# POSITIVE SOLUTIONS TO SOME NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS VIA A MIN-MAX PROCEDURE

GILLES ÉVÉQUOZ AND MOUHAMED MOUSTAPHA FALL

ABSTRACT. The existence of a positive solution to the following fractional semilinear equation is proven, in a situation where a ground state solution may not exist. More precisely, we consider for 0 < s < 1 the equation

$$(-\Delta)^{s}u + V(x)u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^{N}, \ N \ge 1,$$

where the exponent p is superlinear but subcritical, and V > 0,  $Q \ge 0$  are bounded functions converging to 1 as  $|x| \to \infty$ . Using a min-max procedure introduced by Bahri and Li we prove the existence of a positive solution under one-sided asymptotic bounds for V and Q.

#### 1. INTRODUCTION AND MAIN RESULT

Recently Laskin, in [24, 25], derived an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths yielding the fractional Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi, \qquad (1)$$

where  $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ ,  $0 < s \leq 1$ , and  $V \colon \mathbb{R}^N \to \mathbb{R}$  is an external potential function. When s = 1, the Lévy dynamics becomes the Brownian dynamics, and equation (1) reduces to the classical Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi$$

Standing wave solutions to the fractional Schrödinger equation are solutions of the form

$$\psi(x,t) = \mathrm{e}^{-\imath t} u(x),\tag{2}$$

where u solves the elliptic equation

 $(-\Delta)^s u + V(x)u = 0,$ 

for  $s \in (0, 1]$ . The Schrödinger equation with nonlinear source term has also its own interest, especially, when dealing with relativistic particles (s = 1/2), see for example [1, 33, 6, 12, 26, 27, 21].

In the present paper, we consider the following nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$
(3)

with 0 < s < 1 and coefficients satisfying

Date: April 21, 2019.

Key words and phrases. Fractional Schrödinger equation, bound states, variational methods, topological degree, minimax principle.

- $(\mathrm{V}) \ V \in L^\infty(\mathbb{R}^N), \, \mathrm{ess} \inf_{\mathbb{R}^N} V > 0 \ \mathrm{and} \ V(x) \to 1 \ \mathrm{as} \ |x| \to \infty.$
- (Q)  $Q \in L^{\infty}(\mathbb{R}^N), Q \ge 0$  a.e. on  $\mathbb{R}^N$  and  $Q(x) \to 1$  as  $|x| \to \infty$ .

Notice that the condition on the value of the asymptotic limit of V and Q is not restrictive, since the case  $V(x) \to V_{\infty} > 0$  and  $Q(x) \to Q_{\infty} > 0$  can be treated by means of a rescaling.

We now state our main result:

**Theorem 1.1.** Let  $N \ge 1$ , 0 < s < 1 and 2 for <math>N > 2s and  $p \in (2, \infty)$  for  $2s \ge N = 1$ . Suppose (V), (Q) and the following conditions hold:

(H) There exists  $\kappa_1, \kappa_2 \ge 0$  and  $\alpha > N + 2s$  such that for almost every  $x \in \mathbb{R}^N$ 

 $V(x) \leq 1 + \kappa_1 |x|^{-\alpha}$  and  $Q(x) \geq 1 - \kappa_2 |x|^{-\alpha}$ .

(U) The limit problem  $(-\Delta)^s u + u = |u|^{p-2}u$  in  $\mathbb{R}^N$  has (up to translations) a unique positive solution.

Then (3) has a positive solution.

The assumption (U) has been shown to hold in the physically relevant special case of the Benjamin-Ono equation, see [1], where uniqueness was proved for  $s = \frac{1}{2}$ , N = 1 and p = 2. In the general case  $s \in (0, 1)$ , by now only the ground-state (i.e. least energy) solution of the limit problem is known to be unique (up to translations), see [17], [21] and [22], and therefore the assumption (U) needs to be imposed. In the local case s = 1, the celebrated result by Kwong [28] shows that the limit problem has a unique positive solution, but in the fractional setting, the question is much more involved, since ODE methods are not directly applicable anymore. Nevertheless, it is expected that the uniqueness of the positive solution of the limit problem holds for most  $s \in (0, 1)$ .

Concerning the regularity of the solutions given by the preceding theorem, we note that a standard bootstrap argument based on Sobolev embeddings together with [22, Lemma B.1 and Proposition B.3] implies that  $u \in H^{2s}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \cap C^{0,\alpha}(\mathbb{R}^N)$  for any  $0 < \alpha < 2s$ . In particular,  $|u(x)| \to 0$ , as  $|x| \to \infty$ .

Since only recently, intensive work has been devoted to the study of (3) for all  $s \in (0, 1)$ , also with general subcritical right-hand side of the form f(x, u). In [19], existence, asymptotic behavior and symmetry properties of the solutions were studied. In [9] the author gave an existence result for (3) with V = 1 while Q was assumed to be positive only in a set of positive measure. In [32], several existence results were proved for problem (3) with more general nonlinearities on the right hand side, generalizing [8]. For related works about existence and qualitative of solutions, one can also see [14], [17], [20], [22] and [31].

Our main result in Theorem 1.1 is not contained in the afore mentioned papers because of the mild assumptions on V and Q. In addition, in our proof, the tools involved to get positive solution are different. Indeed, our scheme of proof is based on min-max arguments in the spirit of [2, 3, 4] applied to the energy functional associated with (3) on the Nehari manifold (or natural constraint). The polynomial bounds on V and Q are reminiscent of the exponential decay estimates assumed in [3] and [4], and are indeed related to the asymptotic behavior of ground states for the limit problem  $(-\Delta)^s u + u = u^p$ . When s = 1 ground states decay exponentially while for  $s \in (0, 1)$  they decay polynomially (and never exponentially, see [19]!). We point out that, since  $(-\Delta)^s + V$  is positive definite we can work with the usual

 $\mathbf{2}$ 

Nehari manifold, in contrast to [16] where an indefinite problem was treated using the generalized Nehari set.

Finally, we would like to point out that in Theorem 1.1 there is no restriction in assuming that the exponent  $\alpha$  in condition (H) satisfies

$$N + 2s < \alpha < \min\{2(N + 2s), \frac{p}{2}(N + 2s)\},\$$

since p > 2.

The paper is organized as follows. In Section 2 we set up the variational framework for the study of (3). In particular we show that the energy of any weak solution of (3) which changes sign must be higher than twice the ground state energy. In Section 3, we give a splitting result for Palais-Smale sequences in the spirit of [5], and prove the main energy estimate which will allow us in Section 4 to define, using a barycenter map, a min-max critical level and complete the proof of Theorem 1.1.

## 2. Preliminaries

Throughout this paper we shall use the following notation. For a function u on  $\mathbb{R}^N$  and an element  $y \in \mathbb{R}^N$ , we write y \* u for the translate of u by y, *i.e.*,

$$(y * u)(x) := u(x - y), \quad x \in \mathbb{R}^N$$

Since condition (V) holds, the spectrum of the unbounded operator  $S = (-\Delta)^s + V$ acting in  $L^2(\mathbb{R}^N)$  is contained in  $(0, \infty)$  and, therefore,

$$\|u\|_{V} := \left(\int_{\mathbb{R}^{N}} |\xi|^{2s} |\widehat{u}(\xi)|^{2} d\xi + \int_{\mathbb{R}^{N}} V(x) |u(x)|^{2} dx\right)^{\frac{1}{2}}$$

defines a norm on  $H^{s}(\mathbb{R}^{N})$ , equivalent to the standard norm

$$||u||_{s} = \left(\int_{\mathbb{R}^{N}} (|\xi|^{2s} + 1) |\widehat{u}(\xi)|^{2} d\xi\right)^{\frac{1}{2}}.$$

The weak solutions of (3) are critical points of the energy functional  $J: H^s(\mathbb{R}^N) \to \mathbb{R}$  given by

$$J(u) = \frac{1}{2} ||u||_V^2 - \frac{1}{p} \int_{\mathbb{R}^N} Q(x) |u(x)|^p \, dx, \quad u \in H^s(\mathbb{R}^N).$$

We consider the Nehari manifold (or natural constraint)

$$\mathcal{N}=\{u\in H^s(\mathbb{R}^N)\backslash\{0\}\,:\,J'(u)(u)=0\}$$

which contains all critical points of J and set

$$c = \inf_{u \in \mathcal{N}} J(u). \tag{4}$$

For the limit problem

$$(-\Delta)^s u + u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \tag{5}$$

we denote by  $J_{\infty}$  the associated energy functional given by

$$J_{\infty}(u) = \frac{1}{2} \|u\|_{s}^{2} - \frac{1}{p} \int_{\mathbb{R}^{N}} |u|^{p} dx, \quad u \in H^{s}(\mathbb{R}^{N}),$$

consider the corresponding Nehari manifold

$$\mathcal{N}_{\infty} = \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : J'_{\infty}(u)u = 0 \}$$

and let

$$c_{\infty} = \inf_{u \in \mathcal{N}_{\infty}} J_{\infty}(u).$$

We start by giving some properties of the Nehari manifold  $\mathcal{N}$  and study the behavior of J on it. Some of the following results can be found in [34].

