

Orbital stability of standing waves of a class of fractional Schrödinger equations with a general Hartree-type integrand

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Abstract

This article is concerned with the mathematical analysis of a class of a nonlinear fractional Schrödinger equations with a general Hartree-type integrand. We prove existence and uniqueness of global-in-time solutions to the associated Cauchy problem. Under suitable assumptions, we also prove the existence of standing waves using the method of concentration-compactness by studying the associated constrained minimization problem. Finally we show the orbital stability of standing waves which are the minimizers of the associate variational problem.

Keywords: Fractional Schrödinger equation, Hartree type nonlinearity, standing waves, orbital stability

1. Introduction

A partial differential equation is called fractional when it involves derivatives or integrals of fractional order. Various physical phenomena and applications require the use of fractional derivatives, for instance quantum mechanics, pseudo-chaotic dynamics, dynamics in porous media, kinetic theories of systems with chaotic dynamics. The latter application is based on the so called fractional Schrödinger equation. This equation was derived using the

path integral over a kind of Lévy quantum mechanical path approach by Laskin in Ref. [14, 15, 16]. The mathematical analysis of the fractional non-linear Schrödinger equation has been growing continually during the last few decades. Many results have been obtained and we refer for instance to [5] and references therein.

This paper deals with the analysis of the following Cauchy problem

$$\mathcal{S} : \begin{cases} i\partial_t \phi + (-\Delta)^s \phi = (G(|\phi|) \star V(|x|)) G'(\phi), \\ \phi(t=0, x) = \phi_0. \end{cases}$$

In the system \mathcal{S} , $\phi(t, x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^N$ and ϕ_0 is a prescribed initial data in $H^s(\mathbb{R}^N)$. The operator $(-\Delta)^s$ denotes the fractional Laplacian of power $0 < s < 1$. It is defined as a pseudo-differential operator $\mathcal{F}[(-\Delta)^s \phi](\xi) = |\xi|^{2s} \mathcal{F}[\phi](\xi)$ with \mathcal{F} being the Fourier transform. The symbol \star denotes the convolution operator in \mathbb{R}^N with the potential $V(|x|) = |x|^{\beta-N}$ where $\beta > 0$ is such that $\beta > N - 2s$. The function G is a differentiable function from $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, $G'(\phi) := \frac{dG}{d\phi} := F(|\phi|)\phi$, where $F : \mathbb{R} \rightarrow \mathbb{R}$.

The above Cauchy problem reduces to the massless boson Schrödinger equation in three dimensions when $G(\phi) = |\phi|^2$, $V(|x|) = |x|^{-1}$ and $s = \frac{1}{2}$. In this case, standing waves of the system \mathcal{S} , i.e. solutions of the form $\phi(t, x) = u(x)e^{-i\kappa t}$, satisfy the following semilinear partial differential equation

$$(-\Delta)^{1/2} u - (|x|^{-1} \star u^2)u + \kappa u = 0. \quad (1)$$

The associated variational problem

$$\mathcal{I}_\lambda = \inf \left\{ \left\| |\xi|^{\frac{1}{2}} \mathcal{F}[u](\xi) \right\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy, \right. \\ \left. u \in H^{\frac{1}{2}}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u(x)|^2 dx = \lambda \right\}, \quad (2)$$

has played a fundamental role in the mathematical theory of gravitational collapse of boson stars, [18]. In Ref. [12], the authors studied the associated

variational problem

$$\mathcal{I}_\lambda^G = \inf \left\{ \|\xi|^s \mathcal{F}[u](\xi)\|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N \times \mathbb{R}^N} G(u(x))V(|x-y|)G(u(y))dxdy, \right. \\ \left. u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} |u(x)|^2 dx = \lambda \right\}, \quad (3)$$

for a general nonlinearity G , a kernel $V(|x|) = |x|^{\beta-N}$ and dimension N , where here and the following

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^{N^N}), \|\xi|^s \mathcal{F}[u](\xi)\|_{L^2(\mathbb{R}^N)}^2 < \infty\}.$$

In the critical case $2s = N - \beta$, they were able to extend the results of [18]. Moreover, in the subcritical $2s > N - \beta$, they have also proved the existence and symmetry of all minimizers of (3) by using rearrangement techniques. More precisely, they showed that under suitable assumptions on G , one can always take a radial and radial by decreasing minimizing sequence of problem (3).

Another very important issue related to the nonlinear fractional Schrödinger equation \mathcal{S} is the orbital stability of standing waves. For such an issue, it is essential to show that all the minimizing sequences are relatively compact in $H^s(\mathbb{R}^N)$. This is the gist of the breakthrough paper [4]. The line of attack consists of:

1. Prove the uniqueness of the solutions of \mathcal{S} .
2. Prove the conservation of energy and mass of the solutions.
3. Prove the relative compactness of all minimizing sequences of the problem (3).

Our first result concerns the well-posedness of the system \mathcal{S} . Before stating it, we need to fix some conditions on G . We assume that G is nonnegative and differentiable such that $G(0) = 0$ and for all $\psi \in \mathbb{R}_+$

$$\mathcal{A}_0 : \exists \mu \in \left[2, 1 + \frac{2s + \beta}{N}\right) \text{ s.t. } \begin{cases} G(\psi) \leq \eta(|\psi|^2 + |\psi|^\mu), \\ |G'(\psi)| \leq \eta(|\psi| + |\psi|^{\mu-1}). \end{cases}$$

We have obtained the following

Theorem 1.1. *Let $N \geq 1, 0 < s < 1, 0 < \beta < N, N - 2s \leq \beta, \phi_0 \in H^s(\mathbb{R}^N)$ and G such that \mathcal{A}_0 holds true. Then, there exists a weak global-in-time solution $\phi(t, x)$ to the system \mathcal{S} such that*

$$\phi \in L^\infty(\mathbb{R}; H^s(\mathbb{R}^N)) \cap W^{1,\infty}(\mathbb{R}; H^{-s}(\mathbb{R}^N)).$$

Moreover, if $N = 1$ and $\frac{1}{2} < s < 1$ or if $N \geq 3, \frac{N}{2(N-1)} < s < 1, N - s + \frac{1}{2} < \beta < \min(N, \frac{3N}{2} - s - \frac{N}{4s})$ and μ (in \mathcal{A}_0) is such that

$$\max\left(2, 1 + \frac{2\beta - N}{N - 2s}\right) < \mu < 2 + \frac{N}{N - 2s} \frac{2s - 1 - 2N + 2\beta}{2s - 1 + N},$$

then the solution is unique.

The particular case $\mu = 2$ and $2s = N - \beta$ was treated in Ref. [5] and for lightness of the proofs, we shall sometimes omit it and focus on the case $\mu \in (2, 1 + \frac{2s+\beta}{N})$. The proof of the existence part of Theorem 1.1 is based on a classical contraction argument and the conservation laws associated to the dynamics of the system \mathcal{S} . The uniqueness part for $N = 1$ of Theorem 1.1 readily follows from the embedding $H^s \hookrightarrow L^\infty$ for all $s > \frac{1}{2}$. The part for $N \geq 3$ is obtained using mixed norms to be defined later and weighted Strichartz and convolution inequalities, which require $N \geq 3$. It would be very interesting to find estimates to handle the uniqueness for $N = 2$. Let us mention that in Ref. [9] the authors showed the orbital stability of standing waves in the case of power nonlinearities by assuming energy conservation and time continuity without proving uniqueness, which is an inescapable and quite hard step, especially in the fractional setting.

As mentioned before, if $\phi(t, x) = e^{i\kappa t}u(x)$ with $\kappa \in \mathbb{R}$ is a solution of the system \mathcal{S} , then it is called a standing wave solution and $u(x)$ solves the following bifurcation problem

$$\tilde{\mathcal{S}} : \quad (-\Delta)^s u - \kappa u = (G(|u|) \star V(|x|)) G'(u).$$

In order to study the existence of a solution (κ, u) to the stationary equation $\tilde{\mathcal{S}}$, we use a variational method based on the following minimization problem

$$\mathcal{I}_\lambda = \inf \left\{ \mathcal{E}(u), \quad u \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u(x)|^2 dx = \lambda \right\}, \quad (4)$$

where λ is a positive prescribed number and

$$\begin{aligned}\mathcal{E}(u) &= \frac{1}{2} \|\nabla_s u\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} G(|u(x)|) V(|x-y|) G(|u(y)|) dx dy, \\ &:= \frac{1}{2} \|\nabla_s u\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2} \mathcal{D}(G(|u|), G(|u|)).\end{aligned}$$

The kinetic energy is precisely expressed by the formula for all function u in the Schwarz class

$$\|\nabla_s u\|_{L^2(\mathbb{R}^N)}^2 = C_{N,s} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad (5)$$

with $C_{N,s}$ being a positive normalization constant. In order to prove the existence of critical points to the functional \mathcal{E} and thereby solutions to the problem $\tilde{\mathcal{J}}$, we will need some extra growth condition on G : for all $\psi \in \mathbb{R}_+$

$$\mathcal{A}_1 : \begin{cases} \exists 0 < \alpha < 1 + \frac{2s+\beta}{N} \text{ s.t. } \forall \psi, 0 < \psi \ll 1, & G(\psi) \geq \eta \psi^\alpha, \\ G(\theta \psi) \geq \theta^{1+\frac{2s+\beta}{2N}} G(\psi). \end{cases}$$

Our next main result is contained in the following

Theorem 1.2. *Let $0 < s < 1, 0 < \beta < N, N - \beta \leq 2s$ and G such that \mathcal{A}_0 and \mathcal{A}_1 hold true. Then, for all $\lambda > 0$, problem (4) has a minimizer $u_\lambda \in H^s(\mathbb{R}^N)$ such that $I_\lambda = \mathcal{E}(u_\lambda)$.*

In fact we will show that any minimizing sequence of problem 4 is –up to suitable translations– relatively compact in $H^s(\mathbb{R}^N)$. The proof of Theorem 1.2 is based on the concentration-compactness method of P-L. Lions [17].

The last part of the paper deals with the stability of the standing waves. For that purpose, we introduce the following problem

$$\hat{\mathcal{I}}_\lambda = \inf \left\{ \mathcal{J}(z), \quad z \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |z|^2 dx = \lambda \right\},$$

where $z = u + i v$ and

$$\begin{aligned}\mathcal{J}(z) &= \frac{1}{2} \|\nabla_s z\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2} \mathcal{D}(G(|z(x)|), G(|z(x)|)), \\ &= \frac{1}{2} \|\nabla_s u\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \|\nabla_s v\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{2} \mathcal{D}(G((u^2 + v^2)^{\frac{1}{2}}), G((u^2 + v^2)^{\frac{1}{2}})), \\ &:= \mathcal{J}(u, v).\end{aligned}$$

We have obviously $\mathcal{E}(u) = \mathcal{J}(u, 0)$. Following Ref. [4], we introduce the following set

$$\hat{\mathcal{O}}_\lambda = \left\{ z \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |z|^2 dx = \lambda \quad : \quad \mathcal{J}(z) = \hat{\mathcal{I}}_\lambda \right\}.$$

The set $\hat{\mathcal{O}}_\lambda$ is the so called orbit of the standing waves of \mathcal{S} with mass $\sqrt{\lambda}$. We define the stability of $\hat{\mathcal{O}}_\lambda$ as follows

Definition 1.3. *Let $\phi_0 \in H^s(\mathbb{R}^N)$ be an initial data and $\phi(t, x) \in H^s(\mathbb{R}^N)$ the associated solution of problem \mathcal{S} . We say that $\hat{\mathcal{O}}_\lambda$ is $H^s(\mathbb{R}^N)$ -stable with respect to the system \mathcal{S} if*

- $\hat{\mathcal{O}}_\lambda \neq \emptyset$.
- *For all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\phi_0 \in H^s(\mathbb{R}^N)$ satisfying $\inf_{z \in \hat{\mathcal{O}}_\lambda} |\phi_0 - z| < \delta$, we have $\inf_{z \in \hat{\mathcal{O}}_\lambda} |\phi(t, x) - z| < \varepsilon$ for all $t \in \mathbb{R}$.*

The notion of stability depends then intimately on the well-posedness of the Cauchy problem \mathcal{S} and the existence of standing waves. Therefore, having in hand Theorems 1.1 and 1.2, we prove the following

Theorem 1.4. *Let $N \geq 3$, $\frac{N}{2(N-1)} < s < 1$, $N - s + \frac{1}{2} < \beta < \min(N, \frac{3N}{2} - s - \frac{N}{4s})$ and let G satisfying \mathcal{A}_0 and \mathcal{A}_1 with μ (in \mathcal{A}_0) such that*

$$\max \left(2, 1 + \frac{2\beta - N}{N - 2s} \right) < \mu < 2 + \frac{N}{N - 2s} \frac{2s - 1 - 2N + 2\beta}{2s - 1 + N}.$$

Let $\phi_0 \in H^s(\mathbb{R}^N)$ and $\phi(t, x) \in H^s(\mathbb{R}^N)$ the associated solution to the problem \mathcal{S} . Then $\hat{\mathcal{O}}_\lambda$ is $H^s(\mathbb{R}^N)$ -stable with respect to the system \mathcal{S} .

