# REGULARITY RESULTS FOR NONLOCAL EQUATIONS AND APPLICATIONS 

MOUHAMED MOUSTAPHA FALL


#### Abstract

We introduce the concept of $C^{m, \alpha}$-nonlocal operators, extending the notion of second order elliptic operator in divergence form with $C^{m, \alpha}$-coefficients. We then derive the nonlocal analogue of the key existing results for elliptic equations in divergence form, notably the Hölder continuity of the gradient of the solutions in the case of $C^{0, \alpha}$-coefficients and the classical Schauder estimates for $C^{m+1, \alpha}$-coefficients. We further apply the regularity results for $C^{m, \alpha}$-nonlocal operators to derive optimal higher order regularity estimates of Lipschitz graphs with prescribed Nonlocal Mean Curvature. Applications to nonlocal equation on manifolds are also provided.


## 1. Introduction

We are concerned with a class of (not necessarily translation invariant) elliptic equations driven by nonlocal operators of fractional order. We extend in the nonlocal setting some key existing results for elliptic equations in divergence form with $C^{m, \alpha}$-coefficients. For a better description of how far the results in this paper extend to the fractional setting those available in the classical case, we start by recalling some main results of the classical local theory. We consider a weak solution $u \in H^{1}(\Omega)$ to the equation

$$
\begin{equation*}
\sum_{i, j=1}^{N} \partial_{i}\left(a_{i j}(x) \partial_{j} u\right)=f \quad \text { in } \Omega \tag{1.1}
\end{equation*}
$$

where, $\Omega$ is a bounded open subset of $\mathbb{R}^{N}, f \in L_{l o c}^{p}(\Omega), p>N / 2$, and the matrix coefficients $a_{i j}$ are measurable functions and satisfy, for every $x \in \Omega$, the following properties:

$$
\begin{array}{ll}
\text { (i) } a_{i j}(x)=a_{j i}(x) & \text { for all } i, j=1, \ldots, N \\
(i i) \kappa \delta_{i j} \leq a_{i j}(x) \leq \frac{1}{\kappa} \delta_{i j} & \text { for all } i, j=1, \ldots, N \tag{1.2}
\end{array}
$$

In the regularity theory for elliptic equations in divergence form with measurable coefficients, the De Giorgi-Nash-Moser theory provides a priori $C^{0, \alpha_{0}}(\Omega)$ estimates for weak solutions to (1.1), for some $\alpha_{0}=\alpha_{0}(N, p, \kappa)$, see e.g. 38. The range or value of the largest Hölder exponent $\alpha_{0}$ is known in general once the coefficients are sufficiently regular. For instance, if $a_{i j} \in C(\Omega)$ then $u \in C_{l o c}^{0, \beta}(\Omega)$ for all $\beta<\min (2-N / p, 1)$. Now Hölder continuous coefficients $a_{i j}$ yield Hölder continuity of the gradient of $u$. Namely, if $a_{i j} \in C^{0, \alpha}(\Omega)$, for some $\alpha \in(0,1)$, then $u \in C^{1, \min \left(1-\frac{N}{p}, \alpha\right)}(\Omega)$, provided $2-N / p>1$. Moreover the Schauder theory states that if $a_{i j} \in C^{m+1, \alpha}(\Omega)$ and $f \in C^{m, \alpha}(\Omega)$, then $u \in C^{m+2, \alpha}(\Omega)$ for $m \in \mathbb{N}$. We refer the reader to [38, 61]. Notable applications are the smoothness character of variational solutions, including the regularity of critical points of the integral functional

$$
\begin{equation*}
\mathcal{J}(u):=\int_{\Omega} G(\nabla u(x)) d x \tag{1.3}
\end{equation*}
$$

for some twice differentiable function $G$. As a matter of fact, the above regularity results provides a systematic proof of the Hilbert's $19^{\text {th }}$ problem stating that if $G \in C^{\infty}\left(\mathbb{R}^{N}\right)$, then the minimizer of $\mathcal{J}$ is of class $C^{\infty}$ as well. This was solved by de Giorgi in 23. Indeed, given a critical point $u \in H^{1}(\Omega)$ to $\mathcal{J}$, we have that $\frac{u(x+h)-u(x)}{|h|}$ solves an equation as in (1.1) with coefficients

[^0]$a_{i j}(x)=\int_{0}^{1} \partial_{i j}^{2} G(\varrho \nabla u(x+h)+(1-\varrho) \nabla u(x)) d \varrho$ satisfying (1.2) as soon as $D^{2} G$ is uniformly bounded from above and below on $\mathbb{R}^{N}$. Therefore the fact that $a_{i j}$ is as smooth as $\nabla u$ immediately implies that $u$ smooth, thanks to above regularity results for divergence type operators. On the other if $G(\zeta)=\sqrt{1+|\zeta|^{2}}$, then (1.3) becomes the area functional, and in this case $D^{2} G$ is not bounded from below. However, this gap can be filled by assuming that $u$ Lipschitz.
The aim of this paper is to extend all the above regularity results to equations driven by $C^{m, \alpha}$-nonlocal operators of fractional order which we describe below. Our notion of $C^{m, \alpha}$-nonlocal operators can be seen as a nonlocal version of second order partial differential equations in divergence form. On the other hand, as in the local case, since our notion of $C^{m, \alpha}$-nonlocal operators is stable under $C^{m, 1}$ local change of coordinates, our results apply to nonlocal equations on manifolds nonlocal geometric problems such as the prescribed nonlocal mean curvature problems.

We start introducing the class of kernels (defining nonlocal operators) we will use in the remaining of the paper. We consider $s \in(0,1), N \geq 1$ and $K: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow[-\infty,+\infty]$ such that

$$
\begin{array}{cc}
(i) K(x, y)=K(y, x) & \text { for all }(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} \\
(i i)|K(x, y)| \leq \frac{1}{\kappa}|x-y|^{-N-2 s} & \text { for all }(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} \\
\left(i i^{\prime}\right) \kappa|x-y|^{-N-2 s} \leq K(x, y) & \text { for all }(x, y) \in B_{\delta} \times B_{\delta} \tag{1.4}
\end{array}
$$

for some constants $\kappa, \delta>0$. We call $L_{s}\left(\mathbb{R}^{N}\right)$ the space of function $u \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ such that

$$
\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}:=\int_{\mathbb{R}^{N}}|u(y)|(1+|y|)^{-N-2 s} d y<\infty
$$

A kernel $K$ satisfying (1.4) $(i)-(i i)$ induces a linear nonlocal operator $\mathcal{L}_{K}: H^{s}(\Omega) \cap L_{s}\left(\mathbb{R}^{N}\right) \rightarrow \mathcal{D}^{\prime}(\Omega)$ given by

$$
\left\langle\mathcal{L}_{K} u, \psi\right\rangle:=\frac{1}{2} \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}}(u(x)-u(y))(\psi(x)-\psi(y)) K(x, y) d x d y \quad \text { for all } \psi \in C_{c}^{\infty}(\Omega)
$$

The weight in the definition of the space $L_{s}\left(\mathbb{R}^{N}\right)$ is determined by (1.4)-(ii) and can be modified accordingly. Given $f \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$, we say that $u \in H^{s}(\Omega) \cap L_{s}\left(\mathbb{R}^{N}\right)$ is a (weak) solution to the equation

$$
\begin{equation*}
\mathcal{L}_{K} u=f \quad \text { in } \Omega \tag{1.5}
\end{equation*}
$$

if $\mathcal{L}_{K} u=f$ in $\mathcal{D}^{\prime}(\Omega)$.
The class of operators $\mathcal{L}_{K}$ induced by the kernels $K$ satisfying (1.4) are the nonlocal analogue of second order elliptic operators in divergence form with measurable coefficients on $B_{\delta}$. In this case the de Giorgi-Nash-Moser a priori Hölder estimates is well developed, see [13, 21, 24, 27, 46, 47, 49, 51, In particular, it follows from [27] that, if $f \in L^{p}\left(B_{\delta}\right)$, for some $p>N /(2 s)$, then $u \in C_{l o c}^{0, \alpha_{0}}\left(B_{\delta}\right)$, for some $\alpha_{0}=\alpha_{0}(N, s, p)>0$.
The study of nonlocal variational equations involving general kernels, satisfying e.g. (1.4), is currently an intensive research area. In particular several papers deals with existence and, a part in some specific cases e.g. fractional Lapalcian, anisotropic fractional Lapalcian, regional fractional Laplacian, the "smoothness" properties of the nonlocal operators leading to higher order regularity of weak solutions (e.g. $C^{1, \alpha}$ or $C^{2 s+\alpha}$-regularity) remains an open questions. In the case of nonlocal and non-translation invariant operators (say in non-divergence form), different assumptions on the kernels yielding higher order regularity are present in the literature, starting from the work of CaffarelliSilvestre [16, followed by many others e.g. 42, 48, 56]. A first difficulty to address this question in the variational framework is the singular character of the kernel $K$ (satisfying (1.4)) at the diagonal points $x=y$ which encodes also the order of regularity of the solution, regardless the behaviour of the tail. In 30, we attempted to answer this question and introduced a notion of nonlocal operators with "continuous" coefficient, and we proved some optimal interior and boundary regularities of solutions to (1.5). In that paper, we also proved that $u$ is a classical solution provided the operator $\mathcal{L}_{K}$ is smooth enough together with $C^{1, \alpha}$ estimates for translation invariant problems. These results will be
sharpened and generalized in the present work.
Following [30], we now introduce the notion of $C^{m, \alpha}$-nonlocal (or fractional order) operators which, in particular, are the object of study in the present paper.

Definition 1.1. For $\delta>0$, we define $Q_{\delta}:=B_{\delta} \times[0, \delta)$. Let $\alpha \in[0,1), m \in \mathbb{N}$ and $K$ satisfy (1.4).

- We say that the kernel $K$ defines a $C^{m, \alpha}$-nonlocal operator in $Q_{\delta}$, if the function

$$
B_{\delta} \times(0, \delta) \times S^{N-1} \rightarrow \mathbb{R}, \quad(x, r, \theta) \mapsto r^{N+2 s} K(x, x+r \theta)
$$

extends to a map $\mathcal{A}_{K}: Q_{\delta} \times S^{N-1} \rightarrow \mathbb{R}$ satisfying, for some $\kappa>0$, the following properties:

$$
\begin{align*}
& \text { (iii) }\left\|\mathcal{A}_{K}\right\|_{C^{m, \alpha}\left(Q_{\delta} \times S^{N-1}\right)} \leq \frac{1}{\kappa}  \tag{1.6}\\
& \text { (iv) } \mathcal{A}_{K}(x, 0, \theta)=\mathcal{A}_{K}(x, 0,-\theta) \quad \text { for all }(x, \theta) \in B_{\delta} \times S^{N-1}
\end{align*}
$$

- The class of kernels $K$ satisfying (1.4) and (1.6) is denoted by $\mathscr{K}^{s}\left(\kappa, m+\alpha, Q_{\delta}\right)$.

A simple example in the class of kernels in Definition 1.1 is the the one of the fractional Laplacian, where the kernel is given by $K(x, y)=|x-y|^{-N-2 s}$. We remark that the class of operators induced by the kernels in Definition 1.1 provides a naturally extension of second order elliptic operators with $C^{m, \alpha}$-coefficients. Indeed, the computations in [8, Section 5] show, for all $\psi \in C_{c}^{1}\left(B_{\delta}\right)$, that

$$
\begin{equation*}
(1-s) \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}}(\psi(x)-\psi(y))^{2} K(x, y) d x d y \rightarrow \frac{1}{2} \sum_{i, j=1}^{N} \int_{\mathbb{R}^{N}} a_{i j}^{K}(x) \partial_{i} \psi(x) \partial_{j} \psi(x) d x \quad \text { as } s \rightarrow 1 \tag{1.7}
\end{equation*}
$$

where $a_{i j}^{K}(x)=\int_{S^{N-1}} \mathcal{A}_{K}(x, 0, \theta) \theta_{i} \theta_{j} d \theta$. Hence (1.6)-(iv) implies the symmetry of the matrix $\left(a_{i j}^{K}\right)_{1 \leq i, j \leq N}$.
Remark 1.2. In (1.6)-(iii), we impose the regularity of $\mathcal{A}_{K}$ in the angular variable $\theta$. However, this is typically not necessary to derive the accurate local behavior of solutions to (1.5) which parallels those solving (1.1) as stated above. In fact, nonlocal operators provide a wider framework than their local counterpart, since translation invariant nonlocal operators are those given by kernels $K$ of the form $K(x, y)=J(x-y)$, for some even function $J$. In addition, only in this translation invariant setting, regularity theory is already rich enough to include fully nonlinear problems, [15, 17, 42, 48, 56]. This issue on the possible anisotropic regularity of $\mathcal{A}_{K}$ in its variables will be taken into account in our main results stated in Section 1.3 below.

We now start by stating the main results concerning $C^{m, \alpha}$-nonlocal operators. Their generalizations are contained in Section 1.3 below. Our first main result is the following.

Theorem 1.3. Let $s \in(0,1), N \geq 1, \kappa>0$ and $\alpha \in(0,1)$. Let $K \in \mathscr{K}^{s}\left(\kappa, \alpha, Q_{2}\right), u \in H^{s}\left(B_{2}\right) \cap$ $L_{s}\left(\mathbb{R}^{N}\right)$ and $V, f \in L^{p}\left(B_{2}\right)$, for some $p>N /(2 s)$, satisfy

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} .
$$

(i) If $2 s \leq 1$, then there exists $C=C\left(s, N, \kappa, \alpha, p,\|V\|_{L^{p}\left(B_{2}\right)}\right)>0$ such that

$$
\begin{equation*}
\|u\|_{C^{0,2 s-\frac{N}{p}}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{L^{p}\left(B_{2}\right)}\right) \tag{1.8}
\end{equation*}
$$

(ii) If $2 s-1>\max \left(\frac{N}{p}, \alpha\right)$, then there exists $C=C\left(s, N, \kappa, \alpha, p,\|V\|_{L^{p}\left(B_{2}\right)}\right)>0$ such that

$$
\begin{equation*}
\|u\|_{C^{1, \min \left(2 s-\frac{N}{p}-1, \alpha\right)}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{L^{p}\left(B_{2}\right)}\right) \tag{1.9}
\end{equation*}
$$

The Hölder continuity of the gradient in (1.9) is the main novelty in the above result. Theorem 1.3 was only known in the translation invariant case, i.e. $K(x, y)=J(x-y)$, see 30 . We mention that the regularity estimate in (1.8) remains valid if $\alpha=0$, see [30, where it was proven that if $K \in \mathscr{K}^{s}\left(\kappa, 0, Q_{2}\right)$ (and for all $s \in(0,1)$ ), then $u \in C^{0, \beta}\left(B_{1}\right)$ for all $\beta<\min \left(2 s-\frac{N}{p}, 1\right)$. In view of (1.7), it will be apparent from the proof that the estimates in Theorem 1.3 remain stable as $s \rightarrow 1$ once we replace $\mathcal{L}_{K}$ by $(1-s) \mathcal{L}_{K}$ and provided $p>\frac{N}{2 s_{0}}$, with $s_{0} \in(0,1)$.
We recall that Hölder continuity of the gradient of solutions to fully nonlinear and non translation invariant integro-differential equations, in the spirit of Cordes and Nirenberg for elliptic equations in
nondivergence form, has been first established by Caffarelli and Silvestre in [16, see also 42, 48, 56] for higher order regularity estimates in nonlocal problems corresponding to elliptic equations in nondivergence form.
Our next results is concerned with $C^{m+2 s+\alpha}$ regularity estimates for solutions to equations driven by $C^{m+(2 s-1)_{+}+\alpha}$-nonlocal operators, provided $2 s+\alpha \notin \mathbb{N}$. Here and in the following, we put $\ell_{+}=$ $\max (\ell, 0)$ for $\ell \in \mathbb{R}$.

Theorem 1.4. Let $N \geq 1, s \in(0,1)$ and $\kappa>0$. Let $m \in \mathbb{N}$ and $\alpha \in(0,1)$, with $2 s+\alpha \notin \mathbb{N}$. Let $K \in \mathscr{K}^{s}\left(\kappa, m+\alpha+(2 s-1)_{+}, Q_{2}\right), u \in H^{s}\left(B_{2}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$ and $f \in C^{m, \alpha}\left(B_{2}\right)$ such that

$$
\mathcal{L}_{K} u=f \quad \text { in } B_{2}
$$

(i) If $2 s+\alpha<1$, then

$$
\|u\|_{C^{m, 2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right)
$$

(ii) If $1<2 s+\alpha<2$ and $2 s \neq 1$, then

$$
\|u\|_{C^{m+1,2 s+\alpha-1}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right)
$$

(iii) If $2<2 s+\alpha$, then

$$
\|u\|_{C^{m+2,2 s+\alpha-2}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right)
$$

(iv) If $2 s=1$, then for all $\beta \in(0, \alpha)$,

$$
\|u\|_{C^{m+1, \beta}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \beta}\left(B_{2}\right)}\right)
$$

Here $C=C(N, s, \kappa, \alpha, \beta, m)$.
It is clear that Theorem 1.4 includes the fractional Laplacian $\mathcal{L}_{K}=(-\Delta)^{s}$, for which it was proven in $[25,40,53,58$.