**Lemma 2.1.** Under the conditions (V) and (Q), the following holds:

(i)  $\mathcal{N} \neq \emptyset$ . More precisely, for every  $u \in H^s(\mathbb{R}^N)$  with  $\int_{\mathbb{R}^N} Q(x) |u(x)|^p dx > 0$ , there exists a unique  $t_u \in (0, \infty)$ , given by

$$t_u = \left(\frac{\|u\|_V^2}{\int_{\mathbb{R}^N} Q(x) |u(x)|^p \, dx}\right)^{\frac{1}{p-2}},\tag{6}$$

such that  $t_u u \in \mathcal{N}$  and  $J(t_u u) > J(tu)$  for all  $t \ge 0, t \ne t_u$ . Consequently,

$$c = \inf_{u \in \mathcal{N}} J(u) = \inf_{\substack{u \in H^s(\mathbb{R}^N) \\ u \neq 0}} \sup_{t > 0} J(tu).$$
(7)

- (ii) c > 0,  $\inf_{u \in \mathcal{N}} \|u\|_{V} > 0$  and  $\inf_{u \in \mathcal{N}} \|u\|_{L^{p}} > 0$ . (iii) J is coercive on  $\mathcal{N}$ , i.e. if  $(u_{n})_{n} \subset \mathcal{N}$  satisfies  $\|u_{n}\|_{V} \to \infty$  as  $n \to \infty$ , then  $J(u_n) \to \infty$  as  $n \to \infty$ .

*Proof.* (i) Let us consider  $u \in H^s(\mathbb{R}^N)$  satisfying  $\int_{\mathbb{R}^N} Q(x) |u(x)|^p dx > 0$ . (Note that the assumption (Q) gives some r > 0 such that  $Q(x) \ge \frac{1}{2}$  for a.e. x with  $|x| \ge r$ , thereby ensuring the existence of such functions u.) For t > 0 there holds

$$\frac{d}{dt}J(tu) = J'(tu)u = t\left(\|u\|_V^2 - t^{p-2}\int_{\mathbb{R}^N} Q(x)|u(x)|^p \, dx\right).$$

Setting  $t_u > 0$  as in (6), we find that the map  $t \mapsto J(tu)$  is strictly increasing for  $0 < t < t_u$  and strictly decreasing for  $t > t_u$ . Thus,  $t_u u \in \mathcal{N}$  and  $J(t_u u)$  is the (unique) strict global maximum of  $t \mapsto J(tu)$  on  $[0,\infty)$ . Remarking in addition that  $J(tu) \to \infty$  as  $t \to \infty$  in the case where  $\int_{\mathbb{R}^N} Q(x) |u(x)|^p dx = 0$ , the assertion follows.

(ii) Let  $\delta > 0$ . For each  $u \in \mathcal{N}$ , it follows from (i), that  $J(\frac{\delta}{\|u\|_V}u) \leq J(u)$  holds. Therefore,  $c = \inf_{u \in \mathcal{N}} J(u) \ge \inf\{J(w) : \|w\|_V = \delta\}$  for any  $\delta > 0$ . Now from the Sobolev embedding  $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ , there is some constant C > 0 such that for all  $w \in H^s(\mathbb{R}^N)$  with  $||w||_V = \delta$ ,

$$J(w) \ge \frac{1}{2} \|w\|_V^2 - \frac{1}{p} C^p \|Q\|_{\infty} \|w\|_V^p = \delta^2 \left(\frac{1}{2} - \delta^{p-2} \frac{C^p \|Q\|_{\infty}}{p}\right).$$

Since the last expression is positive for sufficiently small  $\delta$ , we obtain c > 0. To prove the second and third statements in (ii) as well as the property (iii), it suffices to notice that if  $u \in \mathcal{N}$ , then  $c \leq J(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_V^2 \leq \left(\frac{1}{2} - \frac{1}{p}\right) \|Q\|_{\infty} \|u\|_{L^p}^p$ holds.

Let us point out that the same result holds with  $J_{\infty}$ ,  $c_{\infty}$  and  $\mathcal{N}_{\infty}$  in place of J, cand  $\mathcal{N}$ , respectively. Minimizers of  $J_{\infty}$  on  $\mathcal{N}_{\infty}$  are critical points of  $J_{\infty}$  on  $H^{s}(\mathbb{R}^{N})$ with least possible energy among all nontrivial critical points. For this reason, they are called *ground states* of (5). According to [22, Proposition 3.1], there is (up to translation) a unique ground state solution  $u_{\infty} \in H^{2s+1}(\mathbb{R}^N) \cap \mathcal{C}^{\infty}(\mathbb{R}^N)$  of (5), positive, radially symmetric, radially decreasing and which satisfies the following asymptotic decay properties:

$$\frac{C_1}{1+|x|^{N+2s}} \leqslant u_{\infty}(x) \leqslant \frac{C_2}{1+|x|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N,$$
(8)

where  $0 < C_1 < C_2$ . Our next result states that critical points u of J with  $J(u) \leq 2c$  cannot change sign, where we recall that c is defined in (4).

**Proposition 2.2.** Let  $u \in H^s(\mathbb{R}^N)$  be a nontrivial sign-changing critical point of J (resp.  $J_{\infty}$ ). Then J(u) > 2c (resp.  $J_{\infty}(u) > 2c_{\infty}$ ).

*Proof.* Let  $u \in H^s(\mathbb{R}^N)$  be a nontrivial sign-changing critical point of J and let  $u_{\pm} = \max\{\pm u, 0\}$  denote the positive and negative part of u, respectively. Then  $u_{\pm} \in H^s(\mathbb{R}^N)$  and  $u = u_{+} - u_{-}$ . Moreover, letting  $\langle \cdot, \cdot \rangle_V$  denote the scalar product that induces the norm  $\|\cdot\|_V$ , the identity

$$\langle v, w \rangle_V = C_{N,s} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) v w \, dx,$$

which holds for all  $v, w \in H^s(\mathbb{R}^N)$ , gives

$$\langle u_+, u_- \rangle_V = 2C_{N,s} \ P.V. \int \int_{\{u>0\} \times \{u<0\}} \frac{u(x)u(y)}{|x-y|^{N+2s}} \, dx \, dy < 0,$$

for some  $C_{N,s} > 0$  and where P.V. denotes the fact that the integral is taken in the 'principal value' sense. Now, from  $0 = J'(u)u_{\pm} = \langle u, u_{\pm} \rangle_V \mp \int_{\mathbb{R}^N} Q(x)u_{\pm}^p dx$ , we deduce that  $\int_{\mathbb{R}^N} Q(x)u_{\pm}^p dx = ||u_{\pm}||_V^2 - \langle u_+, u_- \rangle_V > 0$ . Using (6) we can therefore find  $t_{\pm} > 0$ , given by

$$t_{\pm}^{p-2} = \frac{\|u_{\pm}\|_{V}^{2}}{\int_{\mathbb{R}^{N}} Q(x)u_{\pm}^{p} dx} = \frac{\|u_{\pm}\|_{V}^{2}}{\|u_{\pm}\|_{V}^{2} - \langle u_{+}, u_{-} \rangle_{V}} < 1,$$

such that  $t_{\pm}u_{\pm} \in \mathcal{N}$ . Consequently,

$$2c \leqslant J(t_+u_+) + J(t_-u_-) < \left(\frac{1}{2} - \frac{1}{p}\right) \left(\int_{\mathbb{R}^N} Q(x)u_+^p \, dx + \int_{\mathbb{R}^N} Q(x)u_-^p \, dx\right) = J(u),$$

and thus J(u) > 2c holds for any sign-changing critical point u of J. The corresponding assertion for  $J_{\infty}$  follows by replacing V(x) and Q(x) by 1 in the preceding arguments.