The paper is divided into three sections. The first one is dedicated to the analysis of the dynamics of the system \mathcal{S} . More precisely, in this section we prove Theorem 1.1. First of all, we prove a local-in-time existence of solutions. Second we show that under extra assumptions, this solution is actually unique. Eventually, we use the conservation laws to show the global-in-time well-posedness. The second section is devoted to the proof of existence of solution to the problem $\tilde{\mathcal{S}}$. For that purpose, we use the classical concentration compactness method [17] to prove Theorem 1.2. The last section is

dedicated to the proof of stability of standing waves, namely Theorem 1.4. Here, we use ideas and techniques developed in [13].

From this point onward, η will denote variant universal constants that may change from line to line of inequalities. When η depends on some parameter, we will write $\eta(\cdot)$ instead of η . In order to lighten the notation and the calculation, we shall use L^p and H^s instead of $L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$ respectively for real or complex valued functions. Also, we shall use $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\mathbb{R}^N)}$ for all $p \in [1, \infty]$. The exponent p' will denotes the conjugate exponent of p , that is $\frac{1}{p} + \frac{1}{p'} = 1$. For a more detailed account about the Sobolev spaces H^s , we refer the reader to any textbook of functional analysis (see [3] for instance).

2. Well-posedness of the system \mathcal{S}

In this section we consider the local and global well-posedness of the problem \mathcal{S} and prove Theorem 1.1. Let us denote the nonlinear term $[V(|x|) \star G(\phi)]G'(\phi)$ by $\mathcal{N}(\phi)$. Since the well-posedness of the case $\mu = 2, 2s = N - \beta$ was treated in [5], in this paper we consider the initial value problem \mathcal{S} with $\mu \in (2, 1 + \frac{2s+\beta}{N})$. Let $g = G'$, that is, $\int_0^{|z|} g(\alpha) d\alpha = G(z)$, and assume that $g(z) = \frac{z}{|z|}g(|z|), z \neq 0, G(z) \geq 0$. Then, with \mathcal{A}_0 , the function g satisfies obviously

$$|g(z)| + |g'(z)z| \leq C(|z| + |z|^{\mu-1}) \text{ for all } z \in \mathbb{C}. \quad (6)$$

2.1. Weak solutions

We first show existence of weak solutions to \mathcal{S} in H^s . For this purpose we prove that \mathcal{N} is Lipschitz map from $L^{p'}$ to L^r for some $p, r \in [2, \frac{2N}{N-2s})$. Then the rest of the proof is quite straightforward from the Lipschitz map and well-known regularizing arguments and we refer the readers to the book [3].

Proposition 2.1. *Let $N \geq 2, 0 < s < 1, 0 < \beta < N$ and $2s \geq N - \beta$. If g satisfies (6) with $\mu \in (2, 1 + \frac{2s+\beta}{N})$. Then there exists a weak solution ϕ such that*

$$\phi \in L^\infty(-T_{min}, T_{max}; H^s) \cap W^{1,\infty}(-T_{min}, T_{max}; H^{-s}),$$

$$\|\phi(t)\|_2 = \|\phi_0\|_2, \quad \mathcal{J}(\phi(t)) \leq \mathcal{J}(\phi_0).$$

for all $t \in (-T_{min}, T_{max})$, where $(-T_{min}, T_{max})$ is the maximal existence time interval of ϕ for given initial data ϕ_0 .

Proof. Let us introduce the following cut-off for the function g , $g_1(\alpha) = \chi_{\{0 \leq \alpha < 1\}} g(\alpha)$ and $g_2(\alpha) = \chi_{\{\alpha \geq 1\}} g(\alpha)$ and $G_i(z) = \int_0^{|z|} g_i(\alpha) d\alpha$ with obvious definition of the Euler function χ . Then, one can writes

$$\mathcal{N}(\phi) = \sum_{i,j=1,2} \mathcal{N}_{ij}(\phi) \text{ where } \mathcal{N}_{ij}(\phi) = \int_{\mathbb{R}^N} |x-y|^{-(N-\beta)} G_i(|\phi|) dy g_j(\phi).$$

We claim that there exist $p_{ij}, r_{ij} \in [2, \frac{2N}{N-2s})^1$ such that

$$\|\mathcal{N}_{ij}(\phi) - \mathcal{N}_{ij}(\psi)\|_{p'_{ij}} \leq \eta(K) \|\phi - \psi\|_{r_{ij}}, \quad (7)$$

for some constant $\eta(K)$ with $\eta(K) \leq \eta K^{a_{ij}}$, $a_{ij} > 0$ for all $1 \leq i, j \leq 2$, provided $\|\phi\|_{H^s} + \|\psi\|_{H^s} \leq K$. This implies that $\mathcal{N} : H^s \rightarrow H^{-s}$ is a Lipschitz map on a bounded sets of H^s . Indeed, let $\mu_1 = 2$ and $\mu_2 = \mu$. Then we have

$$\begin{aligned} |\mathcal{N}_{ij}(\phi) - \mathcal{N}_{ij}(\psi)| &\leq \eta \int_{\mathbb{R}^N} |x-y|^{-(N-\beta)} (|\phi|^{\mu_i-1} + |\psi|^{\mu_i-1}) |\phi - \psi| dy |\phi|^{\mu_j-1} \\ &\quad + \eta \int_{\mathbb{R}^N} |x-y|^{-(N-\beta)} |\psi|^{\mu_i} dy (|\phi|^{\mu_j-2} + |\psi|^{\mu_j-2}) |\phi - \psi|. \end{aligned}$$

By Hölder's and Hardy-Littlewood-Sobolev inequalities with indices p_{ij}, r_{ij} such that

$$1 - \frac{1}{p_{ij}} = \frac{\mu_i}{r_{ij}} - \frac{\beta}{N} + \frac{\mu_j - 1}{r_{ij}}, \quad \frac{\mu_i}{r_{ij}} > \frac{\beta}{N}, \quad (8)$$

we obtain

$$\begin{aligned} \|\mathcal{N}_{ij}(\phi) - \mathcal{N}_{ij}(\psi)\|_{p'_{ij}} &\leq \eta \left[(\|\phi\|_{r_{ij}}^{\mu_i-1} + \|\psi\|_{r_{ij}}^{\mu_i-1}) \|\phi\|_{r_{ij}}^{\mu_j-1} \right. \\ &\quad \left. + \|\psi\|_{r_{ij}}^{\mu_i} (\|\phi\|_{r_{ij}}^{\mu_j-2} + \|\psi\|_{r_{ij}}^{\mu_j-2}) \right] \|\phi - \psi\|_{r_{ij}}. \end{aligned}$$

Thus if $p_{ij}, r_{ij} \in [2, \frac{2N}{N-2s})$, then Sobolev inequality shows (7). Now we show that there exist $p_{ij}, r_{ij} \in [2, \frac{2N}{N-2s})$ such that the combinations (8) hold true. If p_{ij}, r_{ij} satisfy (8), then they are on the line

$$\frac{1}{r_{ij}} = \frac{1}{\mu_i + \mu_j - 1} \left(1 + \frac{\beta}{N} - \frac{1}{p_{ij}} \right). \quad (9)$$

¹If $N = 1$ and $\frac{1}{2} \leq s < 1$, then $\frac{2N}{N-2s}$ is interpreted as ∞ .

Since $\frac{1}{\mu_i + \mu_j - 1}(1 + \frac{\beta}{N} - \frac{1}{2}) < \frac{1}{2}$ and $\frac{N-2s}{2N} < \frac{1}{\mu_i + \mu_j - 1}(1 + \frac{\beta}{N} - \frac{N-2s}{2N})$, the line (9) of $(\frac{1}{p_{ij}}, \frac{1}{r_{ij}})$ always passes through the open square $(\frac{N-2s}{2N}, \frac{1}{2}) \times (\frac{N-2s}{2N}, \frac{1}{2})$. We have only to find a pair $(\frac{1}{p_{ij}}, \frac{1}{r_{ij}})$ of line (9) such that $\frac{\mu_i}{r_{ij}} > \frac{\beta}{N}$. If $\frac{\mu_i}{r_{ij}} > \frac{\beta}{N}$, then

$$\frac{1}{p_{ij}} < 1 - \frac{\mu_j - 1}{\mu_i} \frac{\beta}{N}.$$

So, it suffices to show that

$$\max \left(\frac{1}{p_0}, \frac{N-2s}{2N} \right) < 1 - \frac{\mu_j - 1}{\mu_i} \frac{\beta}{N}, \quad (10)$$

where $\frac{1}{p_0}$ is the point of line (9) when $\frac{1}{r_{ij}} = \frac{1}{2}$, that is, $\frac{1}{p_0} = 1 + \frac{\beta}{N} - \frac{\mu_i + \mu_j - 1}{2}$. In fact, it is an easy matter to show (10) from the condition $\mu \in (2, 1 + \frac{\beta+2s}{N})$ and we leave the proof to the reader. The proof of Proposition 2.1 follows now by a straightforward application of a contraction argument. \square

2.2. Uniqueness

Since the case $N = 1$ can be treated as in [3], we omit the details. When $N \geq 3$, the uniqueness of weak solutions can be shown by a weighted Strichartz and convolution estimates. For that purpose, we introduce the following mixed norm for all $1 \leq m, \tilde{m} < \infty$

$$\|h\|_{L_\rho^m L_\sigma^{\tilde{m}}} := \left(\int_0^\infty \left(\int_{S^{N-1}} |h(\rho\sigma)|^{\tilde{m}} d\sigma \right)^{\frac{m}{\tilde{m}}} \rho^{n-1} d\rho \right)^{\frac{1}{m}}.$$

The case $m = \infty$ or $\tilde{m} = \infty$ can be defined in a usual way. Then we have the following.

Proposition 2.2. *Let $N \geq 3$, $\frac{N}{2(N-1)} < s < 1$, $N - s + \frac{1}{2} < \beta < \min(N, \frac{3N}{2} - s - \frac{N}{4s})$, and g such that the condition (6) holds true with*

$$\max \left(2, 1 + \frac{2\beta - N}{N - 2s} \right) < \mu < 2 + \frac{N}{N - 2s} \frac{2s - 1 - 2N + 2\beta}{2s - 1 + N}.$$

Then the H^s -weak solution to the problem \mathcal{S} constructed in proposition 2.1 is unique.

The dimension restriction $N \geq 3$ is necessary for $\frac{N}{2(N-1)} < s < 1$ and $N - s + \frac{1}{2} < \beta < \frac{3N}{2} - s - \frac{N}{4s}$, which are needed for the exponents appearing in (14).

Proof. Let $U(t) = e^{it(-\Delta)^s}$, then the solution ϕ constructed in Proposition 2.1 satisfies the integral equation

$$\phi(t) = U(t)\varphi - i \int_0^t U(t-t')\mathcal{N}(\phi(t')) dt' \quad \text{a.e. } t \in (-T_{min}, T_{max}). \quad (11)$$

Before going further, let us recall the following weighted Strichartz estimate (see for instance Lemma 6.2 of [5] and Lemma 2 of [6]).

Lemma 2.3. *Let $N \geq 2$ and $2 \leq q < 4s$. Then, for all $\psi \in L^2$, we have*

$$\| |x|^{-\delta} U(t)\psi \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \leq \eta \|\psi\|_2,$$

where $\delta = \frac{N+2s}{q} - \frac{N}{2}$, $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{1}{N-1} \left(\frac{2s}{q} - \frac{1}{2} \right)$ and η is independent of t_1, t_2 .