Theorem 1.3 and Theorem 1.4 provide regularity of minimizers of integral energy functional e.g. of the form

$$
\begin{equation*}
\mathcal{J}_{s}(u):=(1-s) \int_{\mathbb{R}^{2 N} \backslash\left(\mathbb{R}^{N} \backslash \Omega\right)^{2}} F\left(\frac{u(x)-u(y)}{|x-y|}\right)|x-y|^{-N-2 s+2} d x d y \tag{1.10}
\end{equation*}
$$

for some twice differentiable function $F$. The case $F(t)=t^{2}$ is the well known localized (in $\Omega$ ) Dirichlet energy for equations involving the fractional Laplacian. The minimization of this energy should be subject to exterior boundary data on $\mathbb{R}^{N} \backslash \Omega$, and posses a minimizer on $H^{s}\left(\mathbb{R}^{N}\right)$ under some quadratic and convexity assumption on $F$. To see how $\mathcal{J}_{s}$ is related with (1.3), we assume that $|F(z)| \leq|z|$. Then, for all $u \in C_{c}^{1}\left(\mathbb{R}^{N}\right)$,

$$
\lim _{s \rightarrow 1} \mathcal{J}_{s}(u) \rightarrow \int_{\Omega} G(\nabla u(x)) d x
$$

where $G(\zeta)=\frac{1}{2} \int_{S^{N-1}} F_{e}(\zeta \cdot \theta) d \theta$ and $F_{e}$ is the even part of $F$ i.e., $F_{e}(t)=\frac{F(t)+F(-t)}{2}$. The nonlocal de Giorgi-Nash-Moser provides a priori estimates for minimizers of $\mathcal{J}_{s}$. Indeed, a critical point $u \in$ $H^{s}\left(\mathbb{R}^{N}\right)$ to $\mathcal{J}_{s}$ satisfies
$\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}}\left\{F^{\prime}\left(p_{u}(x, y)\right)-F^{\prime}\left(-p_{u}(x, y)\right)\right\}(\psi(x)-\psi(y))|x-y|^{-N-2 s+1} d x d y=0 \quad$ for all $\psi \in C_{c}^{\infty}(\Omega)$,
where $p_{u}(x, y):=\frac{u(y)-u(x)}{|y-x|}$. Therefore, following the classical de Giorgi's trick and using the fundamental theorem of calculus, we find that the difference quotient $u_{h}(x)=\frac{u(x+h)-u(x)}{|h|}$ solves the equation

$$
\mathcal{L}_{K_{F, u, h}} u_{h}=0 \quad \text { in } \Omega
$$

where

$$
\begin{equation*}
K_{F, u, h}(x, y)=|x-y|^{-N-2 s} \int_{0}^{1} F_{e}^{\prime \prime}\left(\varrho p_{u(\cdot+h)}(x, y)+(1-\varrho) p_{u}(x, y)\right) d \varrho \tag{1.12}
\end{equation*}
$$

and $F_{e}$ is the even part of $F$. We then immediately see that $K_{F, u, h}$ satisfies (1.4) when $F^{\prime \prime}$ is bounded from above and below on $\mathbb{R}$. Consequently, the nonlocal de Giorgi-Nash-Moser theory implies that $u \in C^{1, \alpha_{0}}(\Omega)$, for some $\alpha_{0}>0$. Now an iterative application of Theorem 1.3 and Theorem 1.4 shows, as for the solution to the Hilbert's problem, that if $F \in C^{\infty}(\mathbb{R})$ then $u \in C^{\infty}(\Omega)$. It is worth to mention that in the nonlocal mean curvature problem, nonlocal minimal graphs satisfy an equation as in (1.11), with $F(t)=\int_{t}^{\infty}\left(1+\tau^{2}\right)^{-\frac{N+2 s}{2 s}} d \tau$ (see Section 1.1 below for a more precise statement). Here, $F^{\prime \prime}$ is not uniformly bounded from below and thus $K_{F, u, h}$ does not satisfy (1.4)-(ii $)$. However, as in the classical case, this lack of ellipticity, is recovered once we know that $u$ is Lipschitz.

Beyond their appearances in the mathematical modeling of real-world phenomenon, $C^{m, \alpha}$-nonlocal operators appear naturally in geometric problems. Indeed, we are naturally confronted with nonlocal equation resulting from an initial one after a change coordinates. For instance, consider $K(x, y)=$ $|x-y|^{-N-2 s}$ (the kernel of the fractional Laplacian) and $K_{\Phi}(x, y)=|\Phi(x)-\Phi(y)|^{-N-2 s}$, for some diffeomorphism $\Phi \in C^{m+1, \alpha}\left(\mathbb{R}^{N} ; \mathbb{R}^{N}\right)$ with $D \Phi$ close to the identity matrix, so that (1.4) holds. In this case, apart in dimension $N=1$, we may not have any regularity of $z \mapsto|z|^{N+2 s} K_{\Phi}(x, x+z)$ at $z=0$. However, using polar coordinates, we easily see that the map

$$
(x, r, \theta) \mapsto r^{N+2 s} K_{\Phi}(x, x+r \theta)=\left|\int_{0}^{1} D \Phi(x+t r \theta) \theta d t\right|^{-N-2 s}
$$

extends to a $C^{m, \alpha}$ map on $\mathbb{R}^{N} \times[0, \infty) \times S^{N-1}$ satisfying (1.6)-(ii), so that $K_{\Phi}$ defines a $C^{m, \alpha}$-nonlocal operator. This is also the case for the kernel in (1.12) with $F \in C^{\infty}(\mathbb{R})$ and $u \in C^{m+1, \alpha}(\Omega)$. These facts, among others, motivate the splitting in polar coordinates in our definition of $C^{m, \alpha}$-nonlocal operators. Moreover, it turns out to be useful in the study of prescribed nonlocal mean curvature problems and nonlocal equations on hypersurfaces, see Section 1.1 and Section 1.2 respectively. On the other hand, we remark that in some interesting non-translation invariant cases, the map $z \mapsto$ $|z|^{N+2 s} K(x, x+z)$ can be smooth at $z=0$, and a first nontrivial example is given by the censored fractional Laplacian or the $\Omega$-regional fractional Laplacian, where the kernel is given by $K(x, y)=$ $1_{\Omega}(x) 1_{\Omega}(y)|x-y|^{-N-2 s}$, see e.g. Mou and Yi [52]. An other example arises in problems from image processing, see e.g. Gilboa Osher [39] and Caffarelli, Chan and Vasseur [13], where the kernel depends on the solution $u \in C^{1, \alpha_{0}}$ and, for simplicity, reads as $K(x, y)=1_{\Omega}(x) 1_{\Omega}(y) \phi^{\prime \prime}(u(x)-u(y)) \mid x-$ $\left.y\right|^{-N-2 s}$, for some even and convex function $\phi$. Here one looks at minimizer $u \in H^{s}(\Omega)$ of the energy functional

$$
\mathcal{J}_{s, \Omega}(u):=(1-s) \int_{\Omega \times \Omega} \phi(u(x)-u(y))|x-y|^{-N-2 s} d x d y .
$$

This is also the case for (possibly) sign-changing kernels e.g. $K(x, y)=|x-y|^{-N-2 s_{1}} \pm|x-y|^{-N-2 s_{2}}$, with $s_{1} \in(0,1)$ and $s_{2}<s_{1}$. However the conditions (1.4) and (1.6) are flexible enough to include such cases.

The following two paragraphs are devoted to the application of the above regularity estimates in some nonlocal geometric problems.
1.1. Application I: Graphs with prescribed nonlocal mean curvature. In this section, we assume that $s \in(1 / 2,1)$. Recall that for a set $E \subset \mathbb{R}^{N+1}$ of class $C^{1,2 s-1+\alpha}$, with $\alpha>0$, near a point $X \in \partial E$, the nonlocal (or fractional) mean curvature of the set $E$ (or the hypersurface $\partial E$ ) at the point $X \in \partial E$ is defined as

$$
\begin{equation*}
H_{s}(\partial E ; X):=P V \int_{\mathbb{R}^{N+1}} \frac{1_{E^{c}}(Y)-1_{E}(Y)}{|Y-X|^{N+2 s}} d Y, \tag{1.13}
\end{equation*}
$$

where $E^{c}:=\mathbb{R}^{N+1} \backslash E$ and $1_{D}$ denotes the characteristic function of a set $D \subset \mathbb{R}^{N+1}$. Recall that the notion of nonlocal mean curvature appeared first in the work of Caffarelli and Souganidis in 19 and first studied by Caffarelli, Roquejoffre, and Savin in [14]. As first discovered in [14] (see also [22,36]), the nonlocal mean curvature arises as the first variation of the fractional perimeter. For the convergence of fractional curvature to the classical one as $s \rightarrow 1$, see [2, 22].

Suppose that $\partial E$ is the graph of a function $u \in C^{1,2 s-1+\alpha}(\Omega) \cap C_{l o c}^{0,1}\left(\mathbb{R}^{N}\right)$, then see e.g. [28], by a change of variable, for all $x \in \Omega$, we have

$$
\begin{equation*}
H_{s}(\partial E ;(x, u(x)))=P V \int_{\mathbb{R}^{N}} \frac{\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(p_{u}(y, x)\right)}{|x-y|^{N+2 s-1}} d y \tag{1.14}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{F}_{s}(p):=\int_{p}^{+\infty}\left(1+\tau^{2}\right)^{\frac{-(N+2 s)}{2}} d \tau \tag{1.15}
\end{equation*}
$$

and for a measurable function $w: \mathbb{R}^{N} \rightarrow \mathbb{R}$, we put

$$
\begin{equation*}
p_{w}(x, y)=\frac{w(y)-w(x)}{|x-y|} \tag{1.16}
\end{equation*}
$$

By the fundamental theorem of calculus and (1.14) and noting that $\mathcal{F}_{s}\left(p_{u}(y, x)\right)=\mathcal{F}_{s}\left(-p_{u}(x, y)\right)$, we have

$$
\begin{equation*}
H_{s}(\partial E ;(x, u(x)))=P V \int_{\mathbb{R}^{N}} \frac{u(x)-u(y)}{|x-y|^{N+2 s}} q_{u}(x, y) d y \tag{1.17}
\end{equation*}
$$

where for a measurable function $w: \mathbb{R}^{N} \rightarrow \mathbb{R}$,

$$
q_{w}(x, y):=-\int_{-1}^{1} \mathcal{F}_{s}^{\prime}\left(t p_{w}(x, y)\right) d t=\int_{-1}^{1}\left(1+t^{2} p_{w}(x, y)^{2}\right)^{\frac{-(N+2 s)}{2}} d t
$$

For the following, we define the the nonlocal mean curvature kernel by

$$
\mathcal{K}_{w}(x, y):=\frac{1}{|x-y|^{N+2 s}} q_{w}(x, y) \quad \text { for all } x \neq y \in \mathbb{R}^{N}
$$

Letting $\Omega$ be an open set of $\mathbb{R}^{N}$ and $f \in L_{l o c}^{1}(\Omega)$, we are interested in the regularity of measurable functions $u: \mathbb{R}^{N} \rightarrow \mathbb{R}$ satisfying

$$
\begin{equation*}
\mathcal{L}_{\mathcal{K}_{u}} u=f \quad \text { in } \Omega \tag{1.18}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\frac{1}{2} \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(p_{u}(y, x)\right)}{|x-y|^{N+2 s-1}}(\psi(x)-\psi(y)) d x d y=\int_{\mathbb{R}^{N}} f(x) \psi(x) d x \quad \text { for all } \psi \in C_{c}^{\infty}(\Omega) \tag{1.19}
\end{equation*}
$$

Note that, since $\mathcal{F}_{s} \in L^{\infty}(\mathbb{R})$ and $2 s>1$, the right hand side in (1.19) is well defined. We observe that if $u \in C^{1,2 s-1+\alpha}(\Omega) \cap L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ solves (1.18), then the set $E_{u}:=\left\{(x, t) \in \mathbb{R}^{N} \times \mathbb{R}: u(x)<t\right\}$, satisfies $H_{s}\left(\partial E_{u} ;(x, u(x))\right)=f(x)$, for all $x \in \Omega$, provided the $(N+1)$-dimension Lebesgue measure of $\partial E_{u}$ is equal to zero. This follows by approximating $u$ by a sequence of smooth functions.
We consider next locally Lipschitz graphs with prescribed nonlocal mean curvature in the weak sense of (1.19), and we prove that they are of class $C^{\infty}$ in $\Omega$ as long as $f$ is $C^{\infty}$ in $\Omega$, with quantitative estimates. In the classical case, this is a consequence of the de Giorgi-Nash theorem and the Schauder theory for uniformly elliptic equations in divergence form with $C^{m, \alpha}$-coefficients. See e.g. Figalli and Valdinoci 35, it was hardly believed that the same strategy could be carried out in the nonlocal setting. In 35, the authors used geometric arguments to prove that Lipschitz sets, locally minimizing fractional perimeter are of class $C^{\infty}$. However their argument does not provide quantitative estimates. Here, we shall show that it is indeed possible to proceed as in the prescribed mean curvature problem, thanks to our regularity estimates for $C^{m, \alpha}$-nonlocal operators. It is important to note, in the theorem below, that we do not require any integrability of $u$ in $\mathbb{R}^{N} \backslash B_{2}$. We have the following result.

Theorem 1.5. Let $f \in L_{l o c}^{1}\left(B_{2}\right)$ and $u: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be a measurable function, with $\|u\|_{C^{0,1}\left(B_{2}\right)} \leq c_{0}$, such that

$$
\mathcal{L}_{\mathcal{K}_{u}} u=f \quad \text { in } B_{2}
$$

in the sense of (1.19). Then the following statements hold.
(i) If $f \in C^{0,1}\left(B_{2}\right)$, then

$$
\begin{equation*}
\|u\|_{C^{1, \alpha_{0}}\left(B_{1}\right)} \leq C\left(1+\|f\|_{C^{0,1}\left(B_{2}\right)}\right), \tag{1.20}
\end{equation*}
$$

for some constants $\alpha_{0}, C>0$, only depending on $N, s$ and $c_{0}$. Moreover, for all $\beta \in(0,2 s-1)$,

$$
\|u\|_{C^{2, \beta}\left(B_{1}\right)} \leq C
$$

for some constant $C$, only depending on $N, s, \beta, c_{0}$ and $\|f\|_{C^{0,1}\left(B_{2}\right)}$.
(ii) If $f \in C^{m, \alpha}\left(B_{2}\right)$, for some $\alpha \in(0,1)$ and $m \geq 1$, then

$$
\begin{aligned}
& \|u\|_{C^{m+1,2 s+\alpha-1}\left(B_{1}\right)} \leq C \quad \text { if } 2 s+\alpha<2, \\
& \|u\|_{C^{m+2,2 s+\alpha-2}\left(B_{1}\right)} \leq C \quad \text { if } 2 s+\alpha>2,
\end{aligned}
$$

for some constant $C$, only depending on $N, s, \alpha, m, c_{0}$ and $\|f\|_{C^{m, \alpha}\left(B_{2}\right)}$.
The first quantitative estimates for nonlocal minimal graphs was found recently by Cabré and Cozzi in [10. Indeed, they provide, in [10, quantitative gradient estimates for global graphs that locally minimize the fractional area functional in a cylinder $B_{R} \times \mathbb{R}$, in the spirit of Finn [37] and Bombieri, de Giorgi and Miranda [7]. In this case $f \equiv 0$. Therefore combining their result and Theorem [1.5] we get quantitative estimates of all partial derivatives of such graphs in terms of the oscillation of $u$ in $B_{R}$.
Recall that the smoothness character for fractional perimeter minimizing sets was known, but without quantitative bounds. Indeed, the seminal paper [14 established the first existence and $C^{1, \gamma}$ (except a closed set of zero ( $N-3$ )-Hausdorff measure) regularity for fractional perimeter minimizing sets. In [5, Barrios, Figalli and Valdinoci, proved that fractional perimeter minimizing sets which are of class $C^{1,(2 s-1) / 2-\varepsilon}$ are of class $C^{\infty}$. On the other hand Caffarelli and Valdinoci showed, in [20, that, for $s$ close to 1 , these sets possess the smoothness property of the classical perimeter minimizing regions. It is proven in [26, by Dipierro, Savin and Valdinoci, that the boundary of a fractional perimeter minimizing set, in a reference smooth set $\Omega$, which coincides with a continuous graph $\mathbb{R}^{N} \backslash \bar{\Omega}$ is in fact a global graphs that is continuous in $\Omega$.

