^1(B_1(0))} \geq \|\nabla k w_{B_1(0)}\|_{L^1(B_1(0))} = k$$

Therefore, passing to the limit as $n \to +\infty$ in the above inequality we get that

$$M_\theta(u + kw_0) \geq \theta + r_0^{-1}\|\nabla k w_{B_1(0)}\|_{L^1(B_1(0))} = \theta + r_0^{-1}k,$$

which implies (3.8).

Therefore, from (3.4) it follows that

$$\mathcal{T}u = \frac{\mathbb{G}_\Omega[(kw_0 + u)^p]}{M_\theta(kw_0 + u)} \leq \frac{\mathbb{G}_\Omega[(kw_0 + t_p k^p w_1)^p]}{\theta + r_0^{-1}k} \leq t_p k^p w_1,$$

then

$$\mathcal{T}\mathcal{D}_k \subset \mathcal{D}_k.$$

Note that for $u \in \mathcal{D}_k$, one has that $(u+kw_0)^p \in L^\sigma(\Omega)$ with $\sigma \in (1, \frac{1}{p}\frac{N}{N-2})$, then $\mathcal{T}\mathcal{D}_k \subset W^{2,\sigma}(\Omega)$, where $\sigma \in (1, \frac{1}{p}\frac{N}{N-2})$. Since the embeddings $W^{2,\sigma}(\Omega) \hookrightarrow W^{1,1}(\Omega)$, $L^1(\Omega)$ are compact and then $\mathcal{T}$ is a compact operator.

Observing that $\mathcal{D}_k$ is a closed and convex set in $L^1(\Omega)$, we may apply the Schauder fixed-point theorem to derive that there exists $v_k \in \mathcal{D}_k$ such that

$$\mathcal{T}v_k = v_k.$$

Since $0 \leq v_k \leq t_p k^p w_1$, so v_k is locally bounded in $\Omega \setminus \{0\}$, then $u_k := v_k + kw_0$ satisfies (1.8), and by interior regularity results, u_k is a positive classical solution of (1.3). From Theorem 2.1 we deduce that u_k is a distributional solution of (1.9). $\qquad\square$

4. Solutions with $M_\theta(u) < 0$

For $\theta < 0$ and $M_\theta(u) < 0$, equation (1.9) could be written as

$$-\Delta u + \frac{1}{-M_\theta(u)} u^p = k\delta_0 \quad \text{in} \quad B_1(0), \qquad u = 0 \quad \text{on} \quad \partial B_1(0). \tag{4.1}$$

Lemma 4.1. *Let $p \in (1, p^*)$ and $\lambda > 0$. For any $k > 0$, the problem*

$$-\Delta u + \lambda u^p = k\delta_0 \quad \text{in} \quad B_1(0), \qquad u = 0 \quad \text{on} \quad \partial B_1(0) \tag{4.2}$$

has a unique positive weak solution $u_{\lambda,k}$ verifying that

$$\lim_{|x| \to 0^+} u_{\lambda,k}(x)|x|^{N-2} = c_N k. \tag{4.3}$$

Furthermore, $u_{\lambda,k}$ is radially symmetric and decreasing with to $|x|$ and the map $\lambda \mapsto u_{\lambda,k}$ is decreasing.

Proof. The existence could be seen [36, theorem 3.7] and uniqueness follows by Kato's inequality [36, theorem 2.4]. The radial symmetry of $u_{\lambda,k}$ and decreasing monotonicity with to $|x|$ could be derived by the method of moving plane, see [12, 33] for the details. It follows from Kato's inequality that the map $\lambda \mapsto u_{\lambda,k}$ is decreasing. The proof ends. $\square$ $\square$

Proof of Theorem 1.2. (i) Observe that

$$M_\theta(kw_0) = k \int_{B_1(0)} |\nabla w_0| dx + \theta = k + \theta < 0.$$

From Lemma 4.1 with $\lambda = \lambda_1 := -M_\theta^{-1}(kw_0)$, problem (4.2) with $\lambda = \lambda_1$ has a unique solution v_{λ_1} verifying that

$$0 < v_{\lambda_1} \le kw_0,$$

then it implies that

$$k \int_{B_1(0)} |\nabla v_{\lambda_1}| dx \le \int_{B_1(0)} |\nabla kw_0| \, dx$$