**Corollary 2.3.** Let u be a nontrivial critical point of J such that  $J(u) \leq 2c$ , where c is defined in (4). Then u is positive in  $\mathbb{R}^N$ 

*Proof.* By Proposition 2.2,  $u \ge 0$  in  $\mathbb{R}^N$ . In particular, we have  $(-\Delta)^s u + V(x)u \ge 0$  in  $\mathbb{R}^N$ . Since  $u \ne 0$ , it follows from the strong maximum principle [18, Corollary 3.4 and Remark 3.5] that u > 0 in  $\mathbb{R}^N$ .

We now give a first existence result in the case where the ground state energy level c is strictly less than the ground state level of the limit problem. In this case, we show that a ground state for (3) exists, i.e., there exists a weak solution u of (3) having minimal energy J(u) = c among all nontrivial solutions. Notice that the conditions (V) and (Q) ensure that  $c \leq c_{\infty}$  holds in any case.

**Proposition 2.4.** Suppose that (V) and (Q) are satisfied. If  $c < c_{\infty}$  holds, then J has a critical point u which satisfies J(u) = c. In particular (3) has a positive solution.

Proof. Let  $(v_n)_n \subset \mathcal{N}$  be a minimizing sequence for J. Since J is of class  $C^2$ ,  $J''(v)(v,v) = (2-p)||v||_V^2 < 0$  for all  $v \in \mathcal{N}$  and since 0 is an isolated point of  $\{u \in H^s(\mathbb{R}^N) : J'(u)u = 0\}$ , we find that  $\mathcal{N}$  is a closed  $C^1$ -submanifold of codimension 1 of the Hilbert space  $H^s(\mathbb{R}^N)$  and, from Lemma 2.1, J is bounded below on  $\mathcal{N}$ . From Ekeland's variational principle (see e.g. [15, Theorem 3.1]), we can find a Palais-Smale sequence  $(u_n)_n \subset \mathcal{N}$  for J at level c such that  $||u_n - v_n||_V \to 0$ as  $n \to \infty$ . Lemma 2.1 (iii) implies that  $(u_n)_n$  is bounded and therefore, up to a subsequence, we may assume  $u_n \rightharpoonup u$  weakly in  $H^s(\mathbb{R}^N)$ . Since J' is weakly sequentially continuous, we find J'(u) = 0. Now, if  $u \neq 0$ , then  $u \in \mathcal{N}$  and it follows that

$$c \leq J(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{V}^{2} \leq \liminf_{n \to \infty} \left(\frac{1}{2} - \frac{1}{p}\right) \|u_{n}\|_{V}^{2} = \liminf_{n \to \infty} J(u_{n}) = c.$$

Hence u is a critical point of J at level c. In the case where u = 0 holds, we claim that we can find a sequence  $(y_n)_n \subset \mathbb{R}^N$  and  $\delta > 0$  such that

$$\liminf_{n \to \infty} \int_{B_1(0)} (y_n * u_n)^2 \, dx \ge \delta > 0. \tag{9}$$

Indeed, if this were false, the concentration-compactness Lemma (see [11] and [19, Lemma 2.2]) would imply  $||u_n||_{L^p} \to 0$  as  $n \to \infty$  and therefore

$$c = \lim_{n \to \infty} J(u_n) = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} Q(x) |u_n(x)|^p \, dx = 0,$$

contradicting the fact that c > 0.

Now, we remark that  $(y_n)_n$  must be unbounded, since we are assuming  $u_n \to 0$ . Hence, going to a subsequence, if needed, we can assume  $|y_n| \to \infty$ ,  $y_n * u_n \to w$ and  $u_n(x - y_n) \to w(x)$  for a.e.  $x \in \mathbb{R}^N$ , as  $n \to \infty$ . The compact embedding  $H^s(B_1(0)) \hookrightarrow L^2(B_1(0))$  and (9) together give  $w \neq 0$ . Furthermore, Lemma 2.1 (i) gives for every  $t_n > 0$ ,

$$\begin{aligned} J(u_n) \geqslant J(t_n u_n) &= J_{\infty}(t_n(y_n * u_n)) + \frac{t_n^2}{2} \int_{\mathbb{R}^N} (V(x - y_n) - 1)(y_n * u_n)^2 \, dx \\ &- \frac{t_n^p}{p} \int_{\mathbb{R}^N} (Q(x - y_n) - 1) |y_n * u_n|^p \, dx. \end{aligned}$$

Taking  $t_n > 0$  such that  $t_n(y_n * u_n) \in \mathcal{N}_{\infty}$  holds, we infer from (6) that  $(t_n)_n$  is a bounded sequence, since  $\int_{B_1(0)} |y_n * u_n|^p dx \to \int_{B_1(0)} |w|^p dx > 0$  as  $n \to \infty$ . Since  $V(x - y_n) \to 1$ ,  $Q(x - y_n) \to 1$  and  $u_n(x - y_n) \to w(x)$  for a.e.  $x \in \mathbb{R}^N$ , as  $n \to \infty$ , the dominated convergence theorem implies  $\int_{\mathbb{R}^N} (V(x - y_n) - 1)(y_n * u_n)^2 dx \to 0$  and  $\int_{\mathbb{R}^N} (Q(x - y_n) - 1)|y_n * u_n|^p dx \to 0$ , as  $n \to \infty$ . We therefore conclude that

$$c = \lim_{n \to \infty} J(u_n) \ge \limsup_{n \to \infty} J_{\infty}(t_n(y_n * u_n)) \ge c_{\infty}$$

holds, in contradiction to the assumption  $c < c_{\infty}$ . Thus,  $u \neq 0$  must hold. Finally by Corollary 2.3 we have that u > 0 in  $\mathbb{R}^N$ .

In [19, Theorem 1.2], the authors give conditions under which  $c < c_{\infty}$  is satisfied and therefore a ground state solution exists for (3). On the contrary, if in addition to (V) and (Q) we require  $V(x) \ge 1$  and  $Q(x) \le 1$  for a.e.  $x \in \mathbb{R}^n$  with one of the inequalities being strict in a set of positive measure, then for all  $u \in H^s(\mathbb{R}^N)$ ,  $J(u) \ge J_{\infty}(u)$ , which gives  $c = c_{\infty}$ . Moreover, assuming by contradiction that this energy level is attained, there would exist  $u \in \mathcal{N}$  satisfying  $J(u) = c_{\infty}$ , and choosing  $\tau > 0$  such that  $\tau u \in \mathcal{N}_{\infty}$  we would obtain by Lemma 2.1:

$$c_{\infty} = J(u) \geqslant J(\tau u) \geqslant J_{\infty}(\tau u) \geqslant c_{\infty}$$

and consequently  $\tau = 1$ , *i.e.*,  $u \in \mathcal{N}_{\infty}$  and  $J_{\infty}(u) = c_{\infty}$ . The uniqueness (up to translations) and the positivity of the ground state solution of (5) would then imply u > 0 on  $\mathbb{R}^{N}$  and therefore

$$J(u) - J_{\infty}(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^{N}} (V(x) - 1)u(x)^{2} dx$$
$$= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^{N}} (Q(x) - 1)u(x)^{p} dx$$

Since V(x) > 1 or Q(x) < 1 holds on a set of positive measure we would obtain  $J(u) \neq J_{\infty}(u)$ . This contradiction shows that the ground state level for J is not attained. In the sequel, we shall prove that in such a case a solution can still be found, provided conditions (H) and (U) are satisfied.