The fact that we do not require any integrability of $u$ in $\mathbb{R}^{N} \backslash B_{2}$ makes the proof of Theorem 1.5 particularly nontrivial. In view of the decomposition in (1.19), we split further the double integral in the left hand side to get

$$
\begin{align*}
\left\langle\mathcal{L}_{\mathcal{K}_{u}} u, \psi\right\rangle= & \frac{1}{2} \int_{\Omega \times \Omega}(u(x)-u(y))(\psi(x)-\psi(y)) \mathcal{K}_{u}(x, y) d x d y \\
& +\int_{\Omega} \psi(x) \int_{\mathbb{R}^{N} \backslash \Omega} \frac{\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(p_{u}(y, x)\right)}{|x-y|^{N+2 s-1}} d y d x . \tag{1.21}
\end{align*}
$$

Now the proof of Theorem 1.5 resides on the regularity of the map

$$
\Omega^{\prime} \rightarrow \mathbb{R}, \quad x \mapsto \int_{\mathbb{R}^{N} \backslash \Omega} \frac{\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(p_{u}(y, x)\right)}{|x-y|^{N+2 s-1}} d y,
$$

for $\Omega^{\prime} \subset \subset \Omega$. Surprisingly, the local behavior of this map is completely determined by the one of $u$ only in $\Omega^{\prime}$. In fact we will show, in Lemma 6.2 below, that this function is indeed as smooth as $u$ in $\Omega^{\prime}$. Once this is proved, the above function is sent in the right hand side, so that we can use the argument as in the classical case. Indeed, we apply first the nonlocal de Giorgi-Nash a priori Hölder estimate to the function $\frac{u(x+h)-u(x)}{|h|}$ which satisfies a nonlocal equation of the form (1.5), driven by a kernel $K_{h}^{u}$ satisfying (1.4), to deduce that $\nabla u \in C^{0, \alpha_{0}}$. This will imply that $K_{h}^{u} \in \mathscr{K}^{s}\left(\kappa, \alpha_{0}, Q_{\delta}\right)$, for some $\kappa, \delta>0$. Now Theorem [1.3( $i$ ii) and Theorem [1.4 (ii) kick in and yield the result, since $\mathcal{A}_{K_{h}^{u}}$ will be, locally, as regular as $\nabla u$.
1.2. Application II: Nonlocal equations on manifolds. Let $\Sigma$ be a Lipschitz hypersurface of $\mathbb{R}^{N+1}$, with $0 \in \Sigma$. We define the space $L_{s}(\Sigma)$ given by the set of functions $u \in L_{l o c}^{1}(\Sigma)$ such that

$$
\|u\|_{L_{s}(\Sigma)}:=\int_{\Sigma}|u(\bar{y})|(1+|\bar{y}|)^{-N-2 s} d \sigma(\bar{y})<\infty
$$

where $d \sigma$ denote the volume element on $\Sigma$. We assume that

$$
\begin{equation*}
\|1\|_{L_{s}(\Sigma)}=\int_{\Sigma}(1+|\bar{y}|)^{-N-2 s} d \sigma(\bar{y})<\infty \tag{1.22}
\end{equation*}
$$

We note that this condition always holds when $\Sigma$ has finite diameter. In this section we are interested in the regularity estimates of functions $u \in H_{l o c}^{s}(\Sigma) \cap L_{s}(\Sigma)$ satisfying, for all $\Psi \in C_{c}^{\infty}(\Sigma)$,

$$
\begin{equation*}
\frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(u(\bar{x})-u(\bar{y}))(\Psi(\bar{x})-\Psi(\bar{y}))}{|\bar{x}-\bar{y}|^{N+2 s}} d \sigma(\bar{x}) d \sigma(\bar{y})+\int_{\Sigma} V(\bar{x}) u(\bar{x}) \Psi(\bar{x}) d \sigma(\bar{x})=\int_{\Sigma} f(\bar{x}) \Psi(\bar{x}) d \sigma(\bar{x}) \tag{1.23}
\end{equation*}
$$

where $f, V \in L_{l o c}^{1}(\Sigma)$ and $u V \in L_{l o c}^{1}(\Sigma)$.

Theorem 1.6. Let $s, \gamma \in(0,1), N \geq 1$ and $\Sigma$ be a $C^{1, \gamma}$-hypersurface of $\mathbb{R}^{N+1}$ as above satisfying (1.22). Let $f, V \in L^{p}(\Sigma)$, for some $p>\frac{N}{2 s}$ and $u \in H_{l o c}^{s}(\Sigma) \cap L_{s}(\Sigma)$ satisfy (1.23). Then the following estimates hold.
(i) If $2 s \leq 1$, then

$$
\|u\|_{C^{2 s-N / p}\left(B_{e} \cap \Sigma\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2_{e}} \cap \Sigma\right)}+\|u\|_{L_{s}(\Sigma)}+\|f\|_{L^{p}(\Sigma)}\right)
$$

(ii) If $2 s-1>\max \left(\frac{N}{p}, \gamma\right)$, then

$$
\|u\|_{C^{1, \min \left(2 s-\frac{N}{p}-1, \gamma\right)}\left(B_{\varrho} \cap \Sigma\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2} \varrho \cap\right)}+\|u\|_{L_{s}(\Sigma)}+\|f\|_{L^{p}(\Sigma)}\right)
$$

Here $C, \varrho>0$ are constants only depending on $N, s, \gamma, p,\|V\|_{L^{p}(\Sigma)},\|1\|_{L_{s}(\Sigma)}$ and the bound of the local geometry of $\Sigma$ near 0 .
In the case of higher order regularity, we obtain the
Theorem 1.7. Let $s, \alpha, \gamma \in(0,1), N \geq 1$ and $\Sigma$ be a $C^{1, \gamma}$-hypersurface of $\mathbb{R}^{N+1}$ as above satisfying (1.22). Let $f, V \in C^{0, \alpha}(\Sigma)$ and $u \in H_{l o c}^{s}(\Sigma) \cap L_{s}(\Sigma)$ satisfy (1.23).
(i) If $2 s>1$ and $\gamma \geq \alpha+2 s-1$, then

$$
\|u\|_{C^{1,2 s-1+\alpha}\left(B_{\varrho} \cap \Sigma\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2 \varrho} \cap \Sigma\right)}+\|u\|_{L_{s}(\Sigma)}+\|f\|_{C^{0, \alpha}(\Sigma)}\right)
$$

(ii) If $2 s+\alpha<1$ and $\gamma \geq \alpha$, then

$$
\|u\|_{C^{0,2 s+\alpha}\left(B_{\varrho} \cap \Sigma\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2 \varrho} \cap \Sigma\right)}+\|u\|_{L_{s}(\Sigma)}+\|f\|_{C^{0, \alpha}(\Sigma)}\right)
$$

(iii) If $2 s=1$ and $\gamma>\alpha$, then

$$
\|u\|_{C^{1, \alpha}\left(B_{\varrho} \cap \Sigma\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2_{\varrho}} \cap \Sigma\right)}+\|u\|_{L_{s}(\Sigma)}+\|f\|_{C^{0, \alpha}(\Sigma)}\right)
$$

Here $C, \varrho>0$ are constants only depending on $N, s, \gamma, \alpha,\|V\|_{C^{0, \alpha}(\Sigma)},\|1\|_{L_{s}(\Sigma)}$ and the bound of the local geometry of $\Sigma$ near 0 .
Here, by the bound of the local geometry of $\Sigma$ near 0 , we mean the $C^{1, \gamma}$ norm of a local parameterization of $\Sigma$ flattening $B_{\varrho_{0}} \cap \Sigma$, for some $\varrho_{0}>0$. If $\Sigma$ is of class $C^{m+1, \gamma}$ and $f, V \in C_{l o c}^{m, \alpha}(\Sigma)$, then under the same assumptions on $\gamma$ in Theorem 1.7, we have the estimates of $C^{m+2 s+\alpha}$-norm of $u$ as long as $2 s+\alpha \notin \mathbb{N}$, thanks to Theorem 1.4 .
Theorem 1.6 and Theorem 1.7 are consequences of Theorem 1.3 and Theorem 1.4 respectively, after using a coordinate system that locally flattens $\Sigma$.

For $2 s>1$, Theorem 1.6 and Theorem 1.7 provide regularity estimates for solutions to some nonlocal equation driven by the linearized nonlocal mean curvature operator (i.e. the nonlocal or fractional Jacobi operator) of a set $E$ with constant nonlocal mean curvature (not necessarily bounded). Indeed,
consider $\Sigma:=\partial E$ a $C^{2}$-hypersurface of $\mathbb{R}^{N+1}$ with constant nonlocal mean curvature such that $0 \in \partial E$ and $\|1\|_{L_{s}(\Sigma)}<\infty$. See e.g. 22, 36], the second variation of the fractional perimeter yields the bilinear form $\mathcal{D}_{\Sigma}: H^{s}(\Sigma) \times H^{s}(\Sigma) \rightarrow \mathbb{R}$, given by

$$
\mathcal{D}_{\Sigma}(u, v):=\frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(u(\bar{x})-u(\bar{y}))(v(\bar{x})-v(\bar{y}))}{|\bar{x}-\bar{y}|^{N+2 s}} d \sigma(\bar{y}) d \sigma(\bar{x})-\frac{1}{2} \int_{\Sigma} V_{\Sigma}(\bar{x}) u(\bar{x}) v(\bar{x}) d \sigma(\bar{x}),
$$

where, letting $\nu_{\Sigma}$ be the unit exterior normal vector field of $\Sigma:=\partial E$,

$$
V_{\Sigma}(\bar{x}):=\frac{1}{2} \int_{\Sigma} \frac{\left|\nu_{\Sigma}(\bar{x})-\nu_{\Sigma}(\bar{y})\right|^{2}}{|\bar{x}-\bar{y}|^{N+2 s}} d \sigma(\bar{y}) .
$$

One then defines the fractional Jacobi operator as

$$
\mathcal{J}_{\Sigma}:=\mathbb{L}_{\Sigma}-V_{\Sigma}
$$

where, for $u \in C_{l o c}^{1,2 s-1+\alpha}(\Sigma) \cap L_{s}(\Sigma)$,

$$
\mathbb{L}_{\Sigma} u(\bar{x}):=P V \int_{\Sigma} \frac{u(\bar{x})-u(\bar{y})}{|\bar{x}-\bar{y}|^{N+2 s}} d \sigma(\bar{y})
$$

The fractional Jacobi fields are solutions to $\mathcal{J}_{\Sigma} u=0$, and they play an important role in the study of stability of constant nonlocal mean curvature surfaces or fractional area estimates of such surfaces. We observe that if $\Sigma$ is a $C^{1, \gamma}$-hypersurface for some $\gamma>s$, then $V_{\Sigma} \in C_{l o c}^{\gamma}(\Sigma)$. Moreover we may consider a weak solutions $u \in H_{l o c}^{s}(\Omega) \cap L_{s}(\Sigma)$ to the equation $\mathcal{J}_{\Sigma} u=f$ on open subsets $\Omega$ of $\Sigma$, in the sense of (1.23). Hence Theorem 1.6 and Theorem 1.7 can be used to obtain regularity estimates of $u$. When $\Sigma=S^{N-1}$, then Theorem [1.7( $i$ ) was proved in [11, using the regularity theory of the fractional Laplacian and the Fredholm theory. Recall that besides the nonlocal minimal surfaces, there exist several nontrivial hypersurfaces with nonzero constant nonlocal mean curvature, see e.g. the survey paper 28.
1.3. Anisotropic $C^{m, \alpha}$-nonlocal operators. As mentioned earlier, in many situations, nonlocal equations provide a wider framework than their local counterpart, since $\mathcal{A}_{K}$ may have anisotropic regularity in its variables. Namely, the spatial variable $x$, the singular variable $r$ and the angular variable might have different qualitative properties. This affects the local behavior of the solutions. First note that the class of operators $\mathcal{L}_{K}$ falls in the class of nonlocal operators generated by a Lévy measure $\nu_{x}$. In particular, the map $z \mapsto K(x, x+z)$ is the density of a Lévy measure $\nu_{x}$ and thus does not necessarily posses any regularity. If the Lévy measure is symmetric and stable, then see [53], $\nu_{x}(r E)=r^{N-1} d r a(E)$ for $E \subset S^{N-1}$. Under fairly general assumptions on the spectral measure $a$ on $S^{N-1}$ (not depending on $x$ ), optimal interior and boundary regularity were proved by Ros-Oton and Serra in 53]. The papers [27, 46, 47] obtained also regularity estimates provided $a$ is absolutely continuous with respect to the Lebesgue measure on $S^{N-1}$ only on an open set of positive measure. To capture this possible anisotropic regularity of $\mathcal{A}_{K}$ in its variables, we introduce a new class of fractional order nonlocal operators which are much larger than the class of $C^{m, \alpha}$-nonlocal operators introduced above.
In the following, for $\delta>0$, we define

$$
\begin{equation*}
Q_{\delta}:=B_{\delta} \times[0, \delta) \quad \text { and } \quad Q_{\infty}:=\mathbb{R}^{N} \times[0, \infty) \tag{1.24}
\end{equation*}
$$

We define the space $C^{m, \alpha}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)$ by the set of functions $A \in L^{\infty}\left(Q_{\delta} \times S^{N-1}\right)$ such that, for every $\theta \in S^{N-1}$, the map $(x, r) \mapsto A(x, r, \theta)$ belongs to $C^{m, \alpha}\left(Q_{\delta}\right)$ and

$$
\begin{equation*}
\|A\|_{C^{m, \alpha}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}:=\sup _{\theta \in S^{N-1}}\|A(\cdot, \cdot, \theta)\|_{C^{m, \alpha}\left(Q_{\delta}\right)}<\infty \tag{1.25}
\end{equation*}
$$

For $\tau \geq 0$, the space $\mathcal{C}_{\tau}^{0}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)$ is given by the the set of function $A \in L^{\infty}\left(Q_{\delta} \times S^{N-1}\right)$ such that

$$
\|A\|_{L_{\tau}^{\infty}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}:=\sup _{\theta \in S^{N-1}} \sup _{x \in B_{\delta}, r \in(0, \delta)} \frac{|A(x, r, \theta)|}{r^{\tau}}<\infty
$$

and

$$
[A]_{\mathcal{C}_{\mathcal{T}}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}:=\sup _{\theta \in S^{N-1}} \sup _{x \neq y \in B_{\delta}, r \in(0, \delta)} \frac{|A(x, r, \theta)-A(y, r, \theta)|}{\min (r,|x-y|)^{\tau}}<\infty .
$$

The space $\mathcal{C}_{\tau}^{m}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)$ is defined as the set of functions $A \in C^{m, 0}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)$ such that

$$
\begin{equation*}
\|A\|_{\mathcal{C}_{\tau}^{m}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}:=\sup _{\gamma \in \mathbb{N}^{N},|\gamma| \leq m}\left\|\partial_{x}^{\gamma} A\right\|_{L_{\tau}^{\infty}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}+\sup _{\gamma \in \mathbb{N}^{N},|\gamma| \leq m}\left[\partial_{x}^{\gamma} A\right]_{\mathcal{C}_{\tau}^{0}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}<\infty . \tag{1.26}
\end{equation*}
$$