In [5] it was shown that

$$\| |x|^{-\delta} D_\sigma^{\frac{2s}{q} - \frac{1}{2}} U(t)\psi \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^2)} \leq \eta \|\psi\|_2.$$

Lemma 2.3 can be derived by Sobolev embedding on the unit sphere. Here $D_\sigma = \sqrt{1 - \Delta_\sigma}$ where Δ_σ is the Laplace-Beltrami operator on the unit sphere. Now, let us recall the following weighted convolution inequality we shall use in the sequel

Lemma 2.4 (Lemma 4.3 of [7]). *Let $r \in [1, \infty]$ and $0 \leq \delta \leq \gamma < N - 1$. If $\frac{1}{r} > \frac{\gamma}{N-1}$, then for all f such that $|x|^{-(\gamma-\delta)} f \in L^1$, we have*

$$\| |x|^\delta (|x|^{-\gamma} * f) \|_{L_\rho^\infty L_\sigma^r} \leq \eta \| |x|^{-(\gamma-\delta)} f \|_1.$$

Therefore, using Lemma 2.3 one can readily deduce that

$$\| |x|^{-\delta} \int_0^t U(t-t')f(t') dt' \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \leq \eta \|f\|_{L^1(-t_1, t_2; L^2)}. \quad (12)$$

Thus, if we set $f = \mathcal{N}(\phi) - \mathcal{N}(\psi)$ and $\gamma = N - \beta$. Then from (11) we infer

$$\begin{aligned}
& \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-t_1, t_2; L^q_\rho L^{\tilde{q}}_\sigma)} \\
& \leq \eta \sum_{i,j=1}^2 \int_{-t_1}^{t_2} \|\mathcal{N}_{ij}(\phi) - \mathcal{N}_{ij}(\psi)\|_2 dt', \\
& \leq \eta \sum_{i,j=1}^2 \int_{-t_1}^{t_2} \left\| \int_{\mathbb{R}^N} |x-y|^{-\gamma} (|\phi|^{\mu_i-1} + |\psi|^{\mu_i-1}) |\phi - \psi| dy |\phi|^{\mu_j-1} \right\|_2 dt', \\
& + \eta \sum_{i,j=1}^2 \int_{-t_1}^{t_2} \left\| \int_{\mathbb{R}^N} |x-y|^{-\gamma} |\psi|^{\mu_i} dy (|\phi|^{\mu_j-2} + |\psi|^{\mu_j-2}) |\phi - \psi| \right\|_2 dt', \\
& \equiv \sum_{i,j=1}^2 (\mathcal{T}_{ij}^1 + \mathcal{T}_{ij}^2).
\end{aligned}$$

We first estimate \mathcal{T}_{ij}^1 using Hölder's and Hardy-Littlewood-Sobolev inequalities. On the one side if $(i, j) = (1, 2)$, since $\mu \in (1 + \frac{2\beta-N}{N-2s}, 1 + \frac{\beta+2s}{N})$, $0 < \beta < N$ and $2s > \gamma = N - \beta$, we can find $r \in [2, \frac{2N}{N-2s}]$ such that

$$\frac{\beta}{N} = \frac{1}{r} + \frac{(\mu-1)(N-2s)}{2N}, \quad \frac{1}{r} + \frac{1}{2} > \frac{\beta}{N}.$$

Thus, we can write

$$\begin{aligned}
\mathcal{T}_{12}^1 & \leq \eta \int_{-t_1}^{t_2} (\|\phi\|_r + \|\psi\|_r) \|\phi - \psi\|_2 |\phi|^{\frac{\mu-1}{N-2s}} dt', \\
& \leq \eta(t_1 + t_2) (\|\phi\|_{L^\infty(-t_1, t_2; H^s)}^\mu + \|\psi\|_{L^\infty(-t_1, t_2; H^s)}^\mu) \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)}.
\end{aligned}$$

On the opposite side, if $(i, j) \neq (1, 2)$, then we can choose $r \in [2, \frac{2N}{N-2s}]$ such that

$$\frac{\beta}{N} = \frac{\mu_i-1}{r} + \frac{(\mu_j-1)}{r}, \quad \frac{\mu_i-1}{r} + \frac{1}{2} > \frac{\beta}{N}.$$

Such a combination is always possible thanks to our conditions on μ, β and s . Therefore, we get as above

$$\begin{aligned}
\mathcal{T}_{ij}^1 & \leq \eta \int_{-t_1}^{t_2} (\|\phi\|_r^{\mu_i-1} + \|\psi\|_r^{\mu_i-1}) \|\phi - \psi\|_2 \|\phi\|_{\frac{r}{\mu_j-1}}^{\mu_j-1} dt', \\
& \leq \eta(t_1 + t_2) (\|\phi\|_{L^\infty(-t_1, t_2; H^s)}^{\mu_i+\mu_j-2} + \|\psi\|_{L^\infty(-t_1, t_2; H^s)}^{\mu_i+\mu_j-2}) \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)}.
\end{aligned}$$

We are kept with the estimates of \mathcal{T}_{ij}^2 . If $j = 1$, then we can use Hardy-Sobolev inequality such that for $0 < q < N$ and $2 \leq p < \infty$

$$\| |x|^{-\frac{q}{p}} f \|_p \leq \eta \|f\|_{\dot{H}^{\frac{N}{2} - \frac{N-q}{p}}}. \quad (13)$$

In fact, we have

$$\begin{aligned} \mathcal{T}_{11}^2 &\leq \int_{-t_1}^{t_2} \left\| \int_{\mathbb{R}^N} |x-y|^{-\gamma} |\psi|^2 dy \right\|_{L_x^\infty} \|\phi - \psi\|_2 dt', \\ &\leq \eta \int_{-t_1}^{t_2} \|\psi\|_{\dot{H}^{\frac{N}{2}}}^2 \|\phi - \psi\|_2 dt', \\ &\leq \eta(t_1 + t_2) \|\psi\|_{L^\infty(-t_1, t_2; H^s)}^2 \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)}. \end{aligned}$$

Since $\frac{N}{2} - \frac{\beta}{\mu} \leq s$ we also have

$$\begin{aligned} \mathcal{T}_{21}^2 &\leq \int_{-t_1}^{t_2} \left\| \int_{\mathbb{R}^N} |x-y|^{-\gamma} |\psi|^\mu dy \right\|_{L_x^\infty} \|\phi - \psi\|_2 dt', \\ &\leq C \int_{-t_1}^{t_2} \|\psi\|_{\dot{H}^{\frac{N}{2} - \frac{\beta}{\mu}}}^\mu \|\phi - \psi\|_2 dt', \\ &\leq C(t_1 + t_2) \|\psi\|_{L^\infty(-t_1, t_2; H^s)}^2 \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)}. \end{aligned}$$

When $j = 2$, we use the weighted convolution inequality (Lemma 2.4). The hypothesis on β, μ guarantees the existence of exponents q, \tilde{q} and r satisfying the conditions of Lemmas 2.3, 2.4 and also the following combination

$$\frac{1}{2} = \frac{(\mu - 2)(N - 2s)}{2N} + \frac{1}{q} = \frac{1}{r} + \frac{(\mu - 2)(N - 2s)}{2N} + \frac{1}{\tilde{q}}. \quad (14)$$

Hence, using the Hardy-Sobolev inequality (13) we write

$$\begin{aligned} \mathcal{T}_{i,2}^2 &\leq \int_{-t_1}^{t_2} \| |x|^\delta \int_{\mathbb{R}^N} |x-y|^{-\gamma} |\psi|^{\mu_i} dy \|_{L_\rho^\infty L_\sigma^r} \left(\|\phi\|_{\frac{2N}{N-2s}}^{\mu-2} + \|\psi\|_{\frac{2N}{N-2s}}^{\mu-2} \right) \times \\ &\quad \times \| |x|^{-\delta} (\phi - \psi) \|_{L_\rho^q L_\sigma^{\tilde{q}}} dt', \\ &\leq \eta \int_{-t_1}^{t_2} \| |x|^{(-\gamma-\delta)} |\psi|^{\mu_i} \|_1 \left(\|\phi\|_{H^s}^{\mu-2} + \|\psi\|_{H^s}^{\mu-2} \right) \| |x|^{-\delta} (\phi - \psi) \|_{L_\rho^q L_\sigma^{\tilde{q}}} dt', \\ &\leq \eta \int_{-t_1}^{t_2} \|\psi\|_{\dot{H}^{\frac{N}{2} - \frac{\beta+\delta}{\mu_i}}}^{\mu_i} \left(\|\phi\|_{H^s}^{\mu-2} + \|\psi\|_{H^s}^{\mu-2} \right) \| |x|^{-\delta} (\phi - \psi) \|_{L_\rho^q L_\sigma^{\tilde{q}}} dt', \\ &\leq \eta(t_1 + t_2)^{1-\frac{1}{q}} \left(\|\phi\|_{L^\infty(-t_1, t_2; H^s)}^{\mu_i + \mu - 2} + \|\psi\|_{L^\infty(-t_1, t_2; H^s)}^{\mu_i + \mu - 2} \right) \times \\ &\quad \times \| |x|^{-\delta} (\phi - \psi) \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})}. \end{aligned}$$

Now, if $(-t_1, t_2) \subset [-T_1, T_2]$ and $\|\phi\|_{L^\infty(-T_1, T_2; H^s)} + \|\psi\|_{L^\infty(-T_1, T_2; H^s)} \leq K$, then by combining all the estimates above we infer

$$\begin{aligned} & \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \leq \eta(K^2 + K^{2\mu-2}) \times \\ & \quad \times (t_1 + t_2)^{1-\frac{1}{q}} \left(\|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \right). \end{aligned}$$

Thus, $\phi = \psi$ on $[-t_1, t_2]$ for sufficiently small t_1, t_2 . Let $I = (-a, b)$ be the maximal interval of $[-T_1, T_2]$ with

$$\|\phi - \psi\|_{L^\infty(-c, d; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-c, d; L_\rho^q L_\sigma^{\tilde{q}})} = 0, \quad c < a, d < b.$$

Assume that $a < T_1$ or $b < T_2$. Without loss of generality, we may also assume that $a < T_1$ and $b < T_2$. Then for a small $\varepsilon > 0$ we can find $a < t_1 < T_1, b < t_2 < T_2$ such that

$$\begin{aligned} & \|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \\ & \leq (K^2 + K^{2\mu-2})(t_1 + t_2 - a - b)^{1-\frac{1}{q}} \times \\ & \quad \times \left(\|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \right), \\ & \leq (1 - \varepsilon) \left(\|\phi - \psi\|_{L^\infty(-t_1, t_2; L^2)} + \| |x|^{-\delta}(\phi - \psi) \|_{L^q(-t_1, t_2; L_\rho^q L_\sigma^{\tilde{q}})} \right). \end{aligned}$$

This contradicts the maximality of I . Thus $I = [-T_1, T_2]$. Since $[-T_1, T_2]$ is arbitrarily taken in $(-T_{min}, T_{max})$, we finally get the whole uniqueness and the Proposition 2.2 is now proved. \square

2.3. Global well-posedness

Using the argument of [3], one can show that the uniqueness implies actually well-posedness and conservation laws:

- $\phi \in C(-T_{min}, T_{max}; H^s) \cap C^1(-T_{min}, T_{max}; H^{-s})$,
- ϕ depends continuously on ϕ_0 in H^s ,
- $\|\phi(t)\|_2 = \|\phi_0\|_2$ and $\mathcal{J}(\phi(t)) = \mathcal{J}(\phi_0) \quad \forall t \in (-T_{min}, T_{max})$.

The proofs of these points are standard, we omit them and refer to [3]. Now we remark that the well-posedness is actually global by establishing a uniform bound on the H^s norm of $\phi(t)$ for all $t \in (-T_{min}, T_{max})$.

We first consider the global existence of weak solutions. Suppose ϕ is a weak solution on $(-T_{min}, T_{max})$ as in Proposition 2.1. We show that $\|\phi(t)\|_{H^s}$ is bounded for all $t \in (-T_{min}, T_{max})$. For this purpose let us introduce the following notation

$$\mathcal{D}(G(|\phi|), G(|\phi|)) = \sum_{i,j=1}^2 \mathcal{D}_{i,j}(|\phi|), \quad \mathcal{D}_{i,j}(|\phi|) := \mathcal{D}(G_i(|\phi|), G_j(|\phi|)), \quad (15)$$

where obviously we set $G_i := \int_0^{|\cdot|} g_i(\alpha) d\alpha$ and recall that the g_i are defined as $g_1(\alpha) = \chi_{\{0 \leq \alpha < 1\}} g(\alpha)$ and $g_2(\alpha) = \chi_{\{\alpha \geq 1\}} g(\alpha)$. Using Hardy-Littlewood-Sobolev and the fractional Gagliardo-Nirenberg inequalities and the assumption \mathcal{A}_0 we can write the following estimates

$$\mathcal{D}_{1,1} \leq \eta \|u\|_{\frac{4N}{N+\beta}}^4 \leq \eta \|u\|_2^{4-\frac{N-\beta}{s}} \|u\|_{\dot{H}^s}^{\frac{N-\beta}{s}}, \quad (16)$$

$$\mathcal{D}_{2,2} \leq \eta \|u\|_{\frac{2N\mu}{N+\beta}}^{2\mu} \leq \eta \|u\|_2^{2\mu-\frac{N(\mu-1)-\beta}{s}} \|u\|_{\dot{H}^s}^{\frac{N(\mu-1)-\beta}{s}}, \quad (17)$$

$$\mathcal{D}_{1,2}, \mathcal{D}_{2,1} \leq \eta \|u\|_2^{\mu+2-\frac{N\mu-2\beta}{2s}} \|u\|_{\dot{H}^s}^{\frac{N\mu-2\beta}{2s}}. \quad (18)$$

Since $N-\beta < 2s$, then $0 < \frac{N-\beta}{s} < 2$ and $4-\frac{N-\beta}{s} > 2$. As well since $2 \leq \mu < 1 + \frac{2s+\beta}{N}$, then $0 < \frac{N-\beta}{s} \leq \frac{N(\mu-1)-\beta}{s} < 2$ and $2 < \mu - \frac{N(\mu-1)-\beta}{s}$. Eventually, we have $0 < \frac{N-\beta}{s} \leq \frac{N\mu-2\beta}{2s}$ and $2 \leq \mu < \mu + 1 + \frac{\beta-N}{2s} < \mu + 2 - \frac{N\mu-2\beta}{2s}$. The estimates above can be summarized as follows with $\mu_1 = 2$ and $\mu_2 = \mu$.

$$\begin{aligned} \mathcal{D}_{i,j}(|u|) &\leq \eta \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^{\mu_i} |u(y)|^{\mu_j}}{|x-y|^{N-\beta}} dx dy, \\ &\leq \eta \|u\|_2^{\mu_i+\mu_j-\gamma_{i,j}} \|u\|_{\dot{H}^s}^{\gamma_{i,j}} \end{aligned} \quad (19)$$

where

$$\gamma_{i,j} = \frac{N}{s} \left(1 + \frac{\beta}{N}\right) - \left(\frac{N}{2s} - 1\right) (\mu_i + \mu_j).$$

Thus, we have clearly

$$\begin{aligned} \frac{1}{2} \|\phi\|_{H^s}^2 &= \frac{1}{2} \|\phi\|_2^2 + \mathcal{J}(\phi) + \mathcal{D}(G(|\phi|), G(|\phi|)), \\ &\leq \frac{1}{2} \|\phi_0\|_2^2 + \mathcal{J}(\phi_0) + \eta \sum_{i,j=1,2} \|\phi_0\|_2^{\frac{2\gamma_{i,j}}{2-\mu_i-\mu_j+\gamma_{i,j}}} + \frac{1}{4} \|\phi\|_{H^s}^2. \end{aligned}$$

Thus

$$\|\phi\|_{H^s} \leq \eta(\|\phi_0\|_{H^s}), \quad \text{for all } t \in (-T_{min}, T_{max}).$$

Therefore $T_{min} = T_{max} = \infty$. If s, β, μ satisfy the hypothesis of Proposition 2.2, then we get the global well-posedness. Eventually, combining this fact with the Propositions 2.1 and 2.2 prove Theorem 1.1.