### 3. Asymptotic estimates and Palais-Smale sequences

The first part of this section is devoted to the proof of an energy estimate that will be an essential ingredient in the proof of Theorem 1.1. As in [3], we consider convex combinations of two translates of the ground state solution  $u_{\infty}$  and project them onto the Nehari manifold  $\mathcal{N}$ . We derive estimates concerning the energy of such a convex combination, showing that it can be made smaller than  $2c_{\infty}$ , as each of the translates is moved away from the other and far away from the origin. Pointing out that the convex combination of two translates of  $u_{\infty}$  is an everywhere positive function, we can use (6) to define its projection onto  $\mathcal{N}$ , for which we introduce the following

**Notation.** Let  $y, z \in \mathbb{R}^N$  and  $\lambda \in [0, 1]$ . We denote by  $t_{\infty} = t_{\infty}(\lambda, y, z)$  the unique positive number (see Lemma 2.1) for which  $t_{\infty}[(1-\lambda)(y * u_{\infty}) + \lambda(z * u_{\infty})] \in \mathcal{N}$ holds.

From now on, we will work under the assumptions (V), (Q) and (H) of Theorem 1.1. Furthermore, we will assume without loss of generality that

$$N + 2s < \alpha < \min\{2(N+2s), \frac{p}{2}(N+2s)\}$$
(10)

holds in (H).

Before stating and proving the above mentioned key energy estimate, we start by studying the asymptotic behavior of some integrals of convolution type.

**Lemma 3.1.** Let  $\sigma, \tau \in (N, \infty)$ . Setting  $\mu = \min\{\sigma, \tau\}$ , there holds

$$\sup_{y \in \mathbb{R}^N} |y|^{\mu} \int_{\mathbb{R}^N} (1+|x|)^{-\sigma} (1+|x-y|)^{-\tau} \, dx < \infty.$$

*Proof.* Since  $|y|/2 \leq |x|$  whenever  $|x - y| \leq |y|/2$ , we have

$$|y|^{\mu} \int_{\{|x-y| \leq \frac{|y|}{2}\}} (1+|x|)^{-\sigma} (1+|x-y|)^{-\tau} dx \leq |y|^{\mu} \frac{2^{\sigma}}{(2+|y|)^{\sigma}} \int_{\mathbb{R}^{N}} (1+|x-y|)^{-\tau} dx$$
$$\leq |y|^{\mu} \left(\frac{2}{-\frac{1}{2}}\right)^{\sigma} \int_{\mathbb{R}^{N}} (1+|x|)^{-\tau} dx.$$

$$\leqslant |y|^{\mu} \left(\frac{2}{2+|y|}\right)^{\sigma} \int_{\mathbb{R}^{N}} (1+|x|)$$

Next, we have

$$|y|^{\mu} \int_{\{|x-y| \ge \frac{|y|}{2}\}} (1+|x|)^{-\sigma} (1+|x-y|)^{-\tau} dx \le |y|^{\mu} \frac{2^{\tau}}{(2+|y|)^{\tau}} \int_{\mathbb{R}^N} (1+|x|)^{-\sigma} dx.$$

Since  $\sigma, \tau \ge \mu > N$ , the conclusion follows.

**Lemma 3.2** (Energy estimate). Suppose (V), (Q) and (H) hold. Then there exists  $R_1 > 0$  such that

$$J(t_{\infty}[(1-\lambda)(y * u_{\infty}) + \lambda(z * u_{\infty})]) < 2c_{\infty}$$

for all  $\lambda \in [0,1]$ ,  $R \ge R_1$  and  $y, z \in \mathbb{R}^N$  satisfying  $|y| \ge R$ ,  $|z| \ge R$  and  $\frac{2}{3}R \le |y-z| \le 2R$ .

*Proof.* Let us consider

$$R \ge 1$$
 and  $y, z \in \mathbb{R}^N$  with  $|y| \ge R$ ,  $|z| \ge R$  and  $\frac{2}{3}R \le |y-z| \le 2R$ . (11)

For such y, z and  $\lambda \in [0, 1]$ , we set  $w_{\infty} = (1 - \lambda)(y * u_{\infty}) + \lambda(z * u_{\infty})$  and choose  $t_{\infty} = t_{\infty}(\lambda, y, z)$  as above, *i.e.*, such that  $t_{\infty}w_{\infty} \in \mathcal{N}$ .

In the following, all constants will neither depend on R nor on  $\lambda, y, z$ . In a first step, we will give bounds on various terms related to the energy functional, with respect to the following nonlinear interaction term:

$$A_{y,z} := \int_{\mathbb{R}^N} (y * u_\infty)^{p-1} (z * u_\infty) \, dx = \int_{\mathbb{R}^N} u_\infty^{p-1} ((z - y) * u_\infty) \, dx$$

Let us first remark that, since  $u_{\infty}$  is positive and radially decreasing, the estimate (8) gives

$$A_{y,z} \ge \left(\frac{C_1}{2}\right)^{p-1} \int_{B_1(0)} u_\infty(x - (z - y)) \, dx \ge \zeta_1 |z - y|^{-(N+2s)} \tag{12}$$

for all |z - y| > 1 and some constant  $\zeta_1 > 0$ .

On the other hand, from (8) and Lemma 3.1 with  $\sigma = (p-1)(N+2s)$  and  $\tau = N+2s$ , there exists a constant  $\zeta_2 > 0$  independent of y, z such that

$$A_{y,z} \leqslant \zeta_2 |z - y|^{-(N+2s)} \tag{13}$$

for all |z - y| > 1. Let now  $\alpha$  be as in assumption (H) and satisfy (10). Applying Lemma 3.1 with  $\sigma = \alpha$  and  $\tau = 2(N + 2s)$ , we obtain

$$\int_{\mathbb{R}^N} (1+|x|)^{-\alpha} \{ (y * u_\infty)^2 + (z * u_\infty)^2 \} \, dx \leqslant C \max\{ |y|^{-\alpha}, |z|^{-\alpha} \}$$

for all  $y, z \in \mathbb{R}^N$  with some constant C > 0. Since  $|y|, |z| \ge R \ge \frac{1}{2}|z-y|$ , we have, making  $\zeta_1$  larger if necessary,

$$\int_{\mathbb{R}^N} (1+|x|)^{-\alpha} \{ (y*u_\infty)^2 + (z*u_\infty)^2 \} \, dx \leqslant \zeta_1^{-1} R^{N+2s-\alpha} A_{y,z} \tag{14}$$

for all R,y,z satisfying (11). A further application of Lemma 3.1 with  $\sigma=\alpha,$   $\tau=p(N+2s),$  gives

$$\int_{\mathbb{R}^N} (1+|x|)^{-\alpha} \{ (y * u_\infty)^p + (z * u_\infty)^p \} \, dx \leqslant C' \max\{ |y|^{-\alpha}, |z|^{-\alpha} \}$$

8

for all  $y, z \in \mathbb{R}^N$ ,  $|y|, |z| \ge 1$  with some constant C' > 0. As above, making  $\zeta_1$  again larger if necessary, we can write

$$\int_{\mathbb{R}^N} (1+|x|)^{-\alpha} \{ (y*u_\infty)^p + (z*u_\infty)^p \} \, dx \leqslant \zeta_1^{-1} R^{N+2s-\alpha} A_{y,z} \tag{15}$$

for all R,y,z satisfying (11). Finally, Lemma 3.1 with  $\sigma=\tau=\frac{p}{2}(N+2s)$  yields

$$\int_{\mathbb{R}^N} (y * u_\infty)^{\frac{p}{2}} (z * u_\infty)^{\frac{p}{2}} dx = \int_{\mathbb{R}^N} u_\infty^{\frac{p}{2}} ((y - z) * u_\infty)^{\frac{p}{2}} dx \leqslant C'' |y - z|^{-\frac{p}{2}(N+2s)}$$

for all  $y, z \in \mathbb{R}^N$  with some constant C'' > 0. Therefore, making  $\zeta_1$  again larger if necessary we find, using (10),

$$\int_{\mathbb{R}^{N}} (y * u_{\infty})^{\frac{p}{2}} (z * u_{\infty})^{\frac{p}{2}} dx \leqslant \zeta_{1}^{-1} R^{N+2s-\alpha} A_{y,z}$$
(16)

holds for all R, y, z satisfying (11). We now have all the tools to estimate

$$J(t_{\infty}w_{\infty}) = \frac{t_{\infty}^2}{2} \|w_{\infty}\|_{V}^2 - \frac{t_{\infty}^p}{p} \int_{\mathbb{R}^N} Q(x)(w_{\infty}(x))^p \, dx.$$