and

$$M_\theta(v_{\lambda_1}) = k \int_{B_1(0)} |\nabla v_{\lambda_1}| dx + \theta \le k \int_{B_1(0)} |\nabla w_0| dx + \theta = k + \theta,$$

thus,

$$\theta < M_\theta(v_{\lambda_1}) < k + \theta,$$

that is,

$$\frac{1}{-M_\theta(v_{\lambda_1})} < \lambda_1. \tag{4.4}$$

In terms of Lemma 4.1, let $\lambda_2 = -M_\theta^{-1}(v_{\lambda_1})$ and $\{v_{\lambda_2}\}$ be the solution of problem (4.2) with $\lambda = \lambda_2$. Since $\lambda_2 > \lambda_1$, then

$$v_{\lambda_1} < v_{\lambda_2} < kw_0.$$

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So it follows by Lemma 3.1 that

$$M_\theta(v_{\lambda_1}) < M_\theta(v_{\lambda_2}) < M_\theta(kw_0),$$

that is,

$$\frac{1}{-M_\theta(v_{\lambda_2})} > \lambda_2. \tag{4.5}$$

We claim that the map $\lambda \in [\lambda_2, \lambda_1] \mapsto M_\theta(u_{\lambda,k})$ is continuous.

At this moment, we assume that the above argument is true. Let

$$F(\lambda) = \frac{1}{-M_\theta(v_\lambda)} - \lambda,$$

where v_λ is the solution of (4.2) with $\lambda \in [\lambda_2, \lambda_1]$. Since F is continuous in $[\lambda_2, \lambda_1]$, by (4.4), (4.5) and the mean value theorem, there exists $\lambda_0 \in (\lambda_2, \lambda_1)$ such that $F(\lambda_0) = 0$, that is, (4.1) has a solution u_k with $\frac{1}{-M_\theta(u_k)} = \lambda_0$. From standard regularity, we have that u_k is a classical solution of (1.3) and verifies the corresponding properties in the lemma.

Now we prove that the map $\lambda \in [\lambda_2, \lambda_1] \mapsto M_\theta(u_{\lambda,k})$ is continuous. Let $\lambda_2 \leq \lambda' < \lambda'' \leq \lambda_1$ and $u_{\lambda',k}$ and $u_{\lambda'',k}$ be the solutions of (4.1) with $\lambda = \lambda'$ and $\lambda = \lambda''$ respectively. Then

$$u_{\lambda'',k} < u_{\lambda',k}$$

and

$$M_\theta(u_{\lambda'',k}) < M_\theta(u_{\lambda',k}). \tag{4.6}$$

Let $\bar{u} = u_{\lambda'',k} + (\frac{\lambda'' - \lambda'}{\lambda_2})^{1/p} w_0$. Then

$$
\begin{aligned}
-\Delta \bar{u} + \lambda' \bar{u}^p &\geq -\Delta u_{\lambda'',k} + \left(\frac{\lambda'' - \lambda'}{\lambda_2}\right)^{\frac{1}{p}} (-\Delta) w_0 + \lambda' u_{\lambda'',k}^p + \lambda' \frac{\lambda'' - \lambda'}{\lambda_2} w_0^p \\
&\geq -\Delta u_{\lambda'',k} + \lambda'' u_{\lambda'',k}^p \\
&= k\delta_0.
\end{aligned}
$$

Therefore Kato's inequality implies that

$$u_{\lambda',k} \leq u_{\lambda'',k} + \left(\frac{\lambda'' - \lambda'}{\lambda_2}\right)^{\frac{1}{p}} w_0,$$

which yields that

$$M_\theta(u_{\lambda',k}) \leq M_\theta(u_{\lambda'',k}) + \left(\frac{\lambda'' - \lambda'}{\lambda_2}\right)^{\frac{1}{p}} k.$$

This together with (4.6), give

$$|M_\theta(u_{\lambda',k}) - M_\theta(u_{\lambda'',k})| \leq \left(\frac{\lambda'' - \lambda'}{\lambda_2}\right)^{\frac{1}{p}} k \to 0 \quad \text{as} \quad |\lambda'' - \lambda'| \to 0,$$

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thus, the map $\lambda \in [\lambda_2, \lambda_1] \mapsto M_\theta(u_{\lambda,k})$ is continuous.