3. Existence of standing waves

In this section we study the minimization problem $\tilde{\mathcal{S}}$. We prove the existence of a solution to $\tilde{\mathcal{S}}$ using a variational approach via the concentration-compactness method of P-L. Lions [17]. Indeed, we aim to prove the existence of critical points to the energy functional

$$\mathcal{E}(u) = \frac{1}{2} \|\nabla_s u\|_2^2 - \frac{1}{2} \mathcal{D}(G(|u|), G(|u|)).$$

In other words, we look for a function u_λ such that

$$\mathcal{E}(u_\lambda) = \mathcal{I}_\lambda = \inf \left\{ \mathcal{E}(u), \quad u \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u(x)|^2 dx = \lambda \right\}.$$

As noticed in the introduction of this paper, this problem has been studied in various situation depending on the value of s and the conditions on β and the integrand G in Ref. [5, 12, 18]. In order to prove the existence of critical points to the functional \mathcal{E} , we start with the following claim

Proposition 3.1. *For all $\lambda > 0$ and G such that \mathcal{A}_0 and \mathcal{A}_1 hold true, we have*

- *The functional $\mathcal{E} \in C^1(H^s, \mathbb{R})$ and there exists a constant $\eta > 0$ such that*

$$\|\mathcal{E}'(u)\|_{H^{-s}} \leq \eta \left(\|u\|_{H^s} + \|u\|_{H^s}^{\frac{2s+\beta}{N}} \right).$$

- $-\infty < \mathcal{I}_\lambda < 0$.
- *Each minimizing sequence for the problem \mathcal{I}_λ is bounded in H^s .*

Proof. Let us mention that only assumption \mathcal{A}_0 is needed to prove the C^1 property of the energy functional \mathcal{E} . The proof of this claim is standard and we refer the reader to Ref. [11] for details. Now, we prove the second

assertion. Let $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_2 = \sqrt{\lambda}$ and assume \mathcal{A}_0 . Then, on the one hand, thanks to (16–18), it is rather easy to show using Young's inequality that for all ϵ_1, ϵ_2 and ϵ_3 , there exist $C_{\epsilon_1}, C_{\epsilon_2}, C_{\epsilon_3} > 0$ such that

$$\mathcal{D}_{1,1} \leq \eta (\epsilon_1 \|u\|_{\dot{H}^s}^2 + C_{\epsilon_1} \lambda^{e_1}), \quad e_1 := \frac{4s + \beta - N}{2s + \beta - N}. \quad (20)$$

$$\mathcal{D}_{1,2} \leq \eta (\epsilon_2 \|u\|_{\dot{H}^s}^2 + C_{\epsilon_2} \lambda^{e_2}), \quad e_2 := \frac{2s\mu + \beta - N(\mu - 1)}{2s + \beta - N(\mu - 1)}. \quad (21)$$

$$\mathcal{D}_{1,2}, \mathcal{D}_{2,1} \leq \eta (\epsilon_3 \|u\|_{\dot{H}^s}^2 + C_{\epsilon_3} \lambda^{e_3}), \quad e_3 := 1 + \frac{2s\mu}{4s - N\mu + 2\beta}. \quad (22)$$

Observe that $0 < 2s + \beta - N < 4s + \beta - N$ so that $e_1 > 1$. Also, $0 < 2s + \beta - N(\mu - 1) < 2s\mu + \beta - N(\mu - 1)$ so that $e_2 > 1$. Eventually, $4s - N\mu + 2\beta > 2s + \beta - N > 0$ so that $\frac{2s\mu}{4s - N\mu + 2\beta} > 0$ and $e_3 > 1$. Therefore, for sufficiently small ϵ_1, ϵ_2 and ϵ_3 , one has

$$\begin{aligned} \mathcal{E}(u) &\geq \left(\frac{1}{2} - \eta(\epsilon_1 + \epsilon_2 + \epsilon_3) \right) \|u\|_{H^s}^2 - \frac{1}{2} - \eta(C_{\epsilon_1} \lambda^{e_1} + C_{\epsilon_2} \lambda^{e_2} + C_{\epsilon_3} \lambda^{e_3}), \\ &\geq -\frac{1}{2} \lambda - \eta(C_{\epsilon_1} \lambda^{e_1} + C_{\epsilon_2} \lambda^{e_2} + C_{\epsilon_3} \lambda^{e_3}). \end{aligned}$$

Thus, we obtain $\mathcal{I}_\lambda > -\infty$. On the other hand, let us introduce for all $\kappa \in \mathbb{R}$, the rescaled function $u_\kappa = \kappa^{\frac{1}{2}} u(\kappa^{\frac{1}{N}} \cdot)$. Obviously, one has $\int_{\mathbb{R}^N} |u_\kappa|^2 = \lambda$ and using \mathcal{A}_1

$$\mathcal{E}(u_\kappa) \leq \frac{1}{2} \kappa^{\frac{2s}{N}} \int_{\mathbb{R}^N} |(-\Delta)^s u(x)|^2 dx - \frac{\kappa^{\alpha - (1 + \frac{\beta}{N})}}{2} \mathcal{D}(|u(x)|^\alpha, |u(y)|^\alpha).$$

We have $0 < \alpha - (1 + \frac{\beta}{N}) < \frac{2s}{N}$, therefore we can take κ small enough to get $\mathcal{E}(u_\kappa) < 0$. Thus, $\mathcal{I}_\lambda \leq \mathcal{E}(u_\kappa) < 0$.

We are kept with the proof of the third assertion. Let $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence for the problem \mathcal{I}_λ . Therefore, thanks to (20–22), we have for all $u \in H^s$

$$\mathcal{D}(G(|u|), G(|u|)) \leq \eta(\epsilon_1 + \epsilon_2 + \epsilon_3) \|u\|_{\dot{H}^s}^2 + \eta(C_{\epsilon_1} \lambda^{e_1} + C_{\epsilon_2} \lambda^{e_2} + C_{\epsilon_3} \lambda^{e_3}).$$

Hence

$$\begin{aligned} \|u_n\|_{H^s}^2 &= 2\mathcal{E}(u_n) + \|u_n\|_2^2 + \mathcal{D}(G(|u_n|), G(|u_n|)), \\ &\leq 2\mathcal{I}_\lambda + \lambda + \eta(\epsilon_1 + \epsilon_2 + \epsilon_3) \|u_n\|_{H^s}^2 + \eta(C_{\epsilon_1} \lambda^{e_1} + C_{\epsilon_2} \lambda^{e_2} + C_{\epsilon_3} \lambda^{e_3}). \end{aligned}$$

Eventually, we pick ϵ_1, ϵ_2 and ϵ_3 such that $\eta(\epsilon_1 + \epsilon_2 + \epsilon_3) < 1$, we get immediately that the minimizing sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in H^s .

□

Before going further, let us introduce the so called Lévy concentration function

$$\mathcal{Q}_n(r) = \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n(x)|^2 dx.$$

It is known that each \mathcal{Q}_n is nondecreasing on $(0, +\infty)$. Also, with the Helly's selection Theorem, the sequence $(\mathcal{Q}_n)_{n \in \mathbb{N}}$ has a subsequence that we still denote $(\mathcal{Q}_n)_{n \in \mathbb{N}}$ by abuse of notation, such that there is a nondecreasing function $\mathcal{Q}(r)$ satisfying

$$\mathcal{Q}_n(r) \xrightarrow{n \rightarrow +\infty} \mathcal{Q}(r), \quad \text{for all } r > 0.$$

Since $0 \leq \mathcal{Q}_n(r) \leq \lambda$, there exists $\beta \in \mathbb{R}$ such that $0 \leq \beta \leq \lambda$ such that

$$\mathcal{Q}(r) \xrightarrow{r \rightarrow +\infty} \gamma.$$

Briefly speaking, a minimizing sequence $(u_n)_{n \in \mathbb{N}}$ for the problem \mathcal{I}_λ can only be in one of the following situations:

- Vanishing, i.e. $\gamma = 0$.
- Dichotomy, i.e. $0 < \gamma < \lambda$.
- Compactness, i.e. $\gamma = \lambda$.

In the sequel we shall proceed by elimination and show that vanishing and dichotomy do not occur. Therefore, compactness holds true and we are done. We start with the following

Proposition 3.2. *Let $\lambda > 0$ and $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence of problem \mathcal{I}_λ with G such that \mathcal{A}_0 and \mathcal{A}_1 hold true. Then $\gamma > 0$.*

The proposition claims then that the situation of vanishing does not occurs. In the proof of Proposition 3.2, we shall use, for all subset of $A \subset \mathbb{R}^N$, the notation

$$\mathcal{D}|_A(G(|u|), G(|u|)) := \int_{A \times A} G(|u(x)|) V(|x - y|) G(|u(y)|) dx dy.$$

Proof. Let us first prove that $\mathcal{D}(G(|u_n|), G(|u_n|))$ is lower bounded. In other words, we show that for $n \in \mathbb{N}$ large enough there exists $\delta > 0$ such that

$$\delta < \mathcal{D}(G(|u_n|), G(|u_n|)). \quad (23)$$

We argue by contradiction and assume that there exist no such δ , therefore $\liminf_{n \rightarrow +\infty} \mathcal{D}(G(|u_n|), G(|u_n|)) \leq 0$, thus

$$\begin{aligned} \mathcal{I}_\lambda = \lim_{n \rightarrow +\infty} \mathcal{E}(u_n) &= \lim_{n \rightarrow +\infty} \left(\frac{1}{2} \|\nabla_s u_n\|_2^2 - \frac{1}{2} \mathcal{D}(G(|u_n|), G(|u_n|)) \right) \\ &\geq -\frac{1}{2} \lim_{n \rightarrow +\infty} \mathcal{D}(G(|u_n|), G(|u_n|)) \geq 0. \end{aligned}$$

The inequality above is in contradiction with the fact that $\mathcal{I}_\lambda < 0$. On the other hand, arguing by contradiction and assuming that the minimizing sequence $(u_n)_{n \in \mathbb{N}}$ vanishes, i.e. assume that $\gamma = 0$. Then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ and a radius $\tilde{r} > 0$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{B(y, \tilde{r})} |u_{n_k}(x)|^2 dx \xrightarrow{k \rightarrow +\infty} 0.$$

Next, since the sequence $(u_{n_k})_{k \in \mathbb{N}}$ is bounded in H^s , then one can find $r_\epsilon > 0$ such that

$$\mathcal{D}|_{|x-y| \geq r_\epsilon}(G(|u_{n_k}|), G(|u_{n_k}|)) \leq \frac{\epsilon}{2}.$$