We start by estimating the term  $||w_{\infty}||_{V}^{2}$  which we split in the following way.

$$\begin{split} &\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{w}_{\infty}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} V(x) w_{\infty}^2 \, dx \\ &= \left( (1-\lambda)^2 + \lambda^2 \right) \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}_{\infty}|^2 \, d\xi + \int_{\mathbb{R}^N} u_{\infty}^2 \, dx \right) \\ &+ 2(1-\lambda) \lambda \left( \int_{\mathbb{R}^N} |\xi|^{2s} \operatorname{Re} \left( e^{-i(y-z) \cdot \xi} \widehat{u}_{\infty}(\xi) \overline{\widehat{u}_{\infty}(\xi)} \right) d\xi + \int_{\mathbb{R}^N} (y \ast u_{\infty}) (z \ast u_{\infty}) \, dx \right) \\ &+ \int_{\mathbb{R}^N} (V(x) - 1) w_{\infty}^2 \, dx. \end{split}$$

The property  $J'_{\infty}(u_{\infty}) = 0$  implies that

$$\int_{\mathbb{R}^N} |\xi|^{2s} \operatorname{Re}\left(e^{-i(y-z)\cdot\xi}\widehat{u}_{\infty}(\xi)\overline{\widehat{u}_{\infty}(\xi)}\right) d\xi + \int_{\mathbb{R}^N} (y \ast u_{\infty})(z \ast u_{\infty}) dx$$

$$= \int_{\mathbb{R}^n} (y \ast u_{\infty})^{p-1}(z \ast u_{\infty}) dx = A_{y,z} = A_{z,y}.$$
(17)

We deduce from (14) and condition (H) that

$$\int_{\mathbb{R}^N} (V(x) - 1) |w_{\infty}|^2 dx \leq 2\widetilde{\kappa}_1 \int_{\mathbb{R}^N} (1 + |x|)^{-\alpha} [(y * u_{\infty})^2 + (z * u_{\infty})^2] dx$$
$$\leq 2\widetilde{\kappa}_1 \zeta_1^{-1} R^{N+2s-\alpha} A_{y,z} \quad \text{for all } \lambda \in [0, 1] \text{ and } R, y, z \text{ satisfying (11)},$$

where  $\tilde{\kappa}_1 = 2^{\alpha} \max\{\kappa_1, \|V\|_{\infty}\}$ . In addition, we point out that from the condition (V), there follows

$$\|w_{\infty}\|_{V}^{2} \longrightarrow ((1-\lambda)^{2} + \lambda^{2})\|u_{\infty}\|_{s}^{2} \quad \text{as } R \to \infty,$$
(18)

uniformly in  $\lambda \in [0, 1]$ . Turning to the second integral, we write

$$\int_{\mathbb{R}^N} Q(x)(w_{\infty}(x))^p dx = (\lambda^p + (1-\lambda)^p) \int_{\mathbb{R}^N} u_{\infty}^p dx$$
$$+ \int_{\mathbb{R}^N} w_{\infty}^p - [\lambda^p (y * u_{\infty})^p + (1-\lambda)^p (z * u_{\infty})^p] dx$$
$$+ \int_{\mathbb{R}^N} (Q(x) - 1)(w_{\infty}(x))^p dx.$$

From [3, Lemma 2.1], there exists a constant C=C(p)>0 such that for all  $a,b\geqslant 0$ 

$$(a+b)^{p} \ge a^{p} + b^{p} + p(a^{p-1}b + ab^{p-1}) - Ca^{\frac{p}{2}}b^{\frac{p}{2}}.$$
(19)

Hence, from (16), we obtain

$$\begin{split} &\int_{\mathbb{R}^N} w_{\infty}^p - \left[\lambda^p (y \ast u_{\infty}) + (1-\lambda)^p (z \ast u_{\infty})\right] dx \\ &\geqslant p \lambda^{p-1} (1-\lambda) \int_{\mathbb{R}^N} u_{\infty} (x-y)^{p-1} u_{\infty} (x-z) \, dx \\ &\quad + p \lambda (1-\lambda)^{p-1} \int_{\mathbb{R}^N} u_{\infty} (x-y) u_{\infty} (x-z)^{p-1} \, dx \\ &\quad - C \lambda^{\frac{p}{2}} (1-\lambda)^{\frac{p}{2}} \int_{\mathbb{R}^N} u_{\infty} (x-y)^{\frac{p}{2}} u_{\infty} (x-z)^{\frac{p}{2}} \, dx \\ &\geqslant \left[ p (\lambda^{p-1} (1-\lambda) + \lambda (1-\lambda)^{p-1}) - C \zeta_1^{-1} R^{N+2s-\alpha} \right] A_{y,z} \end{split}$$

for  $\lambda \in [0, 1]$  and R, y, z satisfying (11). Moreover, condition (H) as well as (15) imply that, setting  $\tilde{\kappa}_2 = 2^{\alpha} \max\{\kappa_2, \|Q\|_{\infty}\},$ 

$$\begin{split} \int_{\mathbb{R}^N} (Q(x)-1)(w_{\infty}(x))^p \, dx &\geq -2^{p-1} \widetilde{\kappa}_2 \int_{\mathbb{R}^N} (1+|x|)^{-\alpha} \{ (y \ast u_{\infty})^p + (z \ast u_{\infty})^p \} \, dx \\ &\geq -2^{p-1} \widetilde{\kappa}_2 \zeta_1^{-1} R^{N+2s-\alpha} A_{y,z}, \end{split}$$

for  $\lambda \in [0,1]$  and R, y, z satisfying (11). Let us also remark that the assumption (Q) ensures

$$\int_{\mathbb{R}^N} Q(x)(w_{\infty}(x))^p \, dx \longrightarrow \left( (1-\lambda)^p + \lambda^p \right) \int_{\mathbb{R}^N} u_{\infty}^p \, dx \quad \text{as } R \to \infty, \tag{20}$$

uniformly in  $\lambda \in [0, 1]$ . Combining this last inequality with (6) and (18), we find

$$0 < t_{\infty} = \left(\frac{\|w_{\infty}\|_{V}^{2}}{\int_{\mathbb{R}^{N}} Q(x)(w_{\infty}(x))^{p} dx}\right)^{\frac{1}{p-2}} \longrightarrow \left(\frac{(1-\lambda)^{2} + \lambda^{2}}{(1-\lambda)^{p} + \lambda^{p}}\right)^{\frac{1}{p-2}}, \quad (21)$$

as  $R \to \infty$ , uniformly in  $\lambda \in [0, 1]$ . Since

$$J(t_{\infty}w_{\infty}) = \left(\frac{1}{2} - \frac{1}{p}\right) t_{\infty}^{2} \|w_{\infty}\|_{V}^{2} \leq \left(\frac{1}{2} - \frac{1}{p}\right) t_{\infty}^{2} \left[ ((1-\lambda)^{2} + \lambda^{2}) \|u_{\infty}\|_{s}^{2} + 2\lambda(1-\lambda)A_{y,z} + 2\widetilde{\kappa}_{1}\zeta_{1}^{-1}R^{N+2s-\alpha}A_{y,z} \right]$$

and  $c_{\infty} = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_{\infty}\|_{s}^{2}$ , we can find some  $R_{0} \ge 1$  and  $0 < \delta_{0} < 1$  such that

$$J(t_{\infty}w_{\infty}) \leqslant \frac{3}{2}c_{\infty} \tag{22}$$

for all  $\lambda \in [0, \delta_0) \cup (1 - \delta_0, 1]$  and R, y, z satisfying (11) with  $R \ge R_0$ .