This section is concerned with optimal Hölder estimates for nonlocal equation driven by the operator $\mathcal{L}_{K}$ with coefficient $\mathcal{A}_{K}$ in the spaces defined above.
Definition 1.8. Let $\alpha \in[0,1), \tau \in[0,1], m \in \mathbb{N}$ and $\kappa>0$. For $\delta \in(0, \infty]$, we define $\widetilde{K}_{\tau}^{s}(\kappa, m+$ $\alpha, Q_{\delta}$ ) by the set of kernels $K: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow[-\infty,+\infty]$ satisfying (1.4) and

$$
\begin{aligned}
& \text { (iii) }\left\|\mathcal{A}_{K}\right\|_{C^{m, \alpha}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)}+\left\|\mathcal{A}_{o, K}\right\|_{C_{\tau}^{m}\left(Q_{\delta}\right) \times L^{\infty}\left(S^{N-1}\right)} \leq \frac{1}{\kappa}, \\
& \text { (iv) } \mathcal{A}_{o, K}(x, 0, \theta)=0 \quad \text { for all }(x, \theta) \in B_{\delta} \times S^{N-1},
\end{aligned}
$$

where

$$
\begin{equation*}
\mathcal{A}_{o, K}(x, r, \theta):=\frac{1}{2}\left\{\mathcal{A}_{K}(x, r, \theta)-\mathcal{A}_{K}(x, r,-\theta)\right\} \tag{1.27}
\end{equation*}
$$

and $\mathcal{A}_{K}(\cdot, \cdot, \theta)$ is a continuous extension of $(x, r) \mapsto r^{N+2 s} K(x, x+r \theta)$ on $Q_{\delta}$ for all $\theta \in S^{N-1}$.
The simple model case for the class of operators in Definition 1.8 is the anisotropic fractional Laplace operator, with kernel $K(x, y)=a((x-y) /|x-y|)|x-y|^{-N-2 s}$ and $a \in L^{\infty}\left(S^{N-1}\right)$ is even but not necessarily continuous. In this case, $\mathcal{A}_{K}(x, r, \theta)=a(\theta)$, so that $K \in \widetilde{\mathcal{K}_{\tau}^{s}}\left(\kappa, m, Q_{\delta}\right)$ for all $m \in \mathbb{N}$. As an example, a prototype energy functional can be an anisotropic integral energy functional, generalizing (1.10), given by

$$
\begin{equation*}
\mathcal{J}_{s}(u):=(1-s) \int_{\mathbb{R}^{2 N} \backslash\left(\mathbb{R}^{N} \backslash \Omega\right)^{2}} F\left(\frac{x-y}{|x-y|}, \frac{u(x)-u(y)}{|x-y|}\right)|x-y|^{-N-2 s+2} d x d y, \tag{1.28}
\end{equation*}
$$

where $F: S^{N-1} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\kappa \leq \partial_{z}^{2} F(\theta, z) \leq \frac{1}{\kappa}$. The results in the present section provide smoothness of a critical point $u \in H^{s}\left(\mathbb{R}^{N}\right)$ to $\mathcal{J}_{s}$ defined in (1.28), provided $z \mapsto F(\cdot, z)$ is smooth. Indeed, as above, the difference quotient $\frac{u(\cdot+h)-u(\cdot)}{|h|}$ solves an equation like (1.5), with $K \in \widetilde{K_{\alpha}^{s}}(\kappa, m+$ $\left.\alpha, Q_{\delta}\right)$, provided $u \in C^{m+1, \alpha}(\Omega)$.
We observe that $\mathscr{K}^{s}\left(\kappa, m+\alpha, Q_{\delta}\right) \subset \widetilde{\mathscr{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{\delta}\right)$ for all $\tau \leq \alpha$ and since $\mathcal{A}_{o, K}(x, 0, \theta)=0$, we have that $\widetilde{\mathcal{K}_{0}^{s}}\left(\kappa, m+\alpha, Q_{\delta}\right)=\widetilde{\mathscr{K}_{\alpha}^{s}}\left(\kappa, m+\alpha, Q_{\delta}\right)$. Moreover, we have the following interesting property on the set $\widetilde{\widetilde{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{\infty}\right)$ concerning scaling and translations. Indeed, for $K \in \widetilde{\widetilde{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{\infty}\right)$, $\rho \in(0,1)$ and $z \in \mathbb{R}^{N}$, letting $K_{z, \rho}(x, y):=\rho^{N+2 s} K(\rho x+z, \rho y+z)$, we then have that $\mathcal{A}_{K_{z, \rho}}(x, r, \theta)=$ $\mathcal{A}_{K}(\rho x+z, \rho r, \theta)$ and thus $K_{z, \rho} \in \widetilde{K}_{\tau}^{s}\left(\kappa, m+\alpha, Q_{\infty}\right)$.

The kernels in $\widetilde{\mathcal{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{\delta}\right)$ yield, in many cases, similar regularity estimates as those in $\mathscr{K}^{s}\left(\kappa, m+\alpha, Q_{\delta}\right)$, stated above, provided some global regularity/behavior of $u$ is a priori known. Our first main result in this section is the following.
Theorem 1.9. Let $s \in(0,1), N \geq 1, \kappa>0$ and $\alpha \in(0,1)$. Let $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, \alpha, Q_{2}\right), u \in H^{s}\left(B_{2}\right) \cap$ $L_{s}\left(\mathbb{R}^{N}\right)$ and $V, f \in L^{p}\left(B_{2}\right)$, for some $p>N /(2 s)$, satisfy

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} .
$$

(i) If $2 s \leq 1$, then there exists $C=C\left(s, N, \kappa, \alpha, p,\|V\|_{L^{p}\left(B_{2}\right)}\right)>0$ such that

$$
\|u\|_{C^{0,2 s-\frac{N}{p}}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{L^{p}\left(B_{2}\right)}\right)
$$

(ii) If $2 s-1>\max \left(\frac{N}{p}, \alpha\right)$, then there exists $C=C\left(s, N, \kappa, \alpha, p,\|V\|_{L^{p}\left(B_{2}\right)}\right)>0$ such that

$$
\|u\|_{C^{1, \min \left(2 s-\frac{N}{p}-1, \alpha\right)}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{L^{p}\left(B_{2}\right)}\right) .
$$

Our next result is concerned with $C^{m+2 s+\alpha}$ Schauder estimates.
Theorem 1.10. Let $N \geq 1$ and $s \in(0,1)$. Let $\kappa>0, \alpha \in(0,1)$ and $m \in \mathbb{N}$. Let $K \in \widetilde{\mathscr{K}_{\beta}^{s}}(\kappa, m+$ $\left.\alpha, Q_{2}\right)$, with $\beta=\min \left(\alpha+(2 s-1)_{+}, 1\right)$. Let $u \in H^{s}\left(B_{2}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$ and $f \in C^{m, \alpha}\left(B_{2}\right)$ such that

$$
\mathcal{L}_{K} u=f \quad \text { in } B_{2}
$$

(i) If $u \in C^{m, \alpha}\left(\mathbb{R}^{N}\right)$ and $2 s+\alpha<1$, then

$$
\|u\|_{C^{m, 2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{m, \alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right)
$$

(ii) If $u \in C^{m, \alpha}\left(\mathbb{R}^{N}\right), 2 s \neq 1$ and $1<2 s+\alpha<2$, then

$$
\|u\|_{C^{m+1,2 s+\alpha-1}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{m, \alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right) .
$$

(iii) If $u \in C^{m, \alpha}\left(\mathbb{R}^{N}\right), 2<2 s+\alpha$ and $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, m+2 s-1+\alpha, Q_{2}\right)$, then

$$
\|u\|_{C^{m+2,2 s+\alpha-2}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{m, \alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right)
$$

(iv) If $u \in C^{m, \alpha}\left(\mathbb{R}^{N}\right), 2 s=1$ and $K \in \widetilde{\mathscr{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{2}\right)$, for some $\tau>\alpha$, then

$$
\|u\|_{C^{m+1, \alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{m, \alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m, \alpha}\left(B_{2}\right)}\right)
$$

If moreover $\left\|\mathcal{A}_{K}\right\|_{C^{m, \alpha}\left(Q_{2} \times S^{N-1}\right)} \leq \frac{1}{\kappa}$, then we can replace $\|u\|_{C^{m, \alpha}\left(\mathbb{R}^{N}\right)}$ with $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}$. Here $C=$ $C(N, s, \kappa, \alpha, m, \tau)$.

We point out the remarkable differences between the last assertion in Theorem 1.10 and the results in Theorem 1.4 Indeed, in the former, $\mathcal{A}_{K}$ is only required to be in $C^{m, \alpha}\left(Q_{2} \times S^{N-1}\right)$, when $2 s+\alpha<2$, instead of $C^{m, \alpha+(2 s-1)+}\left(Q_{2} \times S^{N-1}\right)$ which was assumed in the latter. Moreover, Theorem 1.10 ( $i v$ ), for $s=1 / 2$, provides the optimal estimate which covers the case $\mathcal{L}_{K}=(-\Delta)_{a}^{s}$, the anisotropic fractional Laplacian i.e. when $K(x, y)=a((x-y) /|x-y|)|x-y|^{-N-2 s}$, while Theorem 1.4 does not if $a$ is not smooth enough. In fact the results in Theorem 1.10 were known for the anisotropic fractional Laplacian when $a$ is a measure on the unit sphere $S^{N-1}$, see Ros-Oton and Serra [53] and when $a \in C^{\infty}\left(S^{N-1}\right)$, see Grubb 41].
Interior regularity and Harnack inequality for linear and fully nonlinear nonlocal equations have been intensively investigated in last decades by many authors, see e.g. [1, 3, 5, 6, 15, 17, 27, 32, 42, 44, 45, 48, [55, 57, 59 and the references therein.

Next, we observe that Theorem 1.3 and 1.4 are immediate consequences of Theorem 1.9 and 1.10 respectively. The proof of Theorem 1.9 and 1.10 uses a blow up analysis and compactness method for weak and classical solutions, partly inspired by [57] and [30, see also [33, 34, 53, 54, 56 for translation invariant problems. Indeed, we use a fine scaling argument to balance, in an optimal manner, the norm of the right hand side and the homogeneity of the equation. The scaling parameter is chosen so that the limit of the rescaled solution, after subtracting a polynomial, satisfies an equation for which all solutions with such growth are explicitly known, thanks to a Liouville type theorem. To obtain Hölder, gradient and second order derivative estimates, the subtracted polynomial are, respectively given by the projection, with respect to the $L^{2}\left(B_{r}\right)$ scalar product, of the weak solution $u$ on constant functions, affine functions and second order polynomials. More precisely, our primary goal is to show the growth estimates (or Taylor expansion in $L^{2}$-sense)

$$
\left\|u-P_{r}\right\|_{L^{2}\left(B_{r}\right)} \leq C r^{\frac{N}{2}+\operatorname{deg}\left(P_{r}\right)+\gamma}
$$

where $P_{r}$ is a suitable polynomial and the parameter $\gamma \in(0,1)$ is determined by the regularity of the entries $V, f$ and $\mathcal{A}_{K}$ the coefficient of the operator. The above expansion leads to $u \in C^{m, \gamma}$, with $m=\operatorname{deg}\left(P_{r}\right)$.
To carry over the blow up argument and to use compact Sobolev embedding or the Arzelá-Ascoli theorem, after subtracting polynomials, rescaling and normalization, it is necessary to derive a priori Hölder estimate for functions $v$ solving the more general equation

$$
\begin{equation*}
\mathcal{L}_{K} v+\mathcal{L}_{K^{\prime}} U=F \quad \text { in } \Omega \tag{1.29}
\end{equation*}
$$

Actually, in the counter part of (1.29) in the local case reads as

$$
\begin{equation*}
-\operatorname{div}(A(x) \nabla v)+\operatorname{div} U=F \tag{1.30}
\end{equation*}
$$

for some potential $U$. The study of (1.29) is typically essential for the proof of Theorem $1.9 f(i i)$, where $U=p_{r}$ is a first order polynomial and $\mathcal{L}_{K^{\prime}}$ is a non-translation invariant operator. Recall here that obtaining gradient estimates for solutions to equations involving divergence operators is more subtle than those involving operators in non-divergence form, since the latter annihilate affine functions, while the former do not and so the study of (1.30) becomes useful. The same difficulty is of course faced here since we are dealing with non-translation invariant variational solutions.
The core of the paper, from which we derive all the results, is Proposition 3.3 below, where we prove Hölder continuity of the solutions to (1.29), under mild regularity assumptions on $K, K^{\prime}, U$ and $F$, and we believe that the argument of proof and the result itself could be of independent interest.
We finally remark that the Schauder estimates in the present paper remains stable as $s \rightarrow 1$, provided we replace the kernel $K$, with $(1-s) K$ and $\alpha$ is such that $2<2 s_{0}+\alpha$, for some $s_{0} \in(0,1)$.

The paper is organized as follows. In Section 2 we collect some preliminary result and notations. Section 3 contains the regularity estimates for solutions to (1.29). Now Theorem 1.9 (ii) is proved in Section 4 and Theorem 1.10 in Section 5. Finally the proof of the main results are gathered in Section 6

## 2. Notations and preliminary Results

2.1. Notations. In this paper, the ball centred at $z \in \mathbb{R}^{N}$ with radius $r>0$ is denoted by $B(z, r)$ and $B_{r}:=B_{r}(0)$. Here and in the following, we let $\varphi_{1} \in C_{c}^{\infty}\left(B_{2}\right)$ such that $\varphi_{1} \equiv 1$ on $B_{1}$ and $0 \leq \varphi_{1} \leq 1$ on $\mathbb{R}^{N}$. We put $\varphi_{R}(x):=\varphi(x / R)$. For $b \in L^{\infty}\left(S^{N-1}\right)$, we define $\mu_{b}(x, y)=|x-y|^{-N-2 s} b\left(\frac{x-y}{|x-y|}\right)$. Given $\sigma>0$, we define the space

$$
L_{\sigma}\left(\mathbb{R}^{N}\right):=\left\{u \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right):\|u\|_{L_{\sigma}\left(\mathbb{R}^{N}\right)}:=\int_{\mathbb{R}^{N}}|u(x)|\left(1+|x|^{N+2 \sigma}\right)^{-1} d x<\infty\right\}
$$

Throughout this paper, for the seminorm of the fractional Sobolev spaces, we adopt the notation

$$
[u]_{H^{s}(\Omega)}:=\left(\int_{\Omega \times \Omega}|u(x)-u(y)|^{2} \mu_{1}(x, y) d x d y\right)^{1 / 2}
$$

We will, sometimes use the notation

$$
[u]_{H_{K}^{s}(\Omega)}:=\left(\int_{\Omega \times \Omega}|u(x)-u(y)|^{2}|K(x, y)| d x d y\right)^{1 / 2}
$$

for a function $K: \Omega \times \Omega \rightarrow[-\infty,+\infty]$. For the Hölder and Lipschitz seminorm, we write

$$
[u]_{C^{0, \alpha}(\Omega)}:=\sup _{x \neq y \in \Omega} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}
$$

for $\alpha \in(0,1]$. If there is no ambiguity, when $\alpha \in(0,1)$, we will write $[u]_{C^{\alpha}(\Omega)}$ instead of $[u]_{C^{0, \alpha}(\Omega)}$. If $m \in \mathbb{N}$ and $\alpha \in(0,1)$, the Hölder space $\|u\|_{C^{m, \alpha}(\Omega)}$ is given by the set of functions in $C^{m}(\Omega)$ such that

$$
\|u\|_{C^{m+\alpha}(\Omega)}:=\|u\|_{C^{m, \alpha}(\Omega)}=\sup _{\gamma \in \mathbb{N}^{N},|\gamma| \leq m}\left\|\partial^{\gamma} u\right\|_{L^{\infty}(\Omega)}+\sup _{\gamma \in \mathbb{N}^{N},|\gamma|=m}\left\|\partial^{\gamma} u\right\|_{C^{\alpha}(\Omega)}<\infty
$$

Letting $u \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$, the mean value of $u$ in $B_{r}(z)$ is denoted by

$$
u_{B_{r}(z)}=(u)_{B_{r}(z)}:=\frac{1}{\left|B_{r}\right|} \int_{B_{r}(z)} u(x) d x
$$

For $\alpha \in[0,1], h \in \mathbb{R}^{N} \backslash\{0\}$ and $f \in C_{l o c}^{0, \alpha}\left(\mathbb{R}^{N}\right)$, we define

$$
\begin{equation*}
f_{h, \alpha}(x):=\frac{f(x+h)-f(x)}{|h|^{\alpha}} \tag{2.1}
\end{equation*}
$$

2.2. Preliminary results. We gather in this paragraph some results which we will frequently use in the following of the paper. Let $K: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow[0, \infty]$ satisfy the following properties:

$$
\begin{align*}
& \text { (i) } K(x, y)=K(y, x) \quad \text { for all } x, y \in \mathbb{R}^{N} \\
& \text { (ii) } \kappa \mu_{1}(x, y) \leq K(x, y) \leq \frac{1}{\kappa} \mu_{1}(x, y) \quad \text { for all } x, y \in \mathbb{R}^{N} \tag{2.2}
\end{align*}
$$

For $\alpha^{\prime} \geq 0$, we let $K^{\prime}: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow[-\infty,+\infty]$ satisfy

$$
\begin{align*}
& \text { (i) } K^{\prime}(x, y)=K^{\prime}(y, x) \quad \text { for all } x, y \in \mathbb{R}^{N} \\
& \left(\text { ii) }\left|K^{\prime}(x, y)\right| \leq \frac{1}{\kappa}(|x|+|y|+1)^{\alpha^{\prime}} \mu_{1}(x, y) \quad \text { for all } x, y \in \mathbb{R}^{N}\right. \tag{2.3}
\end{align*}
$$

Let $U \in H_{l o c}^{s}(\Omega) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $f \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$. We say that $u \in H_{l o c}^{s}(\Omega) \cap L_{s}\left(\mathbb{R}^{N}\right)$ is a (weak) solution to

$$
\begin{equation*}
\mathcal{L}_{K} u+\mathcal{L}_{K^{\prime}} U=f \quad \text { in } \Omega \tag{2.4}
\end{equation*}
$$

if, for every $\psi \in C_{c}^{\infty}(\Omega)$,

$$
\begin{aligned}
\int_{\mathbb{R}^{2 N}}(u(x)-u(y))(\psi(x)-\psi(y)) K(x, y) d x d y & +\int_{\mathbb{R}^{2 N}}(U(x)-U(y))(\psi(x)-\psi(y)) K^{\prime}(x, y) d x d y \\
& =\int_{\mathbb{R}^{N}} f(x) \psi(x) d x
\end{aligned}
$$

We note that each of the terms in the above identity is finite. For $\beta \in[0,2 s)$, we define the Morrey space $\mathcal{M}_{\beta}$ by the set of functions $f \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ such that

$$
\|f\|_{\mathcal{M}_{\beta}}:=\sup _{\substack{x \in \mathbb{R}^{N} \\ r \in(0,1)}} r^{\beta-N} \int_{B_{r}(x)}|f(y)| d y<\infty
$$

with $\mathcal{M}_{0}:=L^{\infty}\left(\mathbb{R}^{N}\right)$, and we note that $\|f\|_{\mathcal{M}_{N / p}} \leq C(N, p)\|f\|_{L^{p}\left(\mathbb{R}^{N}\right)}$. We have the following coercivity property, see [30],

$$
\begin{equation*}
\left\||f|^{1 / 2} v\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq C(N, s, \beta)\|f\|_{\mathcal{M}_{\beta}}\|v\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \quad \text { for all } v \in H^{s}\left(\mathbb{R}^{N}\right) \tag{2.5}
\end{equation*}
$$

We prove our a priori estimates for right hand in $\mathcal{M}_{\beta}$. Recall that $\mathcal{M}_{N / p}$ contains strictly $\|f\|_{L^{p}\left(\mathbb{R}^{N}\right)}$. The following energy estimate can be seen as a nonlocal Caccioppoli inequality.