Now, we cover \mathbb{R}^N by balls of radius r and centers c_i for $i = 1, 2, \dots$ such that each point of \mathbb{R}^N is contained in at most $N + 1$ ball. Therefore, there

exists N_ϵ ball and a subsequence $(c_{i_l})_{l=1,\dots,N_\epsilon}$ such that

$$\begin{aligned}
\mathcal{D}|_{|x-y|\geq r_\epsilon}(G(|u_{n_k}|), G(|u_{n_k}|)) &\leq \eta \sum_{p,q=1}^2 \mathcal{D}_{p,q}|_{|x-y|\geq r_\epsilon}(|u_{n_k}|), \\
&\leq \eta \sum_{p,q=1}^2 \sum_{l=1}^\infty \sum_{i=1}^{N_\epsilon} \int_{B_x(c_l,r)} \int_{B_y(c_{i_l},r)} \frac{|u_{n_k}(x)|^{\mu_p} |u_{n_k}(y)|^{\mu_q}}{|x-y|^{N-\beta}} dx dy, \\
&\leq \eta \sum_{p,q=1}^2 \sum_{l=1}^\infty \sum_{i=1}^{N_\epsilon} \|u_{n_k}\|_{L^2(B_x(c_l,r))} \left\| \int_{B_y(c_{i_l},r)} \frac{|u_{n_k}(y)|^{\mu_q}}{|x-y|^{N-\beta}} dy |u_{n_k}|^{\mu_p-1} \right\|_{L^2(B_x(c_l,r))}, \\
&\leq N_\epsilon \eta \left(\sum_{l=1}^\infty \|u_{n_k}\|_{L^2(B_x(c_l,r))} \right) \|u_{n_k}\|_r \|u_{n_k}\|^{\frac{\mu-1}{N-2s}} \sup_{y \in \mathbb{R}^N} \|u_{n_k}\|_{L^2(B(y,r))} \\
&+ N_\epsilon \eta \sum_{(p,q) \neq (1,2), p,q=1}^2 \left(\sum_{l=1}^\infty \|u_{n_k}\|_{L^2(B_x(c_l,r))} \right) \|u_{n_k}\|_{r_{pq}}^{\mu_q-1} \|u_{n_k}\|^{\frac{\mu_p-1}{\mu_p-1}} \times \\
&\quad \times \sup_{y \in \mathbb{R}^N} \|u_{n_k}\|_{L^2(B(y,r))},
\end{aligned}$$

where r and r_{pq} are such that

$$\frac{\beta}{N} = \frac{1}{r} + (\mu - 1) \left(\frac{1}{2} - \frac{s}{N} \right), \quad \frac{1}{r} + \frac{1}{2} > \frac{\beta}{N},$$

$$\frac{\mu_p - 1}{r_{pq}} + \frac{\mu_q - 1}{r_{pq}} = \frac{\beta}{N}, \quad \frac{\mu_p - 1}{r_{pq}} + \frac{1}{2} > \frac{\beta}{N}, \quad (p, q) \neq (1, 2).$$

Since $1 + \frac{2\beta-N}{N-2s} < \mu < 1 + \frac{\beta+2s}{N}$, $0 < \beta < N$ and $s > \frac{N-\beta}{2}$, it is rather clear that one can find (as in section 2) $r, r_{p,q} \in [2, \frac{2N}{N-2s}]$. Consequently, we have obviously

$$\begin{aligned}
\mathcal{D}|_{|x-y|\geq r_\epsilon}(G(|u_{n_k}|), G(|u_{n_k}|)) &\leq (N+1) N_\epsilon \eta \|u_{n_k}\|_2 \times \\
&\quad \times \left(\|u_{n_k}\|_{H^s}^\mu + \|u_{n_k}\|_{H^s}^2 + \|u_{n_k}\|_{H^s}^{2(\mu-1)} \right) \left(\sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_{n_k}|^2 \right)^{\frac{1}{2}}. \\
&\quad \xrightarrow{n \rightarrow +\infty} 0.
\end{aligned}$$

This shows that if the minimizing sequence $(u_n)_{n \in \mathbb{N}}$ vanishes, then

$$\mathcal{D}(G(|u_n|), G(|u_n|)) \xrightarrow{n \rightarrow +\infty} 0.$$

This is in contradiction with the property (23), namely for $n \in \mathbb{N}$ large enough there exists $\gamma > 0$ such that $\mathcal{D}(G(|u_n|), G(|u_n|)) > \gamma$. Thus, vanishing does not occurs. \square

Now, we show the following

Proposition 3.3. *Let $0 < \pi < \lambda$ and G such that \mathcal{A}_0 and \mathcal{A}_1 hold true. Then the mapping $\lambda \mapsto \mathcal{I}_\lambda$ is continuous and $\mathcal{I}_\lambda < \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}$.*

Proof. Let $\lambda > 0$ and $(\lambda_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that $\lambda_k \xrightarrow[k \rightarrow +\infty]{} \lambda$. Let $\epsilon > 0$ and $u \in H^s(\mathbb{R}^N)$ such that $\|u\|_2 = \sqrt{\lambda}$ and

$$\mathcal{I}_\lambda \leq \mathcal{E}(u) \leq \mathcal{I}_\lambda + \frac{\epsilon}{2}.$$

For all $k \in \mathbb{N}$, let $u_k = \sqrt{\frac{\lambda_k}{\lambda}} u$. Obviously $u_k \in H^s(\mathbb{R}^N)$ and $\|u_k\|_2^2 = \lambda_k$ so that for all $k \in \mathbb{N}$, $\mathcal{I}_{\lambda_k} \leq \mathcal{E}(u_k)$. Now, we show that $\mathcal{E}(u_k) \xrightarrow[k \rightarrow +\infty]{} \mathcal{E}(u)$. First, for all $k \in \mathbb{N}$

$$\|u_k - u\|_{\dot{H}^s} \leq \|u_k\|_{\dot{H}^s} \left| 1 - \sqrt{\frac{\lambda_k}{\lambda}} \right|.$$

Since any sequence of \mathcal{I}_λ is bounded in $H^s(\mathbb{R}^N)$ and $\lambda_k \xrightarrow[k \rightarrow +\infty]{} \lambda$, then we have obviously $\frac{1}{2} \|\nabla_s u_k\|_2^2 \xrightarrow[k \rightarrow +\infty]{} \frac{1}{2} \|\nabla_s u\|_2^2$. Next, following the first assertion of Proposition 3.1, we have $\mathcal{E}(u) \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$. In particular, one can easily see from the proof of this point that $D(u) := \mathcal{D}(G(|u|), G(|u|)) \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and

$$|\mathcal{D}'(u)| \leq \eta \left(\|u\|_{H^s} + \|u\|_{H^s}^{\frac{2s+\beta}{N}} \right). \quad (24)$$

We refer to Ref. [11] for details. Therefore, we have

$$\begin{aligned} |\mathcal{D}(u_k) - \mathcal{D}(u)| &= \left| \int_0^1 \frac{d}{dt} \mathcal{D}(tu_k + (1-t)u) dt \right|, \\ &\leq \eta \sup_{u \in H^s, \|u\|_{H^s} \leq \eta} \|\mathcal{D}'(u)\|_{H^{-s}} \|u_k - u\|_{H^s}, \\ &\leq \eta \|u_k\|_{H^s} \left| 1 - \sqrt{\frac{\lambda_k}{\lambda}} \right| \xrightarrow[k \rightarrow +\infty]{} 0. \end{aligned}$$

Thus, we have $\mathcal{E}(u_k) \xrightarrow[k \rightarrow +\infty]{} \mathcal{E}(u)$. Consequently, we have $\mathcal{I}_{\lambda_k} \leq \mathcal{I}_\lambda + \epsilon$ for k large enough. Next, for all $k \in \mathbb{N}$, let us choose $\tilde{u}_k \in H^s(\mathbb{R}^N)$ such that $\|\tilde{u}_k\|_2 = \sqrt{\lambda_k}$ and $\mathcal{E}(\tilde{u}_k) \leq \mathcal{I}_{\lambda_k} + \frac{1}{k}$. Moreover, for all $k \in \mathbb{N}$, we set $\bar{u}_k = \sqrt{\frac{\lambda}{\lambda_k}} \tilde{u}_k$. Obviously, since $\bar{u}_k \in H^s(\mathbb{R}^N)$ and $\|\bar{u}_k\|_2^2 = \lambda$, we have $\mathcal{I}_\lambda \leq \mathcal{E}(\bar{u}_k)$. Exactly the same argument as above shows that $\mathcal{E}(\tilde{u}_k) \xrightarrow[k \rightarrow +\infty]{} \mathcal{E}(\bar{u})$ so that for k large enough, we have $\mathcal{I}_\lambda \leq \mathcal{I}_{\lambda_k} + \epsilon$. Whence, $\lambda \mapsto \mathcal{I}_\lambda$ is continuous on \mathbb{R}_+^* . Eventually, using the energy estimates (16-18) or (19), it is rather easy to show that $\mathcal{I}_\lambda \xrightarrow[\lambda \rightarrow 0^+]{} 0$. This shows that the mapping $\lambda \mapsto \mathcal{I}_\lambda$ is continuous.

Let us now prove the strict sub-additivity inequality. For that purpose, we introduce $u_\theta = \theta^\kappa u(\theta^{\frac{\kappa}{N}})$ for all $\kappa > \frac{N}{N+2s}$. Obviously $u_\kappa \in H^s(\mathbb{R}^N)$ and $\|u_\theta\|_{L^2(\mathbb{R}^N)} = \sqrt{\theta\lambda}$. Moreover, using \mathcal{A}_1 , we have

$$\begin{aligned} \mathcal{E}(u_\theta) &= \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_\theta|^2 dx - \frac{1}{2} \mathcal{D}(G(|u_\theta|), G(|u_\theta|)), \\ &\leq \frac{\theta^{\kappa(1+\frac{2s}{N})}}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \mathcal{D}(G(|u|), G(|u|)) \right) = \theta^{\kappa(1+\frac{2s}{N})} \mathcal{E}(u). \end{aligned}$$

Thus, we deduce that $\mathcal{I}_{\theta\lambda} \leq \theta^{\kappa(1+\frac{2s}{N})} \mathcal{I}_\lambda$ for all $\theta > 0$. Now we let $0 < \pi < \lambda$, therefore since $\kappa(1+\frac{2s}{N}) > 1$ we have

$$\begin{aligned} \mathcal{I}_\lambda &\leq \lambda^{\kappa(1+\frac{2s}{N})} \mathcal{I}_1 < \pi^{\kappa(1+\frac{2s}{N})} \mathcal{I}_1 + (\lambda - \pi)^{\kappa(1+\frac{2s}{N})} \mathcal{I}_1, \\ &\leq \pi^{\kappa(1+\frac{2s}{N})} \pi^{-\kappa(1+\frac{2s}{N})} \mathcal{I}_\pi + (\lambda - \pi)^{\kappa(1+\frac{2s}{N})} (\lambda - \pi)^{-\kappa(1+\frac{2s}{N})} \mathcal{I}_{\lambda-\pi}, \\ &= \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}. \end{aligned}$$

In summary, for all $0 < \pi < \lambda$, we have $\mathcal{I}_\lambda < \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}$. \square

Now, we are able to claim the following

Proposition 3.4. *Let $\lambda > 0$ and $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence of problem \mathcal{I}_λ with G such that \mathcal{A}_0 and \mathcal{A}_1 hold true. Then dichotomy does not occur for $(u_n)_{n \in \mathbb{N}}$.*

Proof. Let us introduce ξ and χ in C^∞ such that $0 \leq \xi, \chi \leq 1$ and

$$\xi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases}, \quad \chi(x) = 1 - \xi(x), \quad \|\nabla \xi\|_\infty, \|\nabla \chi\|_\infty \leq 2.$$

For all $r > 0$, let $\xi_r(\cdot) = \xi(\frac{\cdot}{R})$ and $\chi_r(\cdot) = \chi(\frac{\cdot}{R})$. we will show that dichotomy does not occur by contradicting the fact that for all $0 < \pi < \lambda$, we have $\mathcal{I}_\lambda < \mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi}$ proved in Proposition 3.3. Indeed, let $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence of problem \mathcal{I}_λ and assume that dichotomy holds. Then, using the construction of [17], there exist

- $0 < \pi < \lambda$,
- a sequence $(y_n)_{n \in \mathbb{N}}$ of points in \mathbb{R}^N ,
- two increasing sequences of positive real number $(r_{1,n})_{n \in \mathbb{N}}$ and $(r_{2,n})_{n \in \mathbb{N}}$ such that

$$r_{1,n} \xrightarrow{n \rightarrow +\infty} +\infty \quad \text{and} \quad \frac{r_{2,n}}{2} - r_{1,n} \xrightarrow{n \rightarrow +\infty} +\infty,$$

such that the sequences $u_{1,n} = \xi_{r_{1,n}}(\cdot - y_n)u_n$ and $u_{2,n} = \chi_{r_{2,n}}(\cdot - y_n)u_n$ satisfy

$$\left\{ \begin{array}{l} u_n = u_{1,n} \text{ on } B(y_n, r_{1,n}), \\ u_n = u_{2,n} \text{ on } B^c(y_n, r_{2,n}) = \mathbb{R}^N \setminus B(y_n, r_{2,n}), \\ \int_{\mathbb{R}^N} |u_{1,n}|^2 dx \xrightarrow{n \rightarrow +\infty} \pi, \quad \int_{\mathbb{R}^N} |u_{2,n}|^2 dx \xrightarrow{n \rightarrow +\infty} \lambda - \pi, \\ \|u_n - (u_{1,n} + u_{2,n})\|_p \xrightarrow{n \rightarrow +\infty} 0, \text{ for all } 2 \leq p < \frac{2N}{N-2s}, \\ \|u_n\|_{L^p(B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n}))} \xrightarrow{n \rightarrow +\infty} 0, \text{ for all } 2 \leq p < \frac{2N}{N-2s}, \\ \text{dist}(\text{Supp}(u_{1,n}), \text{Supp}(u_{2,n})) \xrightarrow{n \rightarrow +\infty} +\infty. \end{array} \right.$$

We have obviously

$$\begin{aligned} \mathcal{E}(u_n) &= \mathcal{E}(u_{1,n}) + \mathcal{E}(u_{2,n}) + \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 - \frac{1}{2} \mathcal{D}(G(|u_n|), G(|u_n|)) dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_{1,n}|^2 + |(-\Delta)^{\frac{s}{2}} u_{2,n}|^2) dx \\ &\quad + \frac{1}{2} (\mathcal{D}(G(|u_{1,n}|), G(|u_{1,n}|)) + \mathcal{D}(G(|u_{2,n}|), G(|u_{2,n}|))). \end{aligned}$$