(ii) It is well known that for $p \in (1, p^*)$, the problem

$$- \Delta u + u^p = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{4.7}$$

has a positive solution v_p verifying that

$$\lim_{|x| \to 0^+} v_p(x)|x|^{\frac{2}{p-1}} = c_p, \tag{4.8}$$

where $c_p = [\frac{2}{p-1}(\frac{2}{p-1} + 2 - N)]^{\frac{1}{p-1}}$. Furthermore, v_p is the unique solution of (4.7) such that

$$\liminf_{|x| \to 0^+} u(x)|x|^{\frac{2}{p-2}} > 0. \tag{4.9}$$

We observe that

$$v_\lambda := \lambda^{-\frac{1}{p-1}} v_p$$

is the unique solution of

$$- \Delta u + \lambda u^p = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{4.10}$$

in the set of functions satisfying (4.9).

For $p \in (\frac{N+1}{N-1}, p^*)$, we have that $\int_\Omega |\nabla u_p| dx < +\infty$, so that

$$M_\theta(v_\lambda) := \lambda^{-\frac{1}{p-1}} m_2 + \theta < 0 \quad \text{for} \quad \lambda \in (\lambda_0, +\infty),$$

where $m_2 = \int_\Omega |\nabla u_p| dx$ and $\lambda_0 = (m_2/(-\theta))^{p-1}$.

We define

$$F(\lambda) := \frac{1}{\lambda^{-\frac{1}{p-1}} m_2 + \theta} + \lambda, \quad \lambda \in (\lambda_0, +\infty).$$

Observe that F is continuous, increasing and

$$\lim_{\lambda \to \lambda_0^+} F(\lambda) = -\infty, \qquad \lim_{\lambda \to +\infty} F(\lambda) = +\infty.$$

Hence there exists a unique $\bar{\lambda}$ such that

$$- \frac{1}{\bar{\lambda}^{-\frac{1}{p-1}} m_2 + \theta} = \bar{\lambda}.$$

Meaning that $-M_\theta^{-1}(v_{\bar{\lambda}}) = \bar{\lambda}$. We then conclude that (1.3) has a solution $u_p := v_{\bar{\lambda}}$ with $M_\theta(u_p) < 0$. From (4.8) and the definition of v_λ, we know that u_p is not a weak solution of problem (1.9).

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5. In the supercritical case

In the super critical case that $p^* \leq p < 2^* - 1$, we have the following existence results.

Theorem 5.1. (i) Let $N \geq 3$, $p^* \leq p < 2^* - 1$, $\theta \in \mathbb{R}$ and Ω be a bounded smooth domain containing the origin. If
Case 1: $p > 2$, $p \geq p^*$ and $\theta > 0$;
Case 2: $p = 2 \geq p^*$, $\theta > 0$ and $m_2 < 1$;
Case 3: $p^* \leq p < 2$ and $\theta < 0$;
Case 4: $p^* \leq p < 2^* - 1$, $p \neq 2$ and $\theta = 0$,
then problem (1.3) has two positive solutions u_i with $i = 1, 2$ satisfying that

$$M_\theta(u_i) > 0,$$

$$\text{if } p \in \left(p^*, \frac{N+2}{N-2}\right), \quad \lim_{|x|\to 0^+} u_i(x)|x|^{\frac{2}{p-1}} = M_\theta(u_i)^{\frac{1}{p-1}} c_p \tag{5.1}$$

and

$$\text{if } p = p^*, \quad \lim_{|x|\to 0^+} u_i(x)|x|^{N-2}(\ln|x|)^{\frac{N-2}{2}} = M_\theta(u_i)^{\frac{N-2}{2}} c_{p^*}, \tag{5.2}$$

where $c_p = [\frac{2}{p-1}(N - 2 - \frac{2}{p-1})]^{\frac{1}{p-1}}$ and $c_{p^*} = (\frac{N-2}{4})^{N-2}$.
   (ii) Let $N = 4$, 5, $p = 2 \in [p^*, 2^* - 1)$, $\theta = 0$ and Ω be a bounded smooth domain containing the origin. If v is a solution of (5.3) such that $M_\theta(v) = 1$, then for any $\lambda > 0$, $u := \lambda v$ is a solution of problem (1.3) satisfying $M_\theta(u) = \lambda > 0$ and (5.1)–(5.2).

To prove Theorem 5.1, we need the following lemma.