Lemma 2.1. Let $N \geq 1$, $s \in(0,1)$ and $\kappa>0$. We consider $K$ satisfying (2.2) and $K^{\prime}$ satisfying (2.3), for some $\alpha^{\prime} \geq 0$. Let $v \in H^{s}\left(\mathbb{R}^{N}\right)$ and $U \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} v+\mathcal{L}_{K^{\prime}} U=f \quad \text { in } B_{2 R} \tag{2.6}
\end{equation*}
$$

Then for every $\varepsilon>0$, there exist $\bar{C}=\bar{C}(s, N, \kappa, R)$ and $C=C(\varepsilon, s, N, \kappa, R)$ such that

$$
\begin{aligned}
\left\{\kappa-\varepsilon \bar{C}\|f\|_{\mathcal{M}_{\beta}}\right\} & \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y \\
\leq & C\left(\|f\|_{\mathcal{M}_{\beta}}+1\right)\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}+C\|f\|_{\mathcal{M}_{\beta}}\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \\
& +C[U]_{H_{K^{\prime}}^{s}\left(B_{4 R}\right)}^{2}+C \int_{\mathbb{R}^{N}} \varphi_{R}^{2}(y)|v(y)|\left(\int_{\mathbb{R}^{N} \backslash B_{4 R}}\left|U(x)-U(y) \| K^{\prime}(x, y)\right| d x\right) d y
\end{aligned}
$$

Proof. Applying [30, Lemma 9.1], we get

$$
\begin{align*}
(\kappa-\varepsilon) \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) & \mu_{1}(x, y) d x d y \leq \int_{\mathbb{R}^{N}}|f(x) \| v(x)| \varphi_{R}^{2}(x) d x \\
& +C \int_{\mathbb{R}^{2 N}}\left(\varphi_{R}(x)-\varphi_{R}(y)\right)^{2} v^{2}(y) \mu_{1}(x, y) d x d y \\
& +\int_{\mathbb{R}^{2 N}}\left|U(x)-U(y) \| \varphi_{R}^{2}(x) v(x)-\varphi_{R}^{2}(y) v(y)\right|\left|K^{\prime}(x, y)\right| d x d y \tag{2.7}
\end{align*}
$$

We now estimate

$$
\begin{align*}
\int_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \mid U(x)-U(y) \| \varphi_{R}^{2}(x) & v(x)-\varphi_{R}^{2}(y) v(y)| | K^{\prime}(x, y) \mid d x d y \\
& =\int_{B_{4 R} \times B_{4 R}}|U(x)-U(y)|\left|\varphi_{R}^{2}(x) v(x)-\varphi_{R}^{2}(y) v(y)\right|\left|K^{\prime}(x, y)\right| d x d y \\
& +2 \int_{\mathbb{R}^{N}} \varphi_{R}^{2}(y)|v(y)|\left(\int_{\mathbb{R}^{N} \backslash B_{4 R}}|U(x)-U(y)|\left|K^{\prime}(x, y)\right| d x\right) d y \\
& \leq \varepsilon / \kappa(2(4 R)+1)^{\alpha^{\prime}}\left[\varphi_{R}^{2} v\right]_{H^{s}\left(B_{4 R}\right)}^{2}+C[U]_{H_{K^{\prime}}^{s}\left(B_{4 R}\right)}^{2} \\
& +C \int_{\mathbb{R}^{N}} \varphi_{R}^{2}(y)|v(y)|\left(\int_{\mathbb{R}^{N} \backslash B_{4 R}}|U(x)-U(y)|\left|K^{\prime}(x, y)\right| d x\right) d y \tag{2.8}
\end{align*}
$$

We recall that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left(\varphi_{1}(x)-\varphi_{1}(y)\right)^{2} \mu_{1}(x, y) d y \leq C(N, s)\left(1+|x|^{-N-2 s}\right) \quad \text { for every } x \in \mathbb{R}^{N} \tag{2.9}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
{\left[\varphi_{R}^{2} v\right]_{H^{s}\left(B_{4 R}\right)}^{2} } & \leq 2 \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{4}(y) \mu_{1}(x, y) d x d y+2 \int_{\mathbb{R}^{2 N}}\left(\varphi_{R}^{2}(x)-\varphi_{R}^{2}(y)\right)^{2} v^{2}(y) \mu_{1}(x, y) d x d y \\
& \leq 2 \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y+\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}
\end{aligned}
$$

Using this in (2.8), we get

$$
\begin{align*}
& \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}}\left|U(x)-U(y)\left\|\varphi_{R}^{2}(x) v(x)-\varphi_{R}^{2}(y) v(y)\right\| K^{\prime}(x, y)\right| d x d y \\
& \leq \varepsilon \bar{C} \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y+C\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \\
& +C[U]_{H_{K^{\prime}}^{s}}^{2}\left(B_{4 R}\right)+C \int_{\mathbb{R}^{N}} \varphi_{R}^{2}(y)|v(y)|\left(\int_{\mathbb{R}^{N} \backslash B_{4 R}}\left|U(x)-U(y) \| K^{\prime}(x, y)\right| d x\right) d y \tag{2.10}
\end{align*}
$$

Next, from (2.5), Young's inequality and (2.9), we deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}\left|f(x) \| \varphi_{R}(x)\right|^{2}|v(x)| d x & \leq \varepsilon \bar{C}\|f\|_{\mathcal{M}_{\beta}} \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y \\
& +C\|f\|_{\mathcal{M}_{\beta}}\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}+C\|f\|_{\mathcal{M}_{\beta}}\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}
\end{aligned}
$$

Using this and (2.10) in (2.7), we get the result.
We state the following result.
Lemma 2.2. Let $N \geq 1, s \in(0,1)$ and $\kappa>0$. We consider $K$ satisfying (2.2) and $K^{\prime}$ satisfying (2.3), for some $\alpha^{\prime} \geq 0$. Let $v \in H^{s}\left(\mathbb{R}^{N}\right)$ and $U \in H_{\text {loc }}^{s}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ satisfy

$$
\mathcal{L}_{K} v+\mathcal{L}_{K^{\prime}} U=f \quad \text { in } B_{2 R}
$$

Then there exists $C=C\left(N, s, \kappa, \alpha^{\prime}, R\right)$ such that for every $\psi \in C_{c}^{\infty}\left(B_{R}\right)$, we have

$$
\begin{align*}
& \left|\int_{\mathbb{R}^{2 N}}(v(x)-v(y))(\psi(x)-\psi(y)) K(x, y) d x d y\right| \leq C\|f\|_{\mathcal{M}_{\beta}}\left(1+\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right)  \tag{2.11}\\
& +C[U]_{H_{K^{\prime}}^{s}\left(B_{4 R}\right)}[\psi]_{H^{s}\left(B_{4 R}\right)}+C \int_{\mathbb{R}^{N}}|\psi(y)|\left(\int_{\mathbb{R}^{N} \backslash B_{4 R}}|U(x)-U(y)|\left|K^{\prime}(x, y)\right| d x\right) d y \tag{2.12}
\end{align*}
$$

Proof. Using the weak formulation of the equation and (2.5), we get the expression on the left hand side in (2.11). Now expression (2.12) appears after decomposing the domain of integration and using Hölder's inequality as in the beginning of the proof of Lemma 2.1.

REGULARITY RESULTS FOR NONLOCAL EQUATIONS AND APPLICATIONS
We close this section with the following result.
Lemma 2.3. Let $K$ satisfy (1.4) (i)-(ii). Let $v \in H_{l o c}^{s}\left(B_{2 R}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$ and $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{N}\right)$ satisfy

$$
\mathcal{L}_{K} v=f \quad \text { in } B_{2 R}
$$

for some $R>0$. We let $v_{R}:=\varphi_{R} v$. Then

$$
\begin{equation*}
\mathcal{L}_{K} v_{R}=f+G_{K, v, R} \quad \text { in } B_{R / 2} \tag{2.13}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{K, v, R}(x)=\int_{\mathbb{R}^{N}} v(y)\left(\varphi_{R}(x)-\varphi_{R}(y)\right) K(x, y) d y \tag{2.14}
\end{equation*}
$$

Moreover, $G_{K, v, R}$ satisfies the following properties.
(i) There exists $C=C(N, s, R)$ such that

$$
\left\|G_{K, v, R}\right\|_{L^{\infty}\left(B_{R / 2}\right)} \leq C \sup _{x \in B_{R / 2}} \int_{|y| \geq R}|v(y)||K(x, y)| d y
$$

(ii) If $v \in C^{k+\alpha}\left(\mathbb{R}^{N}\right)$ and $\left\|\mathcal{A}_{K}\right\|_{C^{k, \alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)} \leq c_{0}$, then there exists $C=C\left(N, s, \alpha, c_{0}, R, k\right)$ such that

$$
\left\|G_{K, v, R}\right\|_{C^{k+\alpha}\left(B_{R / 2}\right)} \leq C\|v\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}
$$

(iii) If $\left\|\mathcal{A}_{K}\right\|_{C^{k+\alpha}\left(Q_{\infty} \times S^{N-1}\right)} \leq c_{0}$, then there exists $C=C\left(N, s, \alpha, c_{0}, R, k\right)$ such that

$$
\left\|G_{K, v, R}\right\|_{C^{k+\alpha}\left(B_{R / 4}\right)} \leq C\|v\|_{L_{s}\left(\mathbb{R}^{N}\right)}
$$

Proof. For (2.13), see [30, Lemma 9.2]. Statement (i) follows easily, thanks to the definition of $\varphi_{R}$. To prove (ii), we write

$$
G_{K, v, R}(x)=\int_{S^{N-1}} \int_{0}^{\infty} v(x+r \theta)\left(1-\varphi_{R}(x+r \theta)\right) \mathcal{A}_{K}(x, r, \theta) r^{-1-2 s} d r d \theta
$$

Since $1-\varphi_{R}(x+r \theta)=0$ for all $x \in B_{R / 2}, r \in(0, R / 2)$ and $\theta \in S^{N-1}$, then (ii) follows. To prove (iii), we note that

$$
\begin{aligned}
G_{K, v, R}(x) & :=\int_{\mathbb{R}^{N}} v(y)\left(\varphi_{R}(x)-\varphi_{R}(y)\right) K(x, y) d y \\
& =\int_{|y| \geq R} v(y)\left(1-\varphi_{R}(y)\right) \mathcal{A}_{K}(x,|x-y|,(x-y) /|x-y|)|x-y|^{-N-2 s} d y
\end{aligned}
$$

We recall that (see e.g. [31]) for every $x_{1}, x_{2}, y \in \mathbb{R}^{N}, \varrho>0$ and $\alpha \in(0,1)$,

$$
\left|\left|x_{1}-y\right|^{-\varrho}-\left|x_{2}-y\right|^{-\varrho}\right| \leq C(\alpha, \varrho)\left|x_{1}-x_{2}\right|^{\alpha}\left\{\left|x_{1}-y\right|^{-(\varrho+\alpha)}+\left|x_{2}-y\right|^{-(\varrho+\alpha)}\right\} .
$$

Hence for all $x_{1}, x_{2} \in B_{R / 2}, y \in \mathbb{R}^{N} \backslash B_{R}, \varrho \geq N+2 s$ and $\alpha \in(0,1)$, we get

$$
\left|\left|x_{1}-y\right|^{-\varrho}-\left|x_{2}-y\right|^{-\varrho}\right| \leq C(\alpha, \varrho, R)\left|x_{1}-x_{2}\right|^{\alpha}|y|^{-N-2 s}
$$

Using this and the Leibniz formula for higher order derivatives of the product of functions, we get (iii).

## 3. A PRIORI ESTIMATES

In this section, we prove a priori estimates for solutions to (2.4), provided $\mathcal{L}_{K}$ is close to the translation invariant operator $\mathcal{L}_{\mu_{a}}$, with $a: S^{N-1} \rightarrow \mathbb{R}$ satisfies

$$
\begin{equation*}
a(-\theta)=a(\theta) \quad \text { and } \quad \kappa \leq a(\theta) \leq \frac{1}{\kappa} \quad \text { for all } \theta \in S^{N-1} \tag{3.1}
\end{equation*}
$$

We now recall two results from [30] that will be needed in the following of the paper.

Lemma 3.1. Let $b \in L^{\infty}\left(S^{N-1}\right)$. Suppose that there exists a sequence of functions $\left(a_{n}\right)_{n}$ satisfying (3.1) and such that $a_{n} \stackrel{*}{\rightharpoonup} b$ in $L^{\infty}\left(S^{N-1}\right)$. Let $\lambda_{n}: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow\left[0, \kappa^{-1}\right]$, with $\lambda_{n} \rightarrow 0$ pointwise on $\mathbb{R}^{N} \times \mathbb{R}^{N}$. Let $\left(K_{n}\right)_{n}$ be a sequence of symmetric kernels satisfying

$$
\left|K_{n}(x, y)-\mu_{a_{n}}(x, y)\right| \leq \lambda_{n}(x, y) \mu_{1}(x, y) \quad \text { for all } x \neq y \in \mathbb{R}^{N} \text { and for all } n \in \mathbb{N}
$$

If $\left(v_{n}\right)_{n}$ is a bounded sequence in $L_{s}\left(\mathbb{R}^{N}\right) \cap H_{\text {loc }}^{s}\left(\mathbb{R}^{N}\right)$ such that $v_{n} \rightarrow v$ in $L_{s}\left(\mathbb{R}^{N}\right)$, then
$\int_{\mathbb{R}^{N}} v(x) \mathcal{L}_{\mu_{b}} \psi(x) d x=\frac{1}{2} \lim _{n \rightarrow \infty} \int_{\mathbb{R}^{2 N}}\left(v_{n}(x)-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \quad$ for all $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$.
Lemma 3.2. Let $b \in L^{\infty}\left(S^{N-1}\right)$. Suppose that there exists a sequence of functions $\left(a_{n}\right)_{n}$ satisfying (3.1) and such that $a_{n} \stackrel{*}{\rightharpoonup} b$ in $L^{\infty}\left(S^{N-1}\right)$. Consider $u \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$ satisfying

$$
\left\{\begin{array}{l}
\mathcal{L}_{\mu_{b}} u=0 \quad \text { in } \mathbb{R}^{N}, \\
\|u\|_{L^{2}\left(B_{R}\right)}^{2} \leq R^{N+2 \gamma} \quad \text { for some } \gamma<2 s \text { and for every } R \geq 1
\end{array}\right.
$$

Then $u$ is an affine function.
3.1. A priori estimates and consequences. We now state the main result of the present section.