Now we show the existence of $\epsilon > 0$ such that for sufficiently large radius $r_{1,n}$ and $r_{1,n}$ we have

$$\frac{1}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 - |(-\Delta)^{\frac{s}{2}} u_{1,n}|^2 - |(-\Delta)^{\frac{s}{2}} u_{2,n}|^2) dx \geq -\eta\epsilon. \quad (25)$$

Firs of all, it is rather easy to show that by construction of the sequences $u_{i,n}$ for $i = 1, 2$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u_n|^2 - |(-\Delta)^{\frac{s}{2}} u_{1,n}|^2 - |(-\Delta)^{\frac{s}{2}} u_{2,n}|^2) dx \\ \geq - \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy \\ - \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\chi_{r_{2,n}}(x - y_n) - \chi_{r_{2,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Indeed, the estimate above is justified using the definition (5) combined with the following basic fact for $u_{1,n}$

$$\begin{aligned} |u_{1,n}(x) - u_{1,n}(y)|^2 &= |\xi_{r_{1,n}}(x - y_n)u_n(x) - \xi_{r_{1,n}}(y - y_n)u_n(y)|^2 \\ &\leq \frac{1}{2} |\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 (|u_{1,n}(x)|^2 + |u_{1,n}(y)|^2) \\ &\quad + \frac{1}{2} (|\xi_{r_{1,n}}(x - y_n)|^2 + |\xi_{r_{1,n}}(y - y_n)|^2) |u_{1,n}(x) - u_{1,n}(y)|^2. \end{aligned}$$

and equivalently for $u_{2,n}$

$$\begin{aligned} |u_{2,n}(x) - u_{2,n}(y)|^2 &= |\chi_{r_{2,n}}(x - y_n)u_n(x) - \chi_{r_{2,n}}(y - y_n)u_n(y)|^2 \\ &\leq \frac{1}{2} |\chi_{r_{2,n}}(x - y_n) - \chi_{r_{2,n}}(y - y_n)|^2 (|u_{2,n}(x)|^2 + |u_{2,n}(y)|^2) \\ &\quad + \frac{1}{2} (|\chi_{r_{2,n}}(x - y_n)|^2 + |\chi_{r_{2,n}}(y - y_n)|^2) |u_{2,n}(x) - u_{2,n}(y)|^2. \end{aligned}$$

In order to show (25), it suffices to show that there exist $\epsilon > 0$ such that for large radius $r_{1,n}$ and $r_{2,n}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy &\leq \eta\epsilon, \\ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\chi_{r_{2,n}}(x - y_n) - \chi_{r_{2,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy &\leq \eta\epsilon. \end{aligned}$$

We prove the first assertion and the second one follows equivalently. Indeed, we split the sum in two part as follows

$$\begin{aligned}
& \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy \\
&= \int_{|x-y| \leq r_{1,n}} \frac{|\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy \\
&+ \int_{|x-y| > r_{1,n}} \frac{|\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy := \mathcal{T}_1 + \mathcal{T}_2
\end{aligned}$$

Now, we write

$$\begin{aligned}
\mathcal{T}_1 &\leq r_{1,n}^{-2} \int_{|x-y| \leq r_{1,n}} \frac{|u_n(x)|^2}{|x - y|^{N+2s-2}} dx dy \\
&\leq r_{1,n}^{-2} \int_{\mathbb{R}^N} |u_n(x)|^2 dx \int_{|x| \leq r_{1,n}} \frac{1}{|x|^{N+2s-2}} dx \leq \eta r_{1,n}^{-2s} \int_{\mathbb{R}^N} |u_n(x)|^2 dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathcal{T}_2 &\leq r_{1,n}^{-s} \int_{|x-y| > r_{1,n}} \frac{|\xi_{r_{1,n}}(x - y_n) - \xi_{r_{1,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+s}} dx dy \\
&\leq \eta r_{1,n}^{-s} \int_{\mathbb{R}^N} |u_n(x)|^2 dx \int_{|x-y| > r_{1,n}} \frac{1}{|x - y|^{N+s}} dy \leq \eta r_{1,n}^{-s} \int_{\mathbb{R}^N} |u_n(x)|^2 dx.
\end{aligned}$$

Eventually summing up \mathcal{T}_1 and \mathcal{T}_2 and use the same argument in order to handle the term $\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\chi_{r_{2,n}}(x - y_n) - \chi_{r_{2,n}}(y - y_n)|^2 |u_n(x)|^2}{|x - y|^{N+2s}} dx dy$, one ends with

$$\begin{aligned}
& \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} u_n|^2 - |(-\Delta)^{\frac{s}{2}} u_{1,n}|^2 - |(-\Delta)^{\frac{s}{2}} u_{2,n}|^2 \right) dx \\
&\geq -\eta (r_{1,n}^{-2s} + r_{1,n}^{-s} + r_{2,n}^{-2s} + r_{2,n}^{-s}) \int_{\mathbb{R}^N} |u_n(x)|^2 dx.
\end{aligned}$$

The estimate (25) follows for $r_{1,n}$ and $r_{2,n}$ large enough. Next, observe that $|u_n - u_{1,n} - u_{2,n}| \leq 3 \mathbb{1}_{(B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n}))}$ where $\mathbb{1}_{(B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n}))}$ denotes

the characteristic function of $B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n})$. Now, we have

$$\begin{aligned}
& |\mathcal{D}(G(|u_n|), G(|u_n|)) - \mathcal{D}(G(|v_n|), G(|v_n|)) - \mathcal{D}(G(|w_n|), G(|w_n|))| \\
& \leq \int_{B(y_n, 2r) \setminus \bar{B}(y_n, 2r)} \left(\left| \frac{G(|u_n|)G(|u_n|)}{|x-y|^{N-\beta}} \right| + \left| \frac{G(|v_n|)G(|v_n|)}{|x-y|^{N-\beta}} \right| \right. \\
& \quad \left. + \left| \frac{G(|w_n|)G(|w_n|)}{|x-y|^{N-\beta}} \right| \right) dx dy, \\
& \leq \eta \left(\|u\|_{L^2(B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n}))}^{4-\frac{N-\beta}{s}} \|u\|_{H^s}^{\frac{N-\beta}{s}} + \|u\|_{L^2(B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n}))}^{2\mu-\frac{N(\mu-1)-\beta}{s}} \|u\|_{H^s}^{\frac{N(\mu-1)-\beta}{s}} \right) \\
& + \eta \|u\|_{L^2(B(y_n, r_{2,n}) \setminus B(y_n, r_{1,n}))}^{\mu+2-\frac{N\mu-2\beta}{2s}} \|u\|_{H^s}^{\frac{N\mu-2\beta}{2s}} \xrightarrow{n \rightarrow +\infty} 0.
\end{aligned}$$

where we used the estimates (16–18). Thus, for $r_{2,n}$ and $r_{1,n}$ large enough we have

$$-\frac{1}{2} (\mathcal{D}(G(|u_n|), G(|u_n|)) - \mathcal{D}(G(|v_n|), G(|v_n|)) - \mathcal{D}(G(|w_n|), G(|w_n|))) \geq -\eta\epsilon. \quad (26)$$

Summing up (25) and (26), we end up for large $r_{1,n}$ and $r_{2,n}$ with

$$\mathcal{E}(u_n) - \mathcal{E}(u_{1,n}) - \mathcal{E}(u_{2,n}) \geq -\eta\epsilon. \quad (27)$$

Since we have $\int_{\mathbb{R}^N} |u_{1,n}|^2 dx \xrightarrow{n \rightarrow +\infty} \pi$ and $\int_{\mathbb{R}^N} |u_{1,n}|^2 dx \xrightarrow{n \rightarrow +\infty} \lambda - \pi$, there exist two positive real sequences $(\mu_{1,n})_{n \in \mathbb{N}}$ and $(\mu_{2,n})_{n \in \mathbb{N}}$ such that $|\mu_{1,n} - 1|, |\mu_{2,n} - 1| < \epsilon$ and

$$\int_{\mathbb{R}^N} |\mu_{1,n} u_{1,n}|^2 dx = \pi, \quad \int_{\mathbb{R}^N} |\mu_{2,n} u_{2,n}|^2 dx = \lambda - \pi,$$

so that

$$\begin{aligned}
\mathcal{I}_\pi & \leq \mathcal{E}(\mu_{1,n} u_{1,n}) \leq \mathcal{E}(u_{1,n}) + \frac{\eta\epsilon}{2}, \\
\mathcal{I}_{\lambda-\pi} & \leq \mathcal{E}(\mu_{2,n} u_{2,n}) \leq \mathcal{E}(u_{2,n}) + \frac{\eta\epsilon}{2}.
\end{aligned}$$

Thus, with (27), we have and the continuity of the mapping $\lambda \mapsto \mathcal{I}_\lambda$ for all $\lambda > 0$, we have

$$\mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi} - 3\eta\epsilon \leq \mathcal{E}(u_{1,n}) + \mathcal{E}(u_{2,n}) - \eta\epsilon \leq \mathcal{E}(u_n) \xrightarrow{n \rightarrow +\infty} \mathcal{I}_\lambda.$$

In summary, we proved that for all $0 < \pi < \lambda$, we have $\mathcal{I}_\pi + \mathcal{I}_{\lambda-\pi} \leq \mathcal{I}_\lambda$ contradicting the strict sub-additivity inequality proved above. Then, the dichotomy does not occur. \square

Now, we finish the proof of Theorem 1.2. Since vanishing and dichotomy do not occur for any minimizing sequence $(u_n)_{n \in \mathbb{N}}$ for the problem \mathcal{I}_λ , then the compactness certainly occurs. Following the concentration-compactness principle [17], we know that every minimizing sequence $(u_n)_{n \in \mathbb{N}}$ of \mathcal{I}_λ satisfies (up to extraction if necessary)

$$\lim_{r \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n(x)|^2 dx = \lambda.$$

That is, for all $\epsilon > 0$, there exist $r_\epsilon > 0$ and $n_\epsilon \in \mathbb{N}^*$ and $\{y_n\} \subset \mathbb{R}^N$ such that for all $r > r_\epsilon$ and $n \geq n_\epsilon$, we have

$$\int_{B(y_n, r)} |u_n(x)|^2 dx = \lambda - \epsilon$$

Now, let $w_n = u_n(x + y_n)$, we have obviously that $\|w_n\|_{H^s} = \|u_n\|_{H^s}$ is bounded in $H^s(\mathbb{R}^N)$, therefore $(w_n)_{n \in \mathbb{N}}$ (up to extraction if necessary) converges weakly to w in $H^s(\mathbb{R}^N)$. In particular $(w_n)_{n \in \mathbb{N}}$ converges weakly to w in $L^2(\mathbb{R}^N)$ and $\|w_n\|_2 = \sqrt{\lambda}$. Now, let $\tilde{r}_\epsilon > r_\epsilon$ such that $\|w\|_{L^2(B^c(0, \tilde{r}_\epsilon))} < \frac{\epsilon}{2}$. Thus, there exists $\tilde{n}_\epsilon \in \mathbb{N}^*$, $\tilde{n}_\epsilon > n_\epsilon$ such that for all $n \geq \tilde{n}_\epsilon$, we have $\|w_n - w\|_{L^2(B(0, \tilde{r}_\epsilon))} < \frac{\epsilon}{2}$. Therefore, with the triangle inequality, we have

$$\begin{aligned} \|w\|_2 &\geq \|u_n\|_2 - \|w_n - w\|_{L^2(B(0, \tilde{r}_\epsilon))} - \|w_n - w\|_{L^2(B^c(0, \tilde{r}_\epsilon))}, \\ &\geq \|u_n\|_{L^2(B(y_n, \tilde{r}_\epsilon))} - \|w_n - w\|_{L^2(B(0, \tilde{r}_\epsilon))} - \|w\|_{L^2(B^c(0, \tilde{r}_\epsilon))} \geq \sqrt{\lambda} - \epsilon - \epsilon. \end{aligned}$$

Passing to the limit we get $\|w\|_2 \geq \sqrt{\lambda}$. Since the L^2 is lower semi continuous, we obtain that $\|w\|_2 \leq \liminf_{n \rightarrow +\infty} \|w_n\|_2 = \sqrt{\lambda}$. Eventually, we get $\|w\|_2 = \sqrt{\lambda}$, therefore the sequence $(w_n)_{n \in \mathbb{N}}$ converges strongly in $L^2(\mathbb{R}^N)$ to w .