Lemma 5.1. ([29, 30]) Let $N \geq 3$, $p \in [p^*, \frac{N+2}{N-2})$ and Ω be a bounded smooth domain containing the origin. Then the following problem

$$-\Delta u = u^p \quad \text{in} \quad \Omega \setminus \{0\}, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{5.3}$$

has two positive singular solution v_1 and v_2 verifying that

$$\text{if } p \in (p^*, \frac{N+2}{N-2}), \quad \lim_{|x|\to 0^+} v_i(x)|x|^{\frac{2}{p-1}} = c_p \tag{5.4}$$

and

$$\text{if } p = p^*, \quad \lim_{|x|\to 0^+} v_i(x)|x|^{N-2}(\ln|x|)^{\frac{N-2}{2}} = c_{p^*}. \tag{5.5}$$

Proof of Theorem 5.1. From Lemma 5.1, it is known that for $p^* \leq p < 2^* - 1$, problem (5.3) has two positive solutions v_i verifying that (5.4) and (5.5).
   We observe that

$$v_{\lambda,i} = \lambda^{-\frac{1}{p-1}} v_i$$

is a solution of

$$-\Delta u + \lambda u^p = 0 \quad \text{in} \quad \Omega \setminus \{0\}, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{5.6}$$

For $p^* \le p < 2^* - 1$, we have that $\int_\Omega |\nabla u_p| dx < +\infty$, then

$$M_\theta(v_{\lambda,i}) = \lambda^{-\frac{1}{p-1}} m_i + \theta > 0$$

for $\lambda \in (0, \lambda_+)$, where $m_i = \int_\Omega |\nabla v_i| dx$ and

$$\lambda_+ = \begin{cases} +\infty & \text{if} \quad \theta \ge 0, \\ (-m_i/\theta)^{p-1} & \text{if} \quad \theta < 0. \end{cases}$$

Denote

$$F_\theta(\lambda) = \frac{1}{\lambda^{-\frac{1}{p-1}} m_i + \theta} - \lambda, \quad \lambda \in (0, \lambda_+),$$

which is continuous and

$$\lim_{\lambda \to \lambda_+} F_\theta(\lambda) = \begin{cases} -\infty & \text{if} \quad \theta > 0, \\ +\infty & \text{if} \quad \theta < 0. \end{cases}$$

Case 1: $p > 2$, $p \ge p^*$ and $\theta > 0$, then there exists $t > 0$ such that

$$F_\theta(\lambda) > 0.$$

Case 2: $p = 2 \ge p^*$, $\theta > 0$ and $m_i < 1$, then there exists $t > 0$ such that

$$F_\theta(\lambda) > 0.$$

Case 3: $p^* \le p < 2$ and $\theta < 0$, then there exists $t > 0$ such that

$$F_\theta(\lambda) < 0.$$

In the above three cases, there exists a unique $\bar\lambda_i$ such that

$$\frac{1}{\bar\lambda_i^{-\frac{1}{p-1}} m_i + \theta} = \bar\lambda_i,$$

that is, $M_\theta^{-1}(v_{\bar\lambda_i,i}) = \bar\lambda_i$. Therefore, (1.3) has a solution $u_i := v_{\bar\lambda_i,i}$ with $M_\theta(u_i) > 0$.
    When $\theta = 0$,
$$F_0(\lambda) = \lambda^{\frac{1}{p-1}} m_i - \lambda, \quad \lambda \in (0, +\infty),$$
When $N \ge 4$, we have that $1/(p-1) > 1$ for $p^* \le p < 2^* - 1$, or when $N = 3$, $p^* \le p < 2^* - 1$, $p \ne 2$, $\bar\lambda_i = m_i^{(p-1)/(p-2)}$, then (1.3) has a solution $u_i := v_{\bar\lambda_i,i}$.
    When $N = 4, 5$ and $p = 2 \in [p^*, 2^* - 1)$, if $m_i = 1$, then for any $\lambda > 0$, $u := \lambda^{-\frac{1}{p-1}} v_i$ is a solution (1.3) with $M_\theta(u) = \lambda > 0$ and verifying (5.1)–(5.2). $\square$

Remark 5.1. Our method to prove Theorem 5.1 is based on the homogeneous property of the nonlinearity. When the nonlinearity is not a power function, this scaling method fails and it is challenging to provide the existence results of isolated singular solutions.

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