On the other hand, the above estimates together give for R, y, z satisfying (11) and  $\lambda \in [\delta_0, 1 - \delta_0]$ :

$$J(t_{\infty}w_{\infty}) - 2c_{\infty} \leqslant J(t_{\infty}w_{\infty}) - J_{\infty}(\lambda t_{\infty}u_{\infty}) - J_{\infty}((1-\lambda)t_{\infty}u_{\infty})$$
  
$$\leqslant -t_{\infty}^{p}\lambda(1-\lambda)\Big\{(1-\lambda)^{p-2} + \lambda^{p-2} - t_{\infty}^{2-p} - \zeta_{3}^{-1}R^{N+2s-\alpha}\Big\}A_{y,z},$$
(23)

for some constant  $\zeta_3 > 0$ , where we used the fact that  $t_{\infty}$  is bounded below away from 0, uniformly in y, z and  $\lambda$ . According to (21) and since  $\alpha > N + 2s$ , there exists  $R_1 \ge R_0$  such that

$$(1-\lambda)^{p-2} + \lambda^{p-2} - t_{\infty}^{2-p} - \zeta_3^{-1} R^{N+2s-\alpha} \ge \delta_0^p$$

for all  $\lambda \in [\delta_0, 1 - \delta_0]$  and  $R \ge R_1$ . With this choice, (23) gives

$$J(t_{\infty}w_{\infty}) \leqslant 2c_{\infty} - \kappa_3 \delta_0^{p+2} A_{y,x}$$

for all  $\lambda \in [\delta_0, 1 - \delta_0]$  and  $R \ge R_1$ , with some constant  $\kappa_3 > 0$ . This, together with (22) shows  $J(t_{\infty}w_{\infty}) < 2c_{\infty}$  for every  $\lambda \in [0,1]$  and R, y, z satisfying (11) with  $R \ge R_1$ . 

The last result in this section describes the behavior of the bounded Palais-Smale sequences for J, and shows that the same kind of splitting as in the case s = 1 studied by Benci and Cerami [5] (see also [4, Proposition II.1] or [34, Theorem 8.4]) also occurs in the fractional case 0 < s < 1.

**Lemma 3.3.** Let  $(u_n)_n \subset H^s(\mathbb{R}^N)$  be a bounded sequence, for which  $J'(u_n) \to 0$ as  $n \to \infty$ . Then, there exist  $\ell \in \mathbb{N} \cup \{0\}$ , sequences  $(x_n^i)_n \subset \mathbb{R}^N$ ,  $1 \leq i \leq \ell$ , and  $\overline{u}, w_1, \ldots, w_\ell \in H^s(\mathbb{R}^N)$  satisfying (up to a subsequence)

- (i)  $J'(\overline{u}) = 0$ ,
- (ii)  $J'_{\infty}(w_i) = 0, \ i = 1, \dots, \ell,$ (ii)  $|x_n^i| \to \infty \text{ and } |x_n^i x_n^j| \to \infty \text{ as } n \to \infty \text{ for } 1 \leq i \neq j \leq \ell,$
- (iii)  $\left\| u_n \left[\overline{u} + \sum_{i=1}^{\ell} x_n^i * w_i\right] \right\|_s \to 0 \text{ as } n \to \infty \text{ and}$
- (iv)  $J(u_n) \to J(\overline{u}) + \sum_{i=1}^{\ell} J_{\infty}(w_i), \text{ as } n \to \infty.$

*Proof.* Since  $(u_n)_n$  is a bounded sequence in  $H^s(\mathbb{R}^N)$  we may assume, up to a subsequence, that  $u_n \rightharpoonup \overline{u}$  for some  $\overline{u} \in H^s(\mathbb{R}^N)$ , and the weak sequential continuity of J' implies  $J'(\overline{u}) = 0$ .

**Step 1**: Let  $v_n^1 := u_n - \overline{u}$  for all  $n \in \mathbb{N}$ . Since  $v_n^1 \to 0$  in  $H^s(\mathbb{R}^N)$  and since  $V(x) \to 1$ , resp.  $Q(x) \to 1$  for  $|x| \to \infty$ , the compact embeddings  $H^s(B_R(0)) \hookrightarrow$  $L^q(B_R(0)), R > 0, 2 \leq q < 2_s^*$  (resp.  $2 \leq q < \infty$  if N = 1 and  $s \geq \frac{1}{2}$ ) gives

$$\int_{\mathbb{R}^N} (V(x) - 1) |v_n^1|^2 \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} (Q(x) - 1) |v_n^1|^p \, dx \to 0 \quad \text{as } n \to \infty,$$
(24)

Therefore, as  $n \to \infty$ , we find (up to a subsequence):  $J_{\infty}(v_n^1) = J(v_n^1) + o(1)$ 

$$= J(u_n) - J(\overline{u}) + \frac{1}{p} \int_{\mathbb{R}^N} Q(x) \left( |u_n|^p - |\overline{u}|^p - |v_n^1|^p \right) dx + o(1) = J(u_n) - J(\overline{u}) + o(1),$$

where in the last step we have used the Brézis-Lieb lemma [7]. For  $\varphi \in H^s(\mathbb{R}^N)$  with  $\|\varphi\| = 1$ , we obtain moreover that

$$\begin{aligned} J_{\infty}'(v_n^1)\varphi - J'(u_n)\varphi + J'(\overline{u})\varphi &= \int_{\mathbb{R}^N} (1 - V(x))v_n^1\varphi \, dx \\ &+ \int_{\mathbb{R}^N} (Q(x) - 1)|v_n^1|^{p-2}v_n^1\varphi \, dx \\ &+ \int_{\mathbb{R}^N} Q(x)[|u_n|^{p-2}u_n - |\overline{u}|^{p-2}\overline{u} - |v_n^1|^{p-2}v_n^1]\varphi \, dx \end{aligned}$$

tends uniformly to 0 as  $n \to \infty$ , using (24) and a similar argument as in [4, Lemma A.2]. Since  $J'(u_n) \to 0$  as  $n \to \infty$  and  $J'(\overline{u}) = 0$ , we find

$$J'_{\infty}(v_n^1) \to 0, \quad \text{ as } n \to \infty.$$

Step 2: Let

$$\zeta = \limsup_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} \left( v_n^1(x) \right)^2 dx \right).$$

If  $\zeta = 0$ , then the concentration-compactness Lemma gives  $\|v_n^1\|_{L^p} \to 0$  as  $n \to \infty$ , and we infer

$$||u_n - \overline{u}||_s^2 = ||v_n^1||_s^2 = J'_{\infty}(v_n^1)v_n^1 + \int_{\mathbb{R}^N} |v_n^1|^p \, dx \to 0, \quad \text{as } n \to \infty.$$

Hence  $u_n \to \bar{u}$  in  $H^s(\mathbb{R}^N)$  as  $n \to \infty$ , and the proof is complete. In the case where  $\zeta > 0$ , passing to a subsequence, we can find a sequence  $(x_n^1)_n \subset \mathbb{R}^N$  satisfying  $|x_n^1| \to \infty$  as  $n \to \infty$  and

$$\int_{B_1(0)} \left( v_n^1(x+x_n^1) \right)^2 dx = \int_{B_1(x_n^1)} \left( v_n^1(x) \right)^2 dx > \frac{\zeta}{2}$$

for all *n*. Since  $((-x_n^1) * v_n^1)_n$  is bounded, going to a further subsequence if necessary, we obtain  $(-x_n^1) * v_n^1 \rightarrow w_1 \neq 0$ , using the compactness of the embedding  $H^s(B_1(0)) \hookrightarrow L^2(B_1(0))$ . Furthermore, the weak sequential continuity of  $J'_{\infty}$  and the invariance under translations of  $J_{\infty}$  give for every  $\varphi \in H^s(\mathbb{R}^N)$ ,

$$J'_{\infty}(w_1)\varphi = \lim_{n \to \infty} J'_{\infty} \left( (-x_n^1) * v_n^1 \right)\varphi = \lim_{n \to \infty} J'_{\infty}(v_n^1)(x_n^1 * \varphi) = 0.$$

Setting  $v_n^2 := v_n^1 - (x_n^1 * w_1)$ , it follows that  $v_n^2 \rightarrow 0$  in  $H^s(\mathbb{R}^N)$ , and the same arguments as above, applied to  $J_{\infty}$ , give

$$J_{\infty}(v_n^2) = J_{\infty}(v_n^1) - J_{\infty}(w_1) + o(1) = J(u_n) - J(\bar{u}) - J_{\infty}(w_1) + o(1),$$

and  $J'_{\infty}(v_n^2)\varphi - J'_{\infty}(v_n^1)\varphi + J'_{\infty}(w_1)(-x_n^1 * \varphi) \to 0$  uniformly for  $\|\varphi\| = 1$ , as  $n \to \infty$ .