Also, we have

$$\begin{aligned} &|\mathcal{D}(G(|w_n|), G(|w_n|)) - \mathcal{D}(G(|w|), G(|w|))| \\ &\leq \left| \int_0^t \frac{d}{dt} \mathcal{D}(tG(|w_n|) + (1-t)G(|w|)) dt \right|, \\ &\leq \eta \sup_{u \in H^s, \|u\|_{H^s} \leq \eta} \|\mathcal{D}'(u)\|_{H^{-s}} \|w_n - w\|_{H^s}, \\ &\leq \eta \|w_n - w\|_2 + \eta \|w_n - w\|_{\frac{2s+\beta}{N}} \xrightarrow[n \rightarrow +\infty]{} 0. \end{aligned}$$

In the last line we used 24-kind inequality and again we refer to [11] for a proof. Using the lower semi-continuity of the $-s$ norm, we have $\|w\|_{H^s} \leq \liminf_{n \rightarrow +\infty} \|w_n\|_{H^s}$. Summing up, we get clearly

$$\mathcal{I}_\lambda \leq \mathcal{E}(w) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}(w_n) = \mathcal{I}_\lambda.$$

This shows that w is a minimizer of \mathcal{I}_λ and $w_n \xrightarrow[n \rightarrow +\infty]{} w$ in $H^s(\mathbb{R}^N)$. Theorem 1.1 is now proved.

4. Stability of standing waves

In this section, we prove the orbital stability of standing waves in the sense of Definition 1.3. That is we prove Theorem 1.4.

We argue par contradiction. Assume that $\hat{\mathcal{O}}_\lambda$ is not stable, then either $\hat{\mathcal{O}}_\lambda$ is empty or there exist $w \in \hat{\mathcal{O}}_\lambda$ and a sequence $\phi_0^n \in H^s$ such that $\|\phi_0^n - w\|_{H^s} \xrightarrow[n \rightarrow +\infty]{} 0$ as $n \rightarrow \infty$ but

$$\inf_{z \in \hat{\mathcal{O}}_\lambda} \|\phi^n(t_n, \cdot) - z\|_{H^s} \geq \varepsilon, \quad (28)$$

for some sequence $t_n \subset \mathbb{R}$, where $\phi^n(t_n, \cdot)$ is the solution of the Cauchy problem \mathcal{S} corresponding to the initial condition ϕ_0^n .

Now let $w_n = \phi^n(t_n, \cdot)$, since $\mathcal{J}(w) = \hat{\mathcal{I}}_\lambda$, it follows from the continuity of the L^2 norm and \mathcal{J} in H^s that $\|\phi_0^n\|_2 \xrightarrow[n \rightarrow +\infty]{} \sqrt{\lambda}$ and $\mathcal{J}(w_n) = \mathcal{J}(\phi_0^n) = \hat{\mathcal{I}}_\lambda$. With the conservation of mass and energy associated with the dynamics of the system \mathcal{S} , we deduce that

$$\|w_n\|_2 = \|\phi_0^n\|_2 \xrightarrow[n \rightarrow +\infty]{} \sqrt{\lambda} \quad \text{and} \quad \mathcal{J}(w_n) = \mathcal{J}(\phi_0^n) \xrightarrow[n \rightarrow +\infty]{} \hat{\mathcal{I}}_\lambda.$$

Therefore if $(w_n)_{n \in \mathbb{N}}$ has a subsequence converging to an element $w \in H^s$: $\|w\|_2 = \sqrt{\lambda}$ and $\mathcal{J}(w) = \hat{\mathcal{I}}_\lambda$. This shows that $w \in \hat{\mathcal{O}}_\lambda$, but

$$\inf_{z \in \hat{\mathcal{O}}_\lambda} \|\phi^n(t_n, \cdot) - z\|_{H^s} \leq \|w_n - w\|_{H^s}$$

contradicting (28).

In summary, to show the orbital stability of $\hat{\mathcal{O}}_\lambda$, one has to prove that $\hat{\mathcal{O}}_\lambda$ is not empty and that any sequence $(w_n)_{n \in \mathbb{N}} \subset H^s$ such that

$$\|w_n\|_2 \xrightarrow[n \rightarrow +\infty]{} \sqrt{\lambda} \quad \text{and} \quad \mathcal{J}(w_n) \xrightarrow[n \rightarrow +\infty]{} \hat{\mathcal{I}}_\lambda, \quad (29)$$

is relatively compact in H^s (up to a translation).

From now on, we consider a sequence $(w_n)_{n \in \mathbb{N}}$ satisfying (29). Our aim is to prove that it admits a convergent subsequence to an element $w \in H^s$.

If $(w_n)_{n \in \mathbb{N}} \subset H^s$, it is easy to see that

$$(|w_n|)_{n \in \mathbb{N}} \subset H^s; \quad w_n = (u_n, v_n).$$

Thanks to \mathcal{A}_0 , we have that $(w_n)_{n \in \mathbb{N}}$ is bounded in H^s and hence by passing to a subsequence, there exists $w = (u, v) \in H^s$ such that

$$\begin{cases} u_n \text{ converges weakly to } u \text{ in } H^s, \\ v_n \text{ converges weakly to } v \text{ in } H^s, \\ \text{the limit when } n \text{ goes to } +\infty \text{ of } \|\nabla_s u_n\|_2 + \|\nabla_s v_n\|_2 \text{ exists.} \end{cases} \quad (30)$$

Now, a straightforward calculation shows that

$$\mathcal{J}(w_n) - \mathcal{E}(|w_n|) = \frac{1}{2} \|\nabla_s w_n\|_2^2 - \frac{1}{2} \|\nabla_s |w_n|\|_2^2 \geq 0. \quad (31)$$

Thus we have

$$\hat{\mathcal{I}} = \lim_{n \rightarrow +\infty} \mathcal{J}(w_n) \geq \limsup_{n \rightarrow +\infty} \mathcal{E}(|w_n|). \quad (32)$$

But

$$\||w_n|\|_2^2 = \|w_n\|_2^2 = \lambda_n \xrightarrow{n \rightarrow +\infty} \lambda. \quad (33)$$

By the continuity of the mapping $\lambda \mapsto \mathcal{I}_\lambda$ (see Proposition 3.3), we obtain

$$\lim_{n \rightarrow +\infty} \mathcal{J}(w_n) \geq \liminf_{n \rightarrow +\infty} \mathcal{I}_{\lambda_n} = \mathcal{I}_\lambda \geq \hat{\mathcal{I}}_\lambda. \quad (34)$$

Hence

$$\lim_{n \rightarrow +\infty} \mathcal{J}(w_n) = \lim_{n \rightarrow +\infty} \mathcal{E}(|w_n|) = \mathcal{I}_\lambda = \hat{\mathcal{I}}_\lambda.$$

The properties (30) and the inequalities (31) and (34) imply that

$$\lim_{n \rightarrow +\infty} \|\nabla_s u_n\|_2^2 - \|\nabla_s v_n\|_2^2 - \|\nabla_s (u_n^2 + v_n^2)^{1/2}\|_2^2 = 0, \quad (35)$$

which is equivalent to say that

$$\lim_{n \rightarrow +\infty} \|\nabla_s w_n\|^2 = \lim_{n \rightarrow +\infty} \|\nabla_s |w_n|\|_2^2. \quad (36)$$

The convergence (33), the inequality (34) and Theorem 1.2 imply that $|w_n|$ is relatively compact in H^s (up to a translation). Therefore, there exists $\varphi \in H^s$ such that

$$(u_n^2 + v_n^2)^{1/2} \rightarrow \varphi \text{ in } H^s \text{ and } \|\varphi\|_2 = \sqrt{\lambda} \text{ with } \mathcal{E}(\varphi) = I_\lambda.$$

Let us prove that $\varphi = |w| = (u^2 + v^2)^{1/2}$. Using (30), it follows that $u_n \xrightarrow{n \rightarrow +\infty} u$ and $v_n \xrightarrow{n \rightarrow +\infty} v$ in $L^2(B(0, R))$

$$|(u_n^2 + v_n^2)^{1/2} - (u^2 + v^2)^{1/2}| \leq |u_n - u|^2 + |v_n - v|^2,$$

$$(u_n^2 + v_n^2)^{1/2} \xrightarrow{n \rightarrow +\infty} (u^2 + v^2)^{1/2} \text{ in } L^2(B(0, R)).$$

Thus we certainly have that $(u^2 + v^2)^{1/2} = |w| = \varphi$. On the other hand $\|w_n\|_2 = \|w_n\|_2 \xrightarrow{n \rightarrow +\infty} \sqrt{\lambda} = \|w\|_2 = \|\varphi\|_2$. Therefore, we are done if we prove that $\lim_{n \rightarrow \infty} \|\nabla_s w_n\|_2^2 = \|\nabla_s w\|_2^2$. From (36), we have that $\lim_{n \rightarrow +\infty} \|\nabla_s w_n\|_2^2 = \lim_{n \rightarrow +\infty} \|\nabla_s |w_n|\|_2^2$ and $\lim_{n \rightarrow +\infty} \|\nabla_s |w_n|\|_2^2 = \|\nabla_s |w|\|_2^2$. Hence by the lower semi-continuity of $\|\nabla_s \cdot\|_2$, we obtain

$$\|\nabla_s w\|_2^2 \leq \lim_{n \rightarrow +\infty} \|\nabla_s |w_n|\|_2^2 = \|\nabla_s |w|\|_2^2. \quad (37)$$

Eventually, using (31), it follows that

$$\|\nabla_s w\|_2^2 \geq \|\nabla_s |w|\|_2^2.$$

Since by (30), we know that w_n converges weakly to w in H^s , it follows that $w_n \xrightarrow{n \rightarrow +\infty} w$ in H^s , which completes the proof.

Now, we turn to the characterization of the Orbit $\hat{\mathcal{O}}_\lambda$. We show the following

Proposition 4.1. *With the same assumptions of Theorem 1.4, we have*

$$\hat{\mathcal{O}}_\lambda = \{e^{i\sigma} w(\cdot + y), \quad \sigma \in \mathbb{R}, y \in \mathbb{R}^N\},$$

w is a minimizer of (4).

Proof. Let $z = (u, v) \in \hat{\mathcal{O}}_\lambda$ and set $\varphi = (u^2 + v^2)^{1/2}$. By the previous section, we know that $\mathcal{E}(\varphi) = I_\lambda$, thus φ satisfies the partial differential equation :

$$(-\Delta)^s \varphi + \kappa \varphi = V \star G(|\varphi|) G'(\varphi), \quad (38)$$

where κ is a Lagrange multiplier. Furthermore the equality $\|\nabla_s w\|_2 = \|\nabla_s |w|\|_2$ implies that

$$u(x)v(y) - v(x)u(y) = 0. \quad (39)$$

By Proposition 4.2, it is plain that $\varphi \in C(\mathbb{R}^N)$ and $V \star G(|\varphi|) \in C(\mathbb{R}^N)$. We can write $(-\Delta)^s \varphi + \kappa \varphi = V \star G(|\varphi|) \frac{G'(\varphi)}{\varphi} \chi_{\{\varphi \neq 0\}} \varphi$, with χ_A being the characteristic function of the set A . Since φ is nontrivial and $V \star G(|\varphi|) \frac{G'(\varphi)}{\varphi} \chi_{\{\varphi \neq 0\}} \in L_{loc}^\infty(\mathbb{R}^N)$, we conclude that $\varphi > 0$ in \mathbb{R}^N by the Harnack inequality (see Lemma 4.9 in [2]) and a standard argument of intersecting balls.

Case 1 : $u \equiv 0$

Case 2 : $v \equiv 0$

Case 3 : $u \neq 0$ and $v \neq 0$ everywhere.

Then (39) implies that

$$\begin{aligned} \frac{u(x)}{v(x)} &= \frac{u(y)}{v(y)} \quad \forall x, y \in \mathbb{R}^N, \\ \Rightarrow \frac{u(x)}{v(x)} &= \alpha \Rightarrow u(x) = \alpha v(x) \quad \forall x \in \mathbb{R}^N, \\ z &= (\alpha + i)v \Rightarrow z = e^{i\sigma} w, w = |z|. \end{aligned}$$

Let us now prove (39). By the fact that $\mathcal{J}(z) = \hat{\mathcal{I}}_\lambda$, we can find a Lagrange multiplier $\alpha \in \mathbb{C}$ such that $\mathcal{J}'(z)(\xi) = \frac{\alpha}{2} \int_{\mathbb{R}^N} z \bar{\xi} + \xi \bar{z}$ for all $\xi \in H^s$. Putting

$\xi = z$, it follows immediately that $\alpha \in \mathbb{R}$ and

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^N} \nabla_s u \nabla_s f - \int_{\mathbb{R}^N \times \mathbb{R}^N} G(u^2 + v^2)^{1/2}(y) V(|x - y|) dy G'(f(x)) dx \\ \int_{\mathbb{R}^N} \nabla_s v \nabla_s f - \int_{\mathbb{R}^N \times \mathbb{R}^N} G(u^2 + v^2)^{1/2}(y) V(|x - y|) dy G'(f(x)) dx \end{array} \right. = \alpha \int_{\mathbb{R}^N} u(x) f(x) dx, \\ \left\{ \begin{array}{l} \int_{\mathbb{R}^N} \nabla_s u \nabla_s f - \int_{\mathbb{R}^N \times \mathbb{R}^N} G(u^2 + v^2)^{1/2}(y) V(|x - y|) dy G'(f(x)) dx \\ \int_{\mathbb{R}^N} \nabla_s v \nabla_s f - \int_{\mathbb{R}^N \times \mathbb{R}^N} G(u^2 + v^2)^{1/2}(y) V(|x - y|) dy G'(f(x)) dx \end{array} \right. = \alpha \int_{\mathbb{R}^N} v(x) f(x) dx,$$