Proposition 3.3. Let $s \in(0,1), \beta \in[0,2 s), \sigma \in(s, 1]$ and $\kappa>0$. Let $\alpha^{\prime} \geq 0$, with $\alpha^{\prime}+\sigma \in(0,2 s)$. Pick

$$
\gamma \in(0,1) \cap(0, \sigma] \cap(0,2 s-\beta] .
$$

Then there exist $\varepsilon_{0}>0$ and $C>0$ such that if

- a satisfies (3.1), $K_{a}$ satisfies (2.2) with

$$
\left|K_{a}-\mu_{a}\right|<\varepsilon_{0} \mu_{1}(x, y) \quad \text { on } B_{2} \times B_{2} \backslash\{x=y\}
$$

- $K^{\prime}$ satisfies (2.3),
- $f \in \mathcal{M}_{\beta}, g \in H^{s}\left(\mathbb{R}^{N}\right), U \in C_{\text {loc }}^{0, \sigma}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ are such that

$$
\mathcal{L}_{K_{a}} g+\mathcal{L}_{K^{\prime}} U=f \quad \text { in } B_{2}
$$

then

$$
\sup _{r>0} r^{-2 \gamma-N}\left\|g-g_{B_{r}}\right\|_{L^{2}\left(B_{r}\right)}^{2} \leq C\left(\|g\|_{L^{2}\left(\mathbb{R}^{N}\right)}+[U]_{C^{0, \sigma}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)^{2}
$$

Proof. Assume that the assertion in the proposition does not hold, then for every $n \in \mathbb{N}$, there exist:

- $a_{n}$ and $K_{a_{n}}$ satisfying (3.1) and (2.2) respectively, with

$$
\begin{equation*}
\left|K_{a_{n}}-\mu_{a_{n}}\right|<\frac{1}{n} \mu_{1}(x, y) \quad \text { on } B_{2} \times B_{2} \backslash\{x=y\} \tag{3.2}
\end{equation*}
$$

- $K_{n}^{\prime}$ satisfying (2.3), $f_{n} \in \mathcal{M}_{\beta}, U_{n} \in C_{l o c}^{0, \sigma}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $g_{n} \in H^{s}\left(\mathbb{R}^{N}\right)$, with $\left\|g_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}}+\left\|U_{n}\right\|_{C^{\sigma}\left(\mathbb{R}^{N}\right)} \leq 1$,

$$
\begin{equation*}
\mathcal{L}_{K_{a_{n}}} g_{n}+\mathcal{L}_{K_{n}^{\prime}} U_{n}=f_{n} \quad \text { in } B_{2} \tag{3.3}
\end{equation*}
$$

with the property that

$$
\sup _{r>0} r^{-N-2 \gamma}\left\|g_{n}-\left(g_{n}\right)_{B_{r}}\right\|_{L^{2}\left(B_{r}\right)}^{2}>n
$$

Consequently, there exists $\bar{r}_{n}>0$ such that

$$
\begin{equation*}
\bar{r}_{n}^{-N-2 \gamma}\left\|g_{n}-\left(g_{n}\right)_{B_{\bar{T}_{n}}}\right\|_{L^{2}\left(B_{\bar{T}_{n}}\right)}^{2}>n / 2 \tag{3.4}
\end{equation*}
$$

We consider the (well defined, because $\left\|g_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)} \leq 1$ ) nonincreasing function $\Theta_{n}:(0, \infty) \rightarrow[0, \infty)$ given by

$$
\Theta_{n}(\bar{r})=\sup _{r \geq \bar{r}} r^{-N-2 \gamma}\left\|g_{n}-\left(g_{n}\right)_{B_{r}}\right\|_{L^{2}\left(B_{r}\right)}^{2}
$$

Obviously, for $n \geq 2$, by (3.4),

$$
\begin{equation*}
\Theta_{n}\left(\bar{r}_{n}\right)>n / 2 \geq 1 \tag{3.5}
\end{equation*}
$$

Hence, provided $n \geq 2$, there exists $r_{n} \in\left[\bar{r}_{n}, \infty\right)$ such that

$$
\Theta_{n}\left(r_{n}\right) \geq r_{n}^{-N-2 \gamma}\left\|g_{n}-\left(g_{n}\right)_{B_{r_{n}}}\right\|_{L^{2}\left(B_{r_{n}}\right)}^{2} \geq \Theta_{n}\left(\bar{r}_{n}\right)-1 / 2 \geq(1-1 / 2) \Theta_{n}\left(\bar{r}_{n}\right) \geq \frac{1}{2} \Theta_{n}\left(r_{n}\right)
$$

where we used the monotonicity of $\Theta_{n}$ for the last inequality, while the first inequality comes from the definition of $\Theta_{n}$. In particular, thanks to (3.5), $\Theta_{n}\left(r_{n}\right) \geq n / 4$. Now since $\left\|g_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)} \leq 1$, we have that $r_{n}^{-N-2 \gamma} \geq n / 8$, so that $r_{n} \rightarrow 0$ as $n \rightarrow \infty$. We now define the sequence of functions

$$
w_{n}(x)=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\gamma}\left\{g_{n}\left(r_{n} x\right)-\frac{1}{\left|B_{1}\right|} \int_{B_{1}} g_{n}\left(r_{n} x\right) d x\right\}
$$

which, satisfies

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{2}, \quad \int_{B_{1}} w_{n}(x) d x=0 \quad \text { for every } n \geq 2 \tag{3.6}
\end{equation*}
$$

Using that, for every $r>0,\left\|g_{n}-\left(g_{n}\right)_{B_{r}}\right\|_{L^{2}\left(B_{r}\right)}^{2} \leq r^{N+2 \gamma} \Theta_{n}(r)$ and the monotonicity of $\Theta_{n}$, by [30, Lemma 3.1], we find that

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \gamma} \quad \text { for every } R \geq 1 \text { and } n \geq 2 \tag{3.7}
\end{equation*}
$$

for some constant $C=C(N, \gamma)>0$.
We define

$$
\bar{K}_{n}(x, y)=r_{n}^{N+2 s} K_{a_{n}}\left(r_{n} x, r_{n} y\right), \quad \bar{K}_{n}^{\prime}(x, y)=r_{n}^{N+2 s} K_{n}^{\prime}\left(r_{n} x, r_{n} y\right)
$$

and

$$
\bar{U}_{n}(x)=U_{n}\left(r_{n} x\right), \quad \bar{f}_{n}(x)=r_{n}^{2 s} f_{n}\left(r_{n} x\right)
$$

It is plain that

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{n}} w_{n}+r_{n}^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1 / 2} \mathcal{L}_{\bar{K}_{n}^{\prime}} \bar{U}_{n}=r_{n}^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1 / 2} \bar{f}_{n} \quad \text { in } B_{2 / r_{n}} \tag{3.8}
\end{equation*}
$$

We fix $M>1$ and let $n \geq 2$ large, so that $1<M<\frac{1}{8 r_{n}}$. Therefore, letting $w_{n, M}:=\varphi_{4 M} w_{n} \in$ $H^{s}\left(\mathbb{R}^{N}\right)$, we apply Lemma 2.3 $(i)$ to get

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{n}} w_{n, M}+r_{n}^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1 / 2} \mathcal{L}_{\bar{K}_{n}^{\prime}} \bar{U}_{n}=r^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1 / 2} \bar{f}_{n}+G_{K_{n}, w_{n}, M} \quad \text { in } B_{2 M} \tag{3.9}
\end{equation*}
$$

with $\left\|G_{\bar{K}_{n}, w_{n}, M}\right\|_{L^{\infty}\left(B_{M / 2}\right)} \leq C\left\|w_{n}\right\|_{L_{s}\left(\mathbb{R}^{N}\right)} \leq C$, by (3.7). We also note that

$$
\begin{equation*}
\left\|\bar{f}_{n}\right\|_{\mathcal{M}_{\beta}} \leq r_{n}^{2 s-\beta} \tag{3.10}
\end{equation*}
$$

Clearly $\bar{K}_{n}$ satisfies (1.4). Applying Lemma 2.1 to the equation (3.9) and using (3.7) together with (3.10), we find a constant $\bar{C}$ such that for every $\varepsilon>0$, there exists $C$ satisfying

$$
\begin{align*}
& \left\{\kappa-\varepsilon \bar{C} \Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{2 s-\beta-\gamma}\right\} \int_{B_{M / 8} \times B_{M / 8}}\left(w_{n, M}(x)-w_{n, M}(y)\right)^{2} \mu_{1}(x, y) d x d y \\
& \leq C\left(\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{2 s-\beta-\gamma}+1\right)+C r_{n}^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1 / 2}\left[\bar{U}_{n}\right]_{H_{K_{n}^{\prime}}^{s}}^{2}\left(B_{4 M}\right) \\
&  \tag{3.11}\\
& \quad+C r_{n}^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1 / 2} \int_{\mathbb{R}^{N}} \varphi_{M}^{2}(y)\left|w_{n}(y)\right|\left(\int_{\mathbb{R}^{N} \backslash B_{4 M}}\left|\bar{U}_{n}(x)-\bar{U}_{n}(y)\right|\left|\bar{K}_{n}^{\prime}(x, y)\right| d x\right) d y
\end{align*}
$$

We observe that

$$
\left|\bar{K}_{n}^{\prime}(x, y)\right| \leq \frac{1}{\kappa}\left(r_{n}|x|+r_{n}|y|+1\right)^{\alpha^{\prime}} \mu_{1}(x, y)
$$

From this and the fact that $\left[\bar{U}_{n}\right]_{C^{\sigma}\left(\mathbb{R}^{N}\right)} \leq r_{n}^{\sigma}$, we have the following estimate:

$$
\begin{align*}
{\left[\bar{U}_{n}\right]_{H_{K_{n}^{\prime}}^{s}\left(B_{4 M}\right)}^{2} } & =\int_{B_{4 M} \times B_{4 M}}\left(\bar{U}_{n}(x)-\bar{U}_{n}(y)\right)^{2} \bar{K}_{n}^{\prime}(x, y) d x d y \\
& \leq C(M) \int_{B_{4 M} \times B_{4 M}}\left(\bar{U}_{n}(x)-\bar{U}_{n}(y)\right)^{2} \mu_{1}(x, y) d x d y \\
& \leq C(M) r_{n}^{2 \sigma} \int_{B_{4 M} \times B_{4 M}}|x-y|^{-N-2 s+2 \sigma} d x d y \\
& \leq C(M) r_{n}^{2 \sigma} \tag{3.12}
\end{align*}
$$

because $\sigma>s$. In addition, since $\alpha^{\prime}+\sigma<2 s$, we get

$$
\begin{align*}
\sup _{y \in B_{M}} \int_{\mathbb{R}^{N} \backslash B_{4 M}}\left|\bar{U}_{n}(x)-\bar{U}_{n}(y)\right|\left|\bar{K}_{n}^{\prime}(x, y)\right| d x & \leq C(M) r_{n}^{\sigma} \int_{|x| \geq 2 M}\left(1+|x|^{\alpha^{\prime}}\right)|x|^{-N-2 s+\sigma} d x \\
& \leq C(M) r_{n}^{\sigma} \tag{3.13}
\end{align*}
$$

Now using (3.12) and (3.13) in (3.11) and the fact that $\gamma \leq \min (2 s-\beta, \sigma)$, we find that

$$
\left\{\kappa-\varepsilon \bar{C} \Theta_{n}\left(r_{n}\right)^{-1 / 2}\right\}\left[w_{n}\right]_{H^{s}\left(B_{M / 8}\right)}^{2} \leq C\left(\Theta_{n}\left(r_{n}\right)^{-1 / 2}+1\right)
$$

Therefore, since $\Theta_{n}\left(r_{n}\right)^{-1} \leq 1$, then provided $\varepsilon$ is small enough, by (3.7), we deduce that $w_{n}$ is bounded in $H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Hence by Sobolev embedding and (3.7), there exists $w \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$ such that, up to a subsequence, $w_{n} \rightarrow w$ in $L_{l o c}^{2}\left(\mathbb{R}^{N}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$. Moreover, by (3.6) we deduce that

$$
\begin{equation*}
\|w\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{2} \quad \text { and } \quad w_{B_{1}}=0 \tag{3.14}
\end{equation*}
$$

In addition by (3.7), we have

$$
\begin{equation*}
\|w\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \gamma} \quad \text { for every } R \geq 1 \tag{3.15}
\end{equation*}
$$

Now applying Lemma 2.2 to the equation (3.8) and using (3.12) together with (3.13), we get

$$
\left|\int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)(\psi(x)-\psi(y)) \bar{K}_{n}(x, y) d x d y\right| \leq C \Theta_{n}\left(r_{n}\right)^{-1 / 2} \quad \text { for all } \psi \in C_{c}^{\infty}\left(B_{M}\right)
$$

Since $\left|\bar{K}_{n}-\mu_{a_{n}}\right| \leq \frac{\mu_{1}(x, y)}{n}$ almost everywhere in $B_{1 / r_{n}} \times B_{1 / r_{n}}$ and $\Theta_{n}\left(r_{n}\right) \rightarrow \infty$ as $n \rightarrow \infty$, we can apply Lemma 3.1 to deduce that $\mathcal{L}_{\mu_{b}} w=0 \quad$ in $\mathbb{R}^{N}$, where $b$ is the weak-star limit of $a_{n}$ (which satisfies (3.1) for all $n \in \mathbb{N}$ ). In view of (3.15), by Lemma 3.2, we deduce that $w$ is equivalent to a constant function, since $\gamma<1$. This is clearly in contradiction with (3.14).

As a consequence, we get the following result.
Corollary 3.4. Let $s \in(0,1), N \geq 1, \beta \in[0,2 s), \sigma \in(s, 1], \alpha^{\prime} \geq 0$, with $\alpha^{\prime}+\sigma \in(0,2 s)$, and $\kappa>0$. Let a satisfy (3.1). Consider $K$ and $K^{\prime}$ satisfying (2.2) and (2.3), respectively. Let $g \in H^{s}\left(B_{2}\right) \cap L_{s}\left(\mathbb{R}^{N}\right), U \in C_{l o c}^{0, \sigma}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} g+\mathcal{L}_{K^{\prime}} U=f \quad \text { in } B_{2} \tag{3.16}
\end{equation*}
$$

Then, for $\gamma \in(0,1) \cap(0,2 s-\beta] \cap(0, \sigma]$, there exist $\varepsilon_{0}, C>0$, such that if $\left|K-\mu_{a}\right|<\varepsilon_{0} \mu_{1}(x, y)$ in $B_{2} \times B_{2} \backslash\{x=y\}$, we have

$$
\|g\|_{C^{0, \gamma}\left(B_{1 / 4}\right)} \leq C\left(\|g\|_{L^{2}\left(B_{2}\right)}+\|g\|_{L_{s}\left(\mathbb{R}^{N}\right)}+[U]_{C^{0, \sigma}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

with $C, \varepsilon_{0}>0$ depend only on $s, N, \beta, \sigma, \alpha^{\prime}, \kappa$ and $\gamma$.
Proof. Without loss of generality, we may assume that

$$
\begin{equation*}
\|g\|_{L^{2}\left(B_{2}\right)}+\|g\|_{L_{s}\left(\mathbb{R}^{N}\right)}+[U]_{C^{0, \sigma}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}} \leq 1 \tag{3.17}
\end{equation*}
$$

Let $z \in B_{1 / 2}$ and define $g_{z}:=g(x+z), K_{z}(x, y)=K(x+z, y+z), K_{z}^{\prime}(x, y)=K(x+z, y+z)$, $f_{z}(x)=f(x+z), f_{z}(x)=f(x+z)$, and $U_{z}(x)=U(x+z)$. We then have

$$
\mathcal{L}_{K_{z}} g_{z}+\mathcal{L}_{K_{z}^{\prime}} U_{z}=f_{z} \quad \text { in } B_{1} .
$$

On the other hand by Lemma 2.3.

$$
\begin{equation*}
\mathcal{L}_{K_{z}}\left(\varphi_{1} g_{z}\right)+\mathcal{L}_{K_{z}^{\prime}} U_{z}\left(\varphi_{1} g_{z}\right)=\widetilde{f}_{z} \quad \text { in } B_{1 / 2} \tag{3.18}
\end{equation*}
$$

for some function $\widetilde{f}_{z}$ satisfying

$$
\begin{equation*}
\left\|\tilde{f}_{z}\right\|_{\mathcal{M}_{\beta}} \leq\left\|f_{z}\right\|_{\mathcal{M}_{\beta}}+C\left\|g_{z}\right\|_{L_{s}\left(\mathbb{R}^{N}\right)} \leq C \tag{3.19}
\end{equation*}
$$

where we used (3.17) for the last inequality. By (3.18) and Proposition 3.3, there exist $\varepsilon_{0}, C>0$, only depending on $s, N, \beta, \sigma, \alpha^{\prime}, \kappa, \sigma$ and $\gamma$, such that if $\left|K_{z}-\mu_{a}\right|<\varepsilon_{0}$ in $B_{2} \times B_{2} \backslash\{x=y\}$, we get

$$
\left\|g_{z}-\left(g_{z}\right)_{B_{r}}\right\|_{L^{2}\left(B_{r}\right)}=\left\|g-(g)_{B_{r}(z)}\right\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+\gamma} \quad \text { for every } r>0
$$