We conclude that  $J'_{\infty}(v_n^2) \to 0$  as  $n \to \infty$  and iterate the above procedure. At each step we choose a sequence  $(x_n^i)_n \subset \mathbb{R}^N$  such that  $|x_n^i| \to \infty$  and  $|x_n^i - x_n^j| \to \infty$ for all  $i \neq j$ , as  $n \to \infty$ , and obtain a critical point  $w_i$  of  $J_{\infty}$  such that with  $v_n^{i+1} := v_n^i - x_n^i * w_i = u_n - \overline{u} - \sum_{j=1}^i x_n^j * w_j$  there holds (up to a subsequence)

$$J_{\infty}(v_n^{i+1}) = J(u_n) - J(\bar{u}) - \sum_{j=1}^i J_{\infty}(w_j) + o(1)$$
, and  $J'_{\infty}(v_n^{i+1}) \to 0$  as  $n \to \infty$ . Since  $J_{\infty}(w) \ge c_n \ge 0$  holds for every nontrivial critical point  $w$  of  $J_{\infty}$  and since the

 $J_{\infty}(w) \ge c_{\infty} > 0$  holds for every nontrivial critical point w of  $J_{\infty}$ , and since the boundedness of  $(u_n)_n$  implies that  $\sup_{n \in \mathbb{N}} J(u_n) < \infty$ , the procedure has to stop after a finite number of steps.

#### 4. EXISTENCE OF A NONTRIVIAL SOLUTION

Assuming the conditions (V), (Q), (H) and (U), we now prove the existence of a nontrivial solution to (3), using the method of Bahri and Li [3] (see also [16]). First note that  $c \leq c_{\infty}$  holds, as can be deduced from (18), (20) and (21) by setting  $\lambda = 0$ . If  $c < c_{\infty}$  then Proposition 2.4 gives the desired conclusion.

In the case  $c = c_{\infty}$ , we consider the barycenter map  $\beta$ :  $H^{s}(\mathbb{R}^{N}) \setminus \{0\} \to \mathbb{R}^{N}$  given by

$$\beta(u) = \frac{1}{\|u\|_{L^p}^p} \int_{\mathbb{R}^N} \frac{x}{|x|} |u(x)|^p \, dx, \quad u \in H^s(\mathbb{R}^N) \setminus \{0\}.$$

This mapping is continuous, and even uniformly continuous on the bounded subsets of  $H^s(\mathbb{R}^N) \setminus \{u \in H^s(\mathbb{R}^N) : \|u\|_{L^p} < r\}$  for any r > 0. Moreover,  $|\beta(u)| < 1$  for every  $u \neq 0$ . For each  $b \in B_1(0) \subset \mathbb{R}^N$  we now set

$$I_b := \inf_{\substack{u \in \mathcal{N} \\ \beta(u) = b}} J(u) \ge c$$

and distinguish two cases.

Case 1:  $c = c_{\infty} = I_b$  for some |b| < 1.

Here, we claim that J has a nontrivial critical point at level  $I_b = c = c_{\infty}$ . Indeed, let  $(v_n)_n \subset \mathcal{N}$  with  $\beta(v_n) = b$  for all  $n \in \mathbb{N}$  be a minimizing sequence for  $I_b$ . Since by Lemma 2.1,  $(v_n)_n$  is bounded and  $\mathcal{N}$  is bounded away from 0, we may choose, by the uniform continuity of  $\beta$  on bounded subsets of  $\mathcal{N}$ , some  $\delta > 0$ such that for every  $n \in \mathbb{N}$ :  $|\beta(v)| < \frac{1+|b|}{2}$  for all  $v \in \mathcal{N}$  with  $||v - v_n||_s < \delta$ . According to Ekeland's variational principle, we can find a Palais-Smale sequence  $(u_n)_n \in \mathcal{N}$  for which  $J(u_n) \to I_b$  and  $||u_n - v_n||_s \to 0$  holds, as  $n \to \infty$ . In particular, we find that  $|\beta(u_n)| < \frac{1+|b|}{2}$  holds for n large enough. Assuming by contradiction that J has no non-trivial critical point, the assumption  $c = c_{\infty}$ , Lemma 3.3 (iv) and the uniqueness of the ground state of (5) allow us to find, going to a subsequence of  $(u_n)_n$  if necessary, a sequence  $(x_n)_n \subset \mathbb{R}^N$  such that  $|x_n| \to \infty$ and  $||u_n - (x_n * u_\infty)||_s \to 0$ , as  $n \to \infty$ . Since, also,  $||x_n * u_\infty||_{L^p} = ||u_\infty||_{L^p} > 0$ holds for all n, and  $|\beta(x_n * u_\infty)| \to 1$  as  $n \to \infty$ , the uniform continuity of  $\beta$  gives

$$1 = \lim_{n \to \infty} |\beta(x_n * u_\infty)| \leq \limsup_{n \to \infty} |\beta(u_n)| \leq \frac{1 + |b|}{2}.$$

This contradicts our assumption |b| < 1, and therefore shows that J has a critical point u at level  $c = c_{\infty}$ . This function u is positive by Corollary 2.3. **Case 2**:  $c = c_{\infty} < I_b$  for every |b| < 1.

In this case we will show that J possesses a critical point at some level  $c_0 \in [I_b, 2c)$ and Corollary 2.3 will yield the conclusion.

For R > 0, let  $y = (0, ..., 0, R) \in \mathbb{R}^N$  and consider the open ball

$$\Omega_R := B_{\frac{4}{2}R}(\frac{y}{3}) = \left\{ (1-\lambda)y + \lambda z \in \mathbb{R}^N : 0 \leq \lambda < 1, \ z \in \partial \Omega_R \right\}.$$

We define a min-max level  $c_0$  as follows. Let  $R \ge R_1$  where  $R_1$  is given in Lemma 3.2, and consider the projection of  $\partial \Omega_R$  onto the Nehari manifold:  $\gamma_0: \partial \Omega_R \to \mathcal{N}$  given by

$$\gamma_0(z) := t_\infty \cdot (z * u_\infty) \quad \text{for all } z \in \partial \Omega_R,$$

where  $t_{\infty} = t_{\infty}(1, 0, z)$  in the notation of Section 3. We set  $\Gamma_R := \{\gamma : \overline{\Omega}_R \to \mathcal{N} : \gamma \text{ continuous and } \gamma|_{\partial\Omega_R} = \gamma_0\}$  and consider the min-max energy level

$$c_0 := \inf_{\gamma \in \Gamma_R} \max_{x \in \overline{\Omega}_R} J(\gamma(x)).$$
(25)

We claim that for b = (0, ..., 0, |b|) with 0 < |b| < 1 fixed, there holds

$$I_b \leqslant c_0 < 2c_\infty \tag{26}$$

for R large enough. To show the left-hand inequality, consider for each  $\gamma \in \Gamma_R$ the homotopy  $\eta$ :  $[0,1] \times \overline{\Omega}_R \to \overline{B_1(0)}$  given by  $\eta(\theta, x) = \theta\beta(\gamma(x)) + (1-\theta)g(x)$ ,  $0 \leq \theta \leq 1, x \in \overline{\Omega}_R$ , where g is the homothetic contraction of  $\overline{\Omega}_R$  onto the closed unit ball in  $\mathbb{R}^N$ :

$$g(x) = \begin{cases} \tau(x)\frac{x}{|x|} & \text{if } x \neq 0\\ & & \\ 0 & \text{if } x = 0 \end{cases} \text{ with } \tau(x) = \frac{1}{5R} \left[ \sqrt{15|x|^2 + x_N^2} - x_N \right], \quad x \in \overline{\Omega_R}.$$

Since  $\gamma|_{\partial\Omega_R} = \gamma_0$  and  $\theta\beta(\gamma_0(z)) + (1-\theta)g(z) \to \frac{z}{|z|}$  uniformly for  $z \in \partial\Omega_R$  and  $0 \leq \theta \leq 1$ , as  $R \to \infty$ , we obtain  $b \notin \eta([0,1] \times \partial\Omega_R)$  for R large enough. The homotopy invariance of the degree then implies  $\deg(\beta \circ \gamma, \Omega_R, b) = \deg(g, \Omega_R, b) = 1$ . Using the existence property, we can therefore find some  $x_b \in \Omega_R$  for which  $\beta(\gamma(x_b)) = b$ , and this gives  $I_b \leq J(\gamma(x_b))$ . Since  $\gamma \in \Gamma_R$  was arbitrarily chosen, we obtain  $I_b \leq c_0$ . Lemma 3.2 gives the second inequality, when we consider  $\gamma_2 \in \Gamma_R$  given by

$$\gamma_2((1-\lambda)y+\lambda z) = t_{\infty}((1-\lambda)(y*u_{\infty}) + \lambda(z*u_{\infty})), \qquad \lambda \in [0,1], \ z \in \partial \Omega_R.$$

In particular, the min-max. level  $c_0$  satisfies

$$c = c_{\infty} < c_0 < 2c_{\infty} \tag{27}$$

for R large enough.