∇_s denotes the fractional gradient, for all $f \in H^s$. It follows that u and v solve the following system

$$\left\{ \begin{array}{l} (-\Delta)^s u + \int G(u^2 + v^2)^{1/2}(y) V(|x - y|) dy G'(u(x)) + \alpha u(x) = 0, \\ (-\Delta)^s v + \int G(u^2 + v^2)^{1/2}(y) V(|x - y|) dy G'(v(x)) + \alpha v(x) = 0. \end{array} \right.$$

By Proposition 4.2, we have that u and $v \in C(\mathbb{R}^N)$ because $(u^2 + v^2)^{1/2} \in H^s(\mathbb{R}^N)$. Let $\Omega = \{x \in \mathbb{R}^N : u(x) = 0\}$, obviously Ω is closed since u is continuous. Let us prove that it is also open. Suppose that $x_0 \in \Omega$. Knowing that $\varphi(x_0) > 0$, we can find a ball B centered in x_0 such that $v(x) \neq 0$ for any $x \in B$. Replacing u and v in (35), we certainly have that

$$u(x)v(y) - v(x)u(y) = 0 \quad \forall x, y \in B.$$

This proves the result. \square

Appendix

In this appendix, we prove the following

Proposition 4.2. *Let $s \in (0, 1)$, $N - 2s \leq \beta < N$, $\beta > 0$, $u, \varphi \in H^s(\mathbb{R}^N)$, G such that \mathcal{A}_0 holds and κ is a real number such that*

$$(-\Delta)^s u - \kappa u = [V \star G(\varphi)]G'(u). \quad (40)$$

Then, there exists $\alpha \in (0, 1)$ depending only on N, κ, s, β such that $u \in C_{loc}^{0, \alpha}(\mathbb{R}^N)$. Moreover, if $\varphi \in L_{loc}^\infty(\mathbb{R}^N)$, then $u \in C_{loc}^{0, \alpha}(\mathbb{R}^N)$ if $\beta \leq 1$ and $u \in C_{loc}^{1, \alpha}(\mathbb{R}^N)$ if $\beta > 1$ and in addition $V \star G(\varphi) \in C_{loc}^{0, \alpha}(\mathbb{R}^N)$.

Proof. We start by recalling the Gagliardo-Nirenberg inequality

$$\|\varphi\|_{L^p(\mathbb{R}^N)} \leq c_{N,s,p} \|\varphi\|_{H^s(\mathbb{R}^N)} \quad \text{for all } \varphi \in H^s(\mathbb{R}^N),$$

for $p \in [2, \frac{2N}{N-2s}]$ if $N > 2s$ and for all $p \in [2, \frac{2N}{N-2s})$ and $2s \geq N$ (here we put $\frac{2N}{N-2s} \equiv +\infty$). Also we recall the Hardy-Littlewood-Sobolev inequality:

$$\|V \star g\|_{L^{\frac{qN}{N-q\beta}}(\mathbb{R}^N)} \leq C_{N,\beta,q} \|g\|_{L^q(\mathbb{R}^N)} \quad \text{for every } g \in L^q(\mathbb{R}^N),$$

for $N - q\beta > 0$.

First of all we focus on the case $N > 2s$. Thus, we have

$$\|G(\varphi)\|_{L^q(\mathbb{R}^N)} \leq \|\varphi^2\|_{L^q(\mathbb{R}^N)} + \|\varphi\|_{L^q(\mathbb{R}^N)}^\mu = \|\varphi\|_{L^{2q}(\mathbb{R}^N)}^2 + \|\varphi\|_{L^{\mu q}(\mathbb{R}^N)}^\mu.$$

Hence, since $\varphi \in H^s(\mathbb{R}^N)$, we infer that $G(\varphi) \in L^q(\mathbb{R}^N)$ provided that $1 \leq q \leq \frac{N}{N-2s}$ and $\frac{2}{\mu} \leq q \leq \frac{1}{\mu} \frac{2N}{N-2s}$, that is $1 \leq q \leq \frac{N}{N-2s}$ and $1 \leq q \leq \frac{2N^2}{(N-2s)(N+2s+\beta)}$. Now, thanks to the fact that $N - 2s \leq \beta < N$, we get $1 < \frac{N}{\beta} \leq \frac{N}{N-2s}$ and $1 < \frac{N}{\beta} \leq \frac{2N^2}{(N-2s)(N+2s+\beta)}$. In particular, we deduce that $G(\varphi) \in L^q(\mathbb{R}^N)$ for all $q \in [1, \frac{N}{\beta}]$. Now, for all $\epsilon > 0$ we let $q_\epsilon = \frac{N}{\beta} - \epsilon > 1$.

Using the Hardy-Littlewood-Sobolev inequality, we get $V \star G(\varphi) \in L^{\frac{Nq_\epsilon}{\epsilon\beta}}(\mathbb{R}^N)$ which in turns with the fact that $\beta \geq N - 2s$ shows that $V \star G(\varphi) \in L^r(\mathbb{R}^N)$ for all $r > \frac{N}{N-\beta} \geq \frac{N}{2s}$. Now, using the notation $b(x) = \frac{G'(u)}{1+|u|}$ and $\text{sign}(u) = \frac{u}{|u|}$, we reformulate the equation (40) as follows

$$\begin{aligned} (-\Delta)^s u(x) - \kappa u(x) &= [V \star G(\varphi)] b(x) (1 + |u|), \\ &= \int_{\mathbb{R}^N} V(|x-y|) G(\varphi(y)) dy b(x) (1 + \text{sign}(u) u). \end{aligned}$$

Observing that $\mu - 2 < \frac{2N}{N-2s} - 2 = \frac{4s}{N-2s}$, then for all $r > \frac{N}{2s}$, we can write

$$\begin{aligned} \|[V \star G(\varphi)]b\|_{L^r(\mathbb{R}^N)} &= \|[V \star G(\varphi)] \frac{G'(u)}{1+|u|}\|_{L^r(\mathbb{R}^N)}, \\ &\leq c \|[V \star G(\varphi)] \frac{|u| + |u|^{\mu-1}}{1+|u|}\|_{L^r(\mathbb{R}^N)}, \\ &\leq c \|V \star G(\varphi)\|_{L^r(\mathbb{R}^N)} + c \|[V \star G(\varphi)]|u|^{\mu-2}\|_{L^r(\mathbb{R}^N)}. \end{aligned}$$

In order to deduce that the right hand side of this estimate is finite, we use Hölder's inequality to get

$$\|[V \star G(\varphi)] |u|^{\mu-2}\|_{L^r(\mathbb{R}^N)}^r \leq \|V \star G(\varphi)\|_{L^{\frac{r\theta}{\theta-1}}(\mathbb{R}^N)} \|u\|_{L^{r(\mu-2)\theta}(\mathbb{R}^N)}^{\mu-2},$$

for all $\theta > 1$. Therefore, we can choose $r > \frac{N}{2s}$ and $\theta > 1$ respectively close to $\frac{N}{2s}$ and 1 so that $1 < r(\mu-2)\theta < \frac{2N}{N-2s}$. Hence using the Gagliardo-Nirenberg inequality and the fact that $V \star G(\varphi) \in L^r(\mathbb{R}^N)$ for all $r > \frac{N}{N-\beta} \geq \frac{N}{2s}$ and $u \in H^s(\mathbb{R}^N)$, we end up with $[V \star G(\varphi)]b \in L^r$ for some $r > \frac{N}{2s}$, hence $u \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$.

Now, we write the equation (40) as follows

$$\begin{aligned} (-\Delta)^s u(x) &= c(x)u(x) + d(x) : \\ c(x) &= \kappa + [V \star G(\varphi)] b(x) \operatorname{sign}(u) \in L^r(\mathbb{R}^N), \\ d(x) &= [V \star G(u)] b(x) \in L^r(\mathbb{R}^N), \end{aligned}$$

for some $r > \frac{N}{2s}$. Thus, using the regularity result of Ref. [20], we conclude that $u \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$ provided $\frac{N}{2s} > 1$. If $N = 1$ and $s > \frac{1}{2}$, then it is well-known that $H^s(\mathbb{R}^N)$ is embedded in $C_{loc}^{0,\alpha}(\mathbb{R}^N)$ with $\alpha = s - \frac{1}{2} - [s - \frac{1}{2}]$ so that $u \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$. Moreover, if $N = 1$ and $s = \frac{1}{2}$, we have obviously $u \in L^p(\mathbb{R}^N)$ for every $p \geq 2$ and classical elliptic regularity yields $u \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$.

In the following, $[\cdot]$ stands for the integer part of \cdot . Let us introduce a cutoff function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $\eta \equiv 1$ in the closed ball B_R of center 0 and radius $R > 0$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2R}$. To alleviate the notation, we denote $f = G(\varphi)$ which belongs to $L_{loc}^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with $1 < q \leq \frac{N}{\beta}$. We define $V_1(\varphi) := V \star (\eta f)$ and $V_2(\varphi) := V \star ((1 - \eta)f)$. Then using Fourier transform, we get $(-\Delta)^{\frac{\beta}{2}} V_1(\varphi) = f$ in the sense of distributions. Now, if $\frac{\beta}{2} \in \mathbb{N}^*$, then it is rather easy to show using classical regularity theory that $V \star G(\varphi) \in C^\beta(\mathbb{R}^N)$. Next, if $0 < \frac{\beta}{2} < 1$, then we apply Proposition 2.1.9 of Ref. [19] to show that $V_1(\varphi) \in C^{0,\alpha}(\mathbb{R}^N)$ for $\beta \leq 1$ and $V_1(\varphi) \in C^{[\beta],\alpha}(\mathbb{R}^N)$ for $\beta > 1$ and some $\alpha \in (0, 1)$. Now, $V_2(\varphi)$ is smooth on B_R since it is $\frac{\beta}{2}$ -harmonic in such a ball, see Ref. [1]. Hence, $V \star G(\varphi) \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$ for $\beta \leq 1$ and $V \star G(\varphi) \in C_{loc}^{[\beta],\alpha}(\mathbb{R}^N)$ for $\beta > 1$ and some $\alpha \in (0, 1)$. Let us now turn to the case of $\frac{\beta}{2} > 1$ and $\frac{\beta}{2} \notin \mathbb{N}$. we let $\sigma = \frac{\beta}{2} - [\frac{\beta}{2}]$. Using Fourier

transform, we have

$$(-\Delta)^{[\frac{\beta}{2}]} V_1(\varphi) = (-\Delta)^{[\frac{\beta}{2}]} ((-\Delta)^\sigma V_1(\varphi)) = \eta f$$

in the sense of distributions. Again, classical regularity theory arguments implies that $(-\Delta)^\sigma V_1(\varphi) \in C^{[\beta]}(\mathbb{R}^N)$ and so $V_1(\varphi) \in C^{[\beta]}(\mathbb{R}^N)$. Similarly, we have

$$(-\Delta)^{\frac{\beta}{2}} V_2(\varphi) = (-\Delta)^\sigma \left((-\Delta)^{[\frac{\beta}{2}]} V_2(\varphi) \right) = (1 - \eta) f$$

in the sense of distributions. Therefore the function $g := (-\Delta)^{[\frac{\beta}{2}]} V_2(\varphi)$ is given by

$$(-\Delta)^{[\frac{\beta}{2}]} V_2(\varphi)(x) = \int_{\mathbb{R}^N} \frac{(1 - \eta(y)) f(y)}{|x - y|^{N-\sigma}} dy.$$

Also, using the Hardy-Littelwood-Sobolev inequality, it is rather straightforward to see that $g \in L^p(\mathbb{R}^N)$ for some $p > 1$. Thus, g belongs to the set $\left\{ u, \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2\sigma}} dx < +\infty \right\}$. Again, since g is σ -harmonic in B_R , we deduce that g is smooth on B_R by Ref. [1]. The radius R being arbitrary, it follows that $V_2(\varphi)$ is smooth on \mathbb{R}^N . In particular, we have $V_2(\varphi) \in C^{[\beta]}(\mathbb{R}^N)$ because $[\frac{\beta}{2}]$ is a positive integer. Recalling that we showed $V_1(\varphi) \in C^{[\beta]}(\mathbb{R}^N)$, we conclude $V \star G(\varphi) \in C^{[\beta]}(\mathbb{R}^N)$.

Let us now summarize and conclude the proof. We considered the partial differential equation (40) and proved that for some $\alpha \in (0, 1)$, we have $V \star G(\varphi) \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$ for $\beta \leq 1$ and $V \star G(\varphi) \in C_{loc}^{[\beta],\alpha}(\mathbb{R}^N)$ for $\beta > 1$. Since G' is locally Lipschitz, we deduce that $u \in C_{loc}^{0,\alpha}(\mathbb{R}^N)$ for $\beta \leq 1$ and $u \in C_{loc}^{1,\alpha}(\mathbb{R}^N)$ for $\beta > 1$ by adapting the proof of Lemma 3.3 of Ref. [8] for $N > 2s$. If $N = 1$ and $2s \geq 1$, we have that $[V \star G(\varphi)] G'(u) \in C_{loc}^{0,\gamma}(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$, thus using Proposition 2.1.8 of Ref. [19], we get $u \in C_{loc}^{1,\alpha}(\mathbb{R}^N)$. \square

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