It then follows, from [30, Lemma 3.1], that

$$
\|g-g(z)\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+\gamma} \quad \text { for every } z \in B_{1} \text { and } r \in(0,1)
$$

This implies that $\|g\|_{C^{\gamma}\left(B_{1 / 4}\right)} \leq C$. The proof is thus finished.
By scaling and covering, we have the
Corollary 3.5. Let $s \in(0,1), N \geq 1, \beta \in[0,2 s), \sigma \in(s, 1], \alpha^{\prime} \geq 0$, with $\alpha^{\prime}+\sigma \in(0,2 s)$, $\kappa>0$ and $\gamma \in(0,1) \cap(0, \sigma] \cap(0,2 s-\beta]$. Consider $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, 0, Q_{\infty}\right)$ and $K^{\prime}$ satisfies (2.3). Let $g \in H^{s}\left(B_{2}\right) \cap L_{s}\left(\mathbb{R}^{N}\right), U \in C_{l o c}^{0, \sigma}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} g+\mathcal{L}_{K^{\prime}} U=f \quad \text { in } B_{2} . \tag{3.20}
\end{equation*}
$$

Then there exists $C>0$, only depending on $N, s, \alpha^{\prime}, \beta, \kappa$ and $\gamma$, such that

$$
\|g\|_{C^{0, \gamma}\left(B_{1}\right)} \leq C\left(\|g\|_{L^{2}\left(B_{2}\right)}+\|g\|_{L_{s}\left(\mathbb{R}^{N}\right)}+[U]_{C^{0, \sigma}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Proof. Pick $x_{0} \in B_{3 / 2}$. By the continuity of $\mathcal{A}_{K}(\cdot, \cdot, \theta)$ (uniformly with respect to $\theta$ ), for every $\varepsilon>0$ there exists $\delta=\delta_{x_{0}, \varepsilon} \in(0,1 / 100)$ such that, for every $x \in B_{4 \delta}\left(x_{0}\right), r \in(0,4 \delta)$ and $\theta \in S^{N-1}$, we have

$$
\left|K(x, x+r \theta)-\mathcal{A}_{K}\left(x_{0}, 0, \theta\right) r^{-N-2 s}\right| \leq \varepsilon r^{-N-2 s} .
$$

Therefore, for every $x \in B_{4 \delta}\left(x_{0}\right)$ and $0<|z|<4 \delta$,

$$
\left.\left.\left|K(x, x+z)-\mathcal{A}_{K}\left(x_{0}, 0, z /|z|\right)\right| z\right|^{-N-2 s}|\leq \varepsilon| z\right|^{-N-2 s}
$$

and thus, for every $x, y \in B_{2 \delta}\left(x_{0}\right)$, with $x \neq y$,

$$
\begin{equation*}
\left|K(x, y)-\mu_{a}(x, y)\right| \leq \varepsilon \mu_{1}(x, y) \tag{3.21}
\end{equation*}
$$

where $a(\theta):=\mathcal{A}_{K}\left(x_{0}, 0, \theta\right)$. By Definition 1.8 $\mathcal{A}_{o, K}\left(x_{0}, 0, \theta\right)=0$ and thus $a$ satisfies (1.2). We now let $K_{\delta}(x, y)=\delta^{N+2 s} K\left(\delta x+x_{0}, \delta y+x_{0}\right)$ and $K_{\delta}^{\prime}(x, y)=\delta^{N+2 s} K^{\prime}\left(\delta x+x_{0}, \delta y+x_{0}\right)$, which satisfy (2.2) and (2.3), respectively.

For $x \in B_{2}$, we define $g_{\delta}(x)=g\left(\delta x+x_{0}\right), U_{\delta}(x)=U\left(\delta x+x_{0}\right)$ and $f_{\delta}(x)=\delta^{2 s} f\left(\delta x+x_{0}\right)$. Since $\delta \in(0,1 / 16)$, by a change of variable in (3.20), we get

$$
\begin{equation*}
\mathcal{L}_{K_{\delta}} g_{\delta}+\mathcal{L}_{K_{\delta}^{\prime}} U_{\delta}=f_{\delta} \quad \text { in } B_{8} \tag{3.22}
\end{equation*}
$$

On the other hand (3.21) becomes

$$
\left|K_{\delta}(x, y)-\mu_{a}(x, y)\right| \leq \varepsilon \mu_{1}(x, y) \quad \text { for } x \neq y \in B_{2}
$$

From this and (3.22), then provided $\varepsilon>0$ small, by Corollary 3.4 and a change of variable, we get

$$
\|g\|_{C^{\alpha}\left(B_{\delta_{x_{0}, \varepsilon}}\left(x_{0}\right)\right)} \leq C\left(x_{0}\right)\left(\|g\|_{L^{2}\left(B_{2}\right)}+\|g\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}+[U]_{C^{0, \sigma}\left(\mathbb{R}^{N}\right)}\right),
$$

where $C\left(x_{0}\right)$ is a constant, only depending on $N, s, c_{0}, \delta_{x_{0}}, \kappa, \tau, \alpha, \alpha^{\prime}, \sigma, \gamma$ and $x_{0}$. Next, we cover $\bar{B}_{1}$ with a finite number of balls $B_{\frac{1}{2} \delta_{x_{i}, \varepsilon}}\left(x_{i}\right) \subset \subset B_{3 / 2}$, for $i=1, \ldots, n$, with $x_{i} \in \bar{B}_{1}$. It then follows that

$$
\|g\|_{C^{\alpha}\left(B_{1}\right)} \leq C^{\prime}\left(\|g\|_{L^{2}\left(B_{2}\right)}+\|g\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}+[U]_{C^{0, \sigma}\left(\mathbb{R}^{N}\right)}\right)
$$

We have the following generalization.

Corollary 3.6. Let $s \in(0,1), \beta \in[0,2 s), \sigma_{i} \in(s, 1]$. Let $\kappa>0$ and

$$
\gamma \in(0,1) \cap\left(0, \min _{1 \leq i \leq \ell} \sigma_{i}\right] \cap(0,2 s-\beta]
$$

Consider $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, 0, Q_{\infty}\right)$ and $K_{i}^{\prime}$ satisfying (2.3), for $i=1, \ldots, \ell$, and for some, $\alpha_{i}^{\prime} \geq 0$, with $\alpha_{i}^{\prime}+\sigma_{i} \in(0,2 s)$. Let $g \in H^{s}\left(B_{2}\right) \cap L_{s}\left(\mathbb{R}^{N}\right), U_{i} \in C_{l o c}^{0, \sigma_{i}}\left(\mathbb{R}^{N}\right) \cap L_{\left(\alpha^{\prime}+2 s\right) / 2}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} g+\sum_{i=1}^{\ell} \mathcal{L}_{K_{i}^{\prime}} U_{i}=f \quad \text { in } B_{2} \tag{3.23}
\end{equation*}
$$

Then, there exists $C>0$, only depending on $s, N, \beta, \alpha_{i}, \sigma_{i}, \kappa, \ell$ and $\gamma$, such that

$$
\|g\|_{C^{0, \gamma}\left(B_{1}\right)} \leq C\left(\|g\|_{L^{2}\left(B_{2}\right)}+\|g\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\sum_{i=1}^{\ell}\left[U_{i}\right]_{C^{0, \sigma_{i}\left(\mathbb{R}^{N}\right)}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

## 4. Gradient estimates

In this section, we consider the fractional parameter $s \in(1 / 2,1)$ and we prove Hölder estimates of $\nabla u$. For $g \in L_{l o c}^{2}\left(\mathbb{R}^{N}\right)$ and $r>0$, we define

$$
\begin{equation*}
\mathbf{P}_{r, g}(x)=g_{B_{r}}+T^{r, g} \cdot x=g_{B_{r}}+\sum_{i=1}^{N} T_{i}^{r, g} x_{i} \tag{4.1}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{i}^{r, g}=\frac{\left\langle g, x_{i}\right\rangle_{L^{2}\left(B_{r}\right)}}{\left\|x_{i}\right\|_{L^{2}\left(B_{r}\right)}^{2}} \tag{4.2}
\end{equation*}
$$

Note that $\mathbf{P}_{r, g}$ is the $L^{2}\left(B_{r}\right)$-projection of $g$ on the space of affine functions.
In view of Corollary 3.5, we know that the solutions $u$ to $\mathcal{L}_{K} u=f$ in $B_{2}$ are of class $C^{1-\varrho}\left(B_{1}\right)$ for every small $\varrho>0$, provided $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, 0, Q_{\infty}\right)$ and $f \in \mathcal{M}_{\beta}$, with $\beta \in[0,2 s)$. In particular $\left|T^{r, u-u(0)}\right| \leq C r^{-\varrho}$. The result below improves this to Hölder regularity estimates of the gradient of $u$ when $2 s-\beta>1$ and $K \in \widetilde{K}_{0}^{s}\left(\kappa, \alpha, Q_{\infty}\right)$, with $\alpha>0$.

Proposition 4.1. Let $N \geq 1, s \in(1 / 2,1), \kappa, c_{0}>0$. Consider $\alpha \in(0,2 s-1), \beta \in[0,2 s), \varrho \in[0,1)$ such that

$$
\gamma:=\min (1-\varrho+\alpha, 2 s-\beta)>1
$$

Then there exists $C=C\left(N, s, \alpha, \beta, \kappa, c_{0}, \varrho\right)>0$ such that if:

- $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$,
- $g \in H^{s}\left(B_{2}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ with

$$
\begin{gathered}
\|g\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}} \leq 1, \\
\left|T^{r, g}\right| \leq c_{0} r^{-\varrho} \quad \text { for all } r>0
\end{gathered}
$$

are such that

$$
\mathcal{L}_{K} g=f \quad \text { in } B_{2}
$$

then we have

$$
\sup _{r>0} r^{-\gamma}\left\|g-\mathbf{P}_{r, g}\right\|_{L^{\infty}\left(B_{r}\right)} \leq C .
$$

Proof. Suppose on the contrary that the assertion in the proposition does not hold. Then as in the proof of Proposition 3.3. for all $n \geq 2$, there exist

- $r_{n}>0, K_{n} \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$,
- $g_{n} \in H^{s}\left(B_{2}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$ and $f_{n} \in \mathcal{M}_{\beta}$ satisfying

$$
\begin{gather*}
\left\|g_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}} \leq 1, \quad\left|T^{r, g_{n}}\right| \leq c_{0} r^{-\varrho} \quad \text { for all } r \in(0, \infty)  \tag{4.3}\\
\mathcal{L}_{K_{n}} g_{n}=f_{n} \quad \text { in } B_{2} \tag{4.4}
\end{gather*}
$$

- a nonincreasing function $\Theta_{n}:(0, \infty) \rightarrow[0, \infty)$ satisfying

$$
\begin{equation*}
\Theta_{n}(r) \geq r^{-\gamma}\left\|g_{n}-\mathbf{P}_{r, g_{n}}\right\|_{L^{\infty}\left(B_{r}\right)} \quad \text { for every } r \in(0, \infty) \text { and } n \geq 2 \tag{4.5}
\end{equation*}
$$

with the properties that $r_{n} \rightarrow 0$ as $n \rightarrow \infty$ and

$$
r_{n}^{-\gamma}\left\|g_{n}-\mathbf{P}_{r_{n}, g_{n}}\right\|_{L^{\infty}\left(B_{r_{n}}\right)} \geq \frac{1}{2} \Theta_{n}\left(r_{n}\right) \geq \frac{n}{4}
$$

We define

$$
v_{n}(x)=\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\gamma}\left[g_{n}\left(r_{n} x\right)-\mathbf{P}_{r_{n}, g_{n}}\left(r_{n} x\right)\right]
$$

so that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2} \tag{4.6}
\end{equation*}
$$

In addition, by a change of variable, we get

$$
\begin{equation*}
\int_{B_{1}} v_{n}(x) d x=\int_{B_{1}} v_{n}(x) x_{i} d x=0 \quad \text { for every } i \in\{1, \ldots, N\} \tag{4.7}
\end{equation*}
$$

Since $\Theta_{n}$ is nonincreasing and $\gamma>1$ then see e.g. 30, [57, inequality (4.5) always implies that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{\gamma} \quad \text { for every } R \geq 1 \tag{4.8}
\end{equation*}
$$

for some constant $C=C(N, \gamma)$.
From (4.4), we deduce that

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{n}} v_{n}+\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{1-\gamma} T^{r_{n}, g_{n}} \cdot \mathcal{L}_{\bar{K}_{n}} x=\bar{f}_{n} \quad \text { in } B_{2 / r_{n}}, \tag{4.9}
\end{equation*}
$$

where $\bar{K}_{n}(x, y):=r_{n}^{N+2 s} K_{n}\left(r_{n} x, r_{n} y\right)$ and $\bar{f}_{n}(x)=r_{n}^{2 s} f_{n}\left(r_{n} x\right)$. Then, since $\mathcal{A}_{\bar{K}_{n}}(x, r, \theta)=\mathcal{A}_{K_{n}}\left(r_{n} x, r_{n} r, \theta\right)$ and $\mathcal{A}_{K_{n}} \in C^{0, \alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, we get

$$
\left|\mathcal{A}_{\bar{K}_{n}}(x, r, \theta)-\mathcal{A}_{K_{n}}(0,0, \theta)\right| \leq C \min \left(r_{n}^{\alpha}(|x|+r)^{\alpha}, 1\right) \quad \text { for all } x \in \mathbb{R}^{N}, \theta \in S^{N-1} \text { and } r>0
$$

Moreover, recalling Definition 1.8, we have $\mathcal{A}_{o, K_{n}}(0,0, \theta)=0$ for all $\theta \in S^{N-1}$. Letting $a_{n}(\theta):=$ $\mathcal{A}_{\bar{K}_{n}}(0,0, \theta)$ and $\bar{K}_{n}^{\prime}(x, y):=r_{n}^{-\alpha}\left(\bar{K}_{n}(x, y)-\mu_{a_{n}}(x, y)\right)$, we immediately see that

$$
\begin{equation*}
\left|\bar{K}_{n}(x, y)-\mu_{a_{n}}(x, y)\right| \leq C \min \left(r_{n}^{\alpha}(|x|+|y|)^{\alpha}, 1\right) \mu_{1}(x, y) \tag{4.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\bar{K}_{n}^{\prime}(x, y)\right| \leq C(|x|+|y|)^{\alpha} \mu_{1}(x, y) \tag{4.11}
\end{equation*}
$$

Since $\mathcal{L}_{\mu_{a_{n}}} x_{i}=0$ on $\mathbb{R}^{N}$, we can rewrite (4.9) as

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{n}} v_{n}+\Theta_{n}\left(r_{n}\right)^{-1} \sum_{i=1}^{N} \bar{T}_{n}^{i} \mathcal{L}_{\bar{K}_{n}^{\prime}} x_{i}=r_{n}^{-\gamma} \Theta_{n}\left(r_{n}\right)^{-1} \bar{f}_{n} \quad \text { in } B_{2 / r_{n}}, \tag{4.12}
\end{equation*}
$$

where (recall (4.1)) $\bar{T}_{n}^{i}:=r_{n}^{1+\alpha-\gamma} T_{i}^{r_{n}, g_{n}}$. Note that $x_{i} \in L_{(\alpha+2 s) / 2}\left(\mathbb{R}^{N}\right)$, provided $\alpha \in(0,2 s-1)$ and $\left[x_{i}\right]_{C^{0,1}\left(\mathbb{R}^{N}\right)} \leq 1$. Clearly by (4.3),

$$
\begin{equation*}
\left\|\bar{f}_{n}\right\|_{\mathcal{M}_{\beta}} \leq r_{n}^{2 s-\beta} \quad \text { and } \quad\left|\bar{T}_{n}^{i}\right| \leq c_{0} r_{n}^{1+\alpha-\gamma-\varrho} \leq c_{0} \tag{4.13}
\end{equation*}
$$

Since $\bar{K}_{n} \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, 0, Q_{\infty}\right)$ and $\Theta_{n}\left(r_{n}\right) \rightarrow \infty$ as $n \rightarrow \infty$, applying Corollary 3.6 to (4.12) and using (4.13) together with (4.11), we find that $v_{n}$ is bounded in $C_{l o c}^{1-\delta}\left(\mathbb{R}^{N}\right)$, for all $\delta \in(0,1)$. Hence, provided $\delta$ is small, there exists $v \in C_{l o c}^{s+\delta}\left(\mathbb{R}^{N}\right)$ such that, up to a subsequence, $v_{n} \rightarrow v$ in $C_{l o c}^{0}\left(\mathbb{R}^{N}\right)$. Hence by (4.8), up to a subsequence, $v_{n}$ converges strongly, in $L_{s}\left(\mathbb{R}^{N}\right)$, to $v \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$. Moreover, by (4.6), we deduce that

$$
\begin{equation*}
\|v\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{2} \quad \text { and } \quad \int_{B_{1}} v(x) d x=\int_{B_{1}} v(x) x_{i} d x=0 \quad \text { for every } i \in\{1, \ldots, N\} \tag{4.14}
\end{equation*}
$$

In addition, passing to the limit in (4.8), we have

$$
\begin{equation*}
\|v\|_{L^{\infty}\left(B_{R}\right)} \leq R^{\gamma} \quad \text { for every } R \geq 1 \tag{4.15}
\end{equation*}
$$

We observe that $a_{n}$ satisfies (3.1) for all $n$. By (4.10), Lemma 2.2 and Lemma 3.1, we can pass to the limit in (4.12), to get $\mathcal{L}_{\mu_{b}} v=0 \quad$ in $\mathbb{R}^{N}$, where $b$ is the weak-star limit of $a_{n}$. Now, since $v \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$ and satisfies (4.15), by Lemma 3.2 we deduce that $v$ is an affine function, because $\gamma<2 s$. This is clearly in contradiction with (4.14).