Next, we point out that  $\mathcal{N}$  is a closed connected  $C^1$ -submanifold of the Banach space  $H^s(\mathbb{R}^N)$ . Moreover, for  $R \ge R_1$ , the the family  $\mathcal{F}_R = \{\gamma(\overline{\Omega}_R) \subset \mathcal{N} : \gamma \in \Gamma_R\}$ of compact subsets of  $\mathcal{N}$  is a homotopy-stable family with boundary  $\gamma_0(\partial\Omega_R) \subset \mathcal{N}$ , in the sense of Ghoussoub [23, Definition 3.1]. Since  $J(\gamma_0(z))$  converges to  $c_\infty$  as  $R \to \infty$ , uniformly for  $z \in \partial\Omega_R$ , we have furthermore

$$\max_{z \in \partial \Omega_R} J(\gamma_0(z)) < c_0 = \inf_{\gamma \in \Gamma_R} \max_{x \in \overline{\Omega}_R} J(\gamma(x)) = \inf_{A \in \mathcal{F}_R} \sup_{v \in A} J(v)$$

for large R, and the min-max principle [23, Theorem 3.2], gives the existence of a Palais-Smale sequence  $(u_n)_n \subset \mathcal{N}$  for J at level  $c_0$ . Since J is coercive on  $\mathcal{N}$ ,  $(u_n)_n$ is bounded in  $H^s(\mathbb{R}^N)$ . Moreover, since  $J_{\infty}(w) > 2c_{\infty}$  holds for every sign-changing critical point w of  $J_{\infty}$  (see Proposition 2.2), the estimate (27), the assumption (U) and Lemma 3.3 imply that, up to a subsequence,  $u_n \to \overline{u}$  as  $n \to \infty$  for some critical point  $\overline{u} \neq 0$  of J which satisfies  $J(\overline{u}) = c_0 < 2c = 2c_{\infty}$ . This concludes the proof.  $\Box$ 

#### References

- C. J. Amick and J. F. Toland, Uniqueness and related analytic properties for the Benjamin-Ono equation—a nonlinear Neumann problem in the plane, Acta Math. 167 (1991) 107–126.
- [2] A. Bahri and J. M. Coron, On an nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.* 41 (1988) 253–294.

- [3] A. Bahri and Y. Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in R<sup>N</sup>, Rev. Mat. Iberoamericana 6 (1990) 1–15.
- [4] A. Bahri and P. L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997) 365–413.
- [5] V. Benci and G. Cerami, Positive solutions of semilinear elliptic problems in exterior domains, Arch. Rat. Mech. Anal. 99 (1987) 283–300.
- [6] J. L. Bona and Y. Li, Decay and analyticity of solitary waves, J. Math. Pures Appl. 76 (1997) 377–430.
- [7] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* 88 (1983) 486–490.
- [8] M. Cheng, Bound state for the fractional Schrödinger equation with unbounded potential, J. Math. Phys. 53 (2012) 043507.
- [9] Y. Cho and T. Ozawa, A note on the existence of a ground state solution to a fractional Schrödinger equation, Kyushu J. Math. 67 (2013), 227–236.
- [10] M. Clapp and T. Weth, Multiple solutions of nonlinear scalar field equations, Comm. Part. Diff. Equations 29 (2004) 1533–1554.
- [12] A. de Bouard and J.-Cl. Saut, Symmetries and decay of the generalized Kadomtsev-Petviashvili solitary waves, SIAM J. Math. Anal. 28 (1997) 1064–1085.
- [13] W. Y. Ding and W.-M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, Arch. Rational Mech. Anal. 91 (1986) 283–308.
- [14] S. Dipierro, G. Palatucci and E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, *Matematiche (Catania)* 68 (2013) 201–216.
- [15] I. Ekeland, On the variational principle, J. Math. Anal. and Appl. 47 (1974) 324–353.
- [16] G. Evéquoz and T. Weth, Entire solutions to nonlinear scalar field equations with indefinite linear part, Adv. Nonlin. Studies 12 (2012) 281–314.
- [17] M. M. Fall and E. Valdinoci, Uniqueness and nondegeneracy of positive solutions of  $(-\Delta)^s u + u = u^p$  in  $\mathbb{R}^N$  when s is close to 1. Preprint (2013) available online at http://arxiv.org/abs/1301.4868v2. To appear in Comm. Math. Physics.
- [18] M. M. Fall and S. Jarohs, Overdetermined problems with fractional laplacian. Preprint (2013) available online at http://arxiv.org/abs/1311.7549v2
- [19] P. Felmer, A. Quaas and J. Tan, Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian, Proc. Roy. Soc. Edinburgh, Sect. A 142 (2012) 1237–1262.
- [20] B. Feng, Ground states for the fractional Schrödinger equation, Electron. J. Differential Equations 127 (2013) 1–11.
- [21] R. L. Frank and E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in R, Acta Mathematica. 210 (2013) 261–318.
- [22] R. L. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian. Preprint (2013) available online at http://arxiv.org/abs/1302.2652
- [23] N. Ghoussoub, Duality and perturbation methods in critical point theory, Cambridge Tracts in Mathematics, 107, Cambridge University Press, Cambridge, 1993.
- [24] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66 (2002) 056108.
- [25] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A 268 (2000) 298-305.
- [26] M. Mariş, On the existence, regularity and decay of solitary waves to a generalized Benjamin-Ono equation, Nonlinear Anal. 51 (2002) 1073–1085.
- [27] C. E. Kenig, Y. Martel and L. Robbiano, Local well-posedness and blow-up in the energy space for a class of L<sup>2</sup> critical dispersion generalized Benjamin-Ono equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011) 853–887.
- [28] M. K. Kwong, Uniqueness of positive solutions of Δ u + u<sup>p</sup> = 0 in ℝ<sup>N</sup>, Arch. Rational Mech. Anal. 105 (1989) 243–266.
- [29] N. Hirano, Multiple existence of sign changing solutions for semilinear elliptic problems on  $\mathbb{R}^N$ , Nonlinear Anal. 46 (2001), 997–1020.
- [30] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145 and 223–283.
- [31] S. Secchi, On fractional Schrödinger equations in  $\mathbb{R}^N$  without the Ambrosetti-Rabinowitz condition. Preprint (2012) available online at http://arxiv.org/abs/1210.0755

- [32] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in  $\mathbb{R}^N$ , J. Math. Phys. 54 (2013) 031501.
- [33] M. I. Weinstein, Solitary waves of nonlinear dispersive evolution equations with critical power nonlinearities, J. Differential Equations 69 (1987) 192–203.
- [34] M. Willem, *Minimax Theorems*, Progr. in Nonlinear Differential Equations Appl., 24, Birkhäuser, Boston, 1996.

INSTITUT FÜR MATHEMATIK, JOHANN WOLFGANG GOETHE-UNIVERSITÄT, ROBERT-MAYER-Str. 10, 60054 Frankfurt am Main, Germany

*E-mail address*: evequoz@math.uni-frankfurt.de

African Institute for Mathematical Sciences (A.I.M.S.) of Senegal, KM 2, Route de Joal, B.P. 1418 Mbour, Sénégal

*E-mail address*: mouhamed.m.fall@aims-senegal.org

16