A first consequence of the previous result is the
Corollary 4.2. Let $s \in(1 / 2,1), \beta \in[0,2 s-1), N \geq 1$ and $\kappa>0$. Let $\alpha \in(0,2 s-1)$ and $\varrho \in[0, \alpha)$. Let $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, \alpha, Q_{\infty}\right), g \in H^{s}\left(B_{2}\right) \cap C^{0,1-\varrho}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}_{\beta}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} g=f \quad \text { in } B_{2} \tag{4.16}
\end{equation*}
$$

Then, there exists $C>0$, only depending on $s, N, \beta, \alpha, \kappa, \varrho$, such that

$$
\|g\|_{C^{1, \min (1+\alpha-\varrho, 2 s-\beta)-1}\left(B_{1}\right)} \leq C\left(\|g\|_{C^{0,1-\varrho\left(\mathbb{R}^{N}\right)}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Proof. Put $A:=\|g\|_{C^{0,1-\varrho}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}$. We define $G_{z}(x):=g(x+z)-g(z)$, for $z \in B_{1}$. Since $G_{z}(0)=0$, we have $\left|T^{r, G_{z}}\right| \leq C(\varrho, N) r^{-\varrho}[g]_{C^{0,1-\varrho\left(\mathbb{R}^{N}\right)}} \leq C A$, for $r>0$. Obviously, $\mathcal{L}_{K_{z}} G_{z}=f(\cdot+z)$ in $B_{1}$, where $K_{z}(x, y)=K(x+z, y+z)$. We then apply Proposition4.1 to get a constant $C>0$, only depending on $s, N, \beta, \alpha, \kappa, \varrho$, such that

$$
\sup _{r>0} r^{-\gamma}\left\|G_{z}-\mathbf{P}_{r, G_{z}}\right\|_{L^{\infty}\left(B_{r}\right)} \leq C A
$$

where $\gamma:=\min (1+\alpha-\varrho, 2 s-\beta)>1$. By a well known iteration argument (see e.g [57]), we find that

$$
|g(x)-g(z)-T(z) \cdot(x-z)| \leq C A|x-z|^{\gamma} \quad \text { for every } x, z \in B_{1 / 2}
$$

for some $T$, satisfying $\|T\|_{L^{\infty}\left(B_{1}\right)} \leq C A$. Since $\gamma>1$, then $\nabla u(z)=T(z)$. By a classical extension theorem (see e.g. 60] [Page 177], we deduce that $u \in C^{1, \gamma-1}\left(\overline{B_{1 / 2}}\right)$. Moreover

$$
\|u\|_{C^{1, \gamma-1}\left(\overline{B_{1 / 2}}\right)} \leq C A
$$

By a bootstrap argument, we have the following result.
Theorem 4.3. Let $s \in(1 / 2,1), \beta \in[0,2 s-1)$, $\alpha \in(0,2 s-1)$ and $\kappa>0$. Let $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$, $u \in H^{s}\left(B_{2}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$ and $V, f \in \mathcal{M}_{\beta}$ such that

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} .
$$

Then

$$
\|u\|_{C^{1, \min (\alpha, 2 s-\beta-1)}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

where $C>0$ only depends on $s, N, \kappa, \alpha, \beta$ and $\|V\|_{\mathcal{M}_{\beta}}$.
Proof. Since $2 s-\beta>1$, by [30], for every $\varrho \in(0,1)$, there exists $C=C\left(N, s, \beta, \alpha,\|V\|_{\mathcal{M}_{\beta}}, \varrho\right)>0$ such that

$$
\|u\|_{C^{1-\varrho}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Using Lemma 2.3 ( $i$, we apply first Corollary 4.2 to get

$$
\left\|\varphi_{1 / 2} u\right\|_{C^{0,1}\left(B_{1 / 8}\right)} \leq C\left(\left\|\varphi_{1 / 2} u\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\varphi_{1 / 2} u\right\|_{C^{1-\varrho\left(\mathbb{R}^{N}\right)}}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\left\|\varphi_{2} V u\right\|_{\mathcal{M}_{\beta}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Therefore,

$$
\|u\|_{C^{0,1}\left(B_{2}-3\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

We then apply once more Corollary 4.2 (with $\varrho=0$ ) and use Lemma $2.3(i)$ to obtain
$\left\|\varphi_{2^{-4}} u\right\|_{C^{1, \min (2 s-\beta-1, \alpha)}\left(B_{2}-8\right)} \leq C\left(\left\|\varphi_{2^{-4}} u\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\varphi_{2^{-4}} u\right\|_{C^{0,1}\left(\mathbb{R}^{N}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\left\|\varphi_{2^{-4}} V u\right\|_{\mathcal{M}_{\beta}}+\|f\|_{\mathcal{M}_{\beta}}\right)$,
so that

$$
\|u\|_{C^{1, \min (2 s-\beta-1, \alpha)}\left(B_{2}-8\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

with $C$ as in the statement of the theorem. After a covering argument, we obtain the result.

REGULARITY RESULTS FOR NONLOCAL EQUATIONS AND APPLICATIONS

## 5. Schauder estimates

Here and in the following, given $u \in C_{l o c}^{2 s+\alpha}\left(\mathbb{R}^{N}\right)$, with $\alpha>0$, we let

$$
\begin{equation*}
\delta^{e} u(x, r, \theta):=\frac{1}{2}(2 u(x)-u(x+r \theta)-u(x-r \theta)), \quad \delta^{o} u(x, r, \theta):=\frac{1}{2}(u(x+r \theta)-u(x-r \theta)) . \tag{5.1}
\end{equation*}
$$

For $A \in C^{m, \alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, we define

$$
\begin{equation*}
\mathcal{E}_{A, u}^{s}(x):=\int_{S^{N-1}} \int_{0}^{\infty} \delta^{e} u(x, r, \theta) A(x, r, \theta) r^{-1-2 s} d r d \theta \tag{5.2}
\end{equation*}
$$

and for $B \in \mathcal{C}_{\min \left(1, \alpha+(2 s-1)_{+}\right)}^{m}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, we define

$$
\begin{equation*}
\mathcal{O}_{B, u}^{s}(x):=\int_{S^{N-1}} \int_{0}^{\infty} \delta^{o} u(x, r, \theta) B(x, r, \theta) r^{-1-2 s} d r d \theta \tag{5.3}
\end{equation*}
$$

 we get

$$
\begin{equation*}
\frac{1}{2} \int_{\mathbb{R}^{2 N}}(u(x)-u(y))(\psi(x)-\psi(y)) K(x, y) d x d y=\int_{\mathbb{R}^{N}} \psi(x) \mathcal{E}_{\mathcal{A}_{e, K}, u}^{s}(x) d x+\int_{\mathbb{R}^{N}} \psi(x) \mathcal{O}_{\mathcal{A}_{o, K}, u}^{s}(x) d x \tag{5.4}
\end{equation*}
$$

where

$$
\mathcal{A}_{e, K}(x, r, \theta):=\frac{1}{2}\left(\mathcal{A}_{K}(x, r, \theta)+\mathcal{A}_{K}(x, r,-\theta)\right), \quad \mathcal{A}_{o, K}(x, r, \theta):=\frac{1}{2}\left(\mathcal{A}_{K}(x, r, \theta)-\mathcal{A}_{K}(x, r,-\theta)\right)
$$

We have the following result which will be proved in Section 7
Lemma 5.1. Let $s \in(0,1)$ and $\alpha \in(0,1)$. Let $\kappa>0, m \in \mathbb{N}, A \in C^{m+\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$ and $B \in \mathcal{C}_{\tau}^{m}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, with $\tau:=\min \left(\alpha+(2 s-1)_{+}, 1\right)$.

- Let $u \in C^{2 s+\alpha+m}\left(\mathbb{R}^{N}\right)$ and $2 s \neq 1$. If $2 s+\alpha<1$ or $1<2 s+\alpha<2$, then

$$
\left\|\mathcal{E}_{A, u}^{s}\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)} \leq C\|A\|_{C^{m+\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha+m}\left(\mathbb{R}^{N}\right)}
$$

and

$$
\left\|\mathcal{O}_{B, u}^{s}\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)} \leq C\|B\|_{\mathcal{C}_{\tau}^{m}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha+m}\left(\mathbb{R}^{N}\right)}
$$

with $C=C(N, s, \alpha, m)$.

- Let $u \in C^{1+\alpha+m+\varepsilon}\left(\mathbb{R}^{N}\right)$, for some $\varepsilon \in(0,1-\alpha)$. If $2 s=1$ and $B \in \mathcal{C}_{\alpha+\varepsilon}^{m}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, then

$$
\left\|\mathcal{E}_{A, u}^{s}\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)} \leq C\|A\|_{C^{m, \alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha+m+\varepsilon}\left(\mathbb{R}^{N}\right)}
$$

and

$$
\left\|\mathcal{O}_{B, u}^{s}\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)} \leq C\|B\|_{\mathcal{C}_{\alpha+\varepsilon}^{m}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha+m+\varepsilon}\left(\mathbb{R}^{N}\right)}
$$

with $C=C(N, \alpha, m, \varepsilon)$.

- Let $u \in C^{2 s+\alpha+m}\left(\mathbb{R}^{N}\right)$ and $2 s+\alpha>2$. If $A, B \in C^{m+2 s-1+\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$ and $B \in \mathcal{C}_{2 s+\alpha-2}^{m+1}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, then

$$
\left\|\mathcal{E}_{A, u}^{s}\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)} \leq C\|A\|_{C^{m+2 s-1+\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha+m}\left(\mathbb{R}^{N}\right)}
$$

and
$\left\|\mathcal{O}_{B, u}^{s}\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)} \leq C\left(\|B\|_{\mathcal{C}_{2 s+\alpha-2}^{m+1}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}+\|B\|_{C^{m+2 s-1+\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\right)\|u\|_{C^{2 s+\alpha+m}\left(\mathbb{R}^{N}\right)}$, with $C=C(N, s, \alpha, m)$.

We remark that under the assumptions on $A$ and $B$, for $2 s=1$, the first assertion of Lemma 5.1 does not in general hold.
5.1. Schauder estimates. The following result is intended to the $C^{2 s+\alpha}$ regularity estimates, for $2 s+\alpha \notin \mathbb{N}$. To deal with the case $2 s+\alpha>2$, we look for optimal growth estimate of the difference between $u$ a second order polynomial that is close to $u$ in the $L^{2}$-norm.
For $g \in L^{2}\left(B_{r}\right)$ and $i, j=1, \ldots, N$, we define

$$
T_{i j}^{r, g}=\frac{1}{\left\|y_{i} y_{j}\right\|_{L^{2}\left(B_{r}\right)}^{2}} \int_{B_{r}} y_{i} y_{j} g(y) d y
$$

and

$$
\mathbf{Q}_{r, g}(x)=\sum_{i, j=1}^{N} T_{i j}^{r, g} x_{i} x_{j}
$$

We note that $\mathbf{Q}_{r, g}$ is nothing but the $L^{2}\left(B_{r}\right)$-projection of $g$ on the space of homogeneous quadratic polynomials. We now state the main result of this section.

Proposition 5.2. Let $N \geq 1, s \in(0,1), \kappa>0, \alpha \in(0,1), \beta \in(0, \alpha)$ and $\varepsilon>0$. Let $K \in$ $\widetilde{\mathscr{K}_{\tau}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$, with $\tau:=\min \left(\alpha+(2 s-1)_{+}, 1\right)$. Let $f \in C^{\alpha}\left(\mathbb{R}^{N}\right)$ and $g \in C^{2 s+\beta}\left(\mathbb{R}^{N}\right)$, for $2 s+\beta \notin \mathbb{N}$, such that

$$
\mathcal{L}_{K} g=f \quad \text { in } B_{2},
$$

(i) If $1<2 s+\alpha<2$ and $2 s \neq 1$, then there exists $C=C(N, s, \kappa, \alpha, \beta)>0$ such that

$$
\sup _{r>0} r^{-(2 s-1+\alpha-\beta)}[\nabla g]_{C^{\beta}\left(B_{r}\right)} \leq C\left(\|g\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)}+[f]_{C^{\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

(ii) If $2 s+\alpha>2>2 s+\beta \geq 1+\alpha, K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, 2 s-1+\alpha, Q_{\infty}\right)$ and $g(0)=|\nabla g(0)|=0$, then there exists $C=C(N, s, \kappa, \alpha, \beta)>0$ such that

$$
\sup _{r>0} r^{-(2 s-1+\alpha-\beta)}\left[\nabla g-\nabla \mathbf{Q}_{r, g}\right]_{C^{\beta}\left(B_{r}\right)} \leq C\left(\|g\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)}+[f]_{C^{\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

(iii) If $2 s=1$ and $K \in \widetilde{\mathscr{K}_{\alpha+\varepsilon}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$, then there exists $C=C(N, s, \kappa, \alpha, \beta, \varepsilon)>0$ such that

$$
\sup _{r>0} r^{-(2 s-1+\alpha-\beta)}[\nabla g]_{C^{\beta}\left(B_{r}\right)} \leq C\left(\|g\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)}+[f]_{C^{\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

(iv) If $2 s+\alpha<1$, then there exists $C=C(N, s, \kappa, \alpha, \beta)>0$ such that

$$
\sup _{r>0} r^{-(\alpha-\beta)}[g]_{C^{2 s+\beta}\left(B_{r}\right)} \leq C\left(\|g\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)}+[f]_{C^{\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

Proof. We start with (i). Assume that the assertion in $(i)$ does not hold, then arguing as in the proof of Proposition 3.3, we can find sequences

- $r_{n}>0, K_{n} \in \widetilde{\mathscr{K}_{\alpha+2 s-1}^{s}}\left(\kappa, \alpha, Q_{\infty}\right) g_{n} \in C^{2 s+\beta}\left(\mathbb{R}^{N}\right)$ and $f_{n} \in C^{\alpha}\left(\mathbb{R}^{N}\right)$ with $\left\|g_{n}\right\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)}+$ $\left[f_{n}\right]_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq 1$,

$$
\begin{equation*}
\mathcal{L}_{K_{n}} g_{n}=f_{n} \quad \text { in } B_{2} \tag{5.5}
\end{equation*}
$$

- $\Theta_{n}:(0, \infty) \rightarrow[0, \infty)$, nonincreasing,
with the properties that $r_{n} \rightarrow 0$ as $n \rightarrow \infty$,

$$
\begin{equation*}
\Theta_{n}(r) \geq r^{-(2 s-1+\alpha-\beta)}\left[\nabla g_{n}\right]_{C^{\beta}\left(B_{r}\right)} \quad \text { for every } r>0 \text { and } n \geq 2 \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{n}^{-(2 s-1+\alpha-\beta)}\left[\nabla g_{n}\right]_{C^{\beta}\left(B_{r_{n}}\right)} \geq \frac{1}{2} \Theta_{n}\left(r_{n}\right) \geq \frac{n}{4} \tag{5.7}
\end{equation*}
$$

We define

$$
u_{n}(x):=\frac{1}{r_{n}^{2 s+\alpha} \Theta_{n}\left(r_{n}\right)} g\left(r_{n} x\right)
$$

By (5.6), for $R \geq 1$, we have

$$
\left[\nabla u_{n}\right]_{C^{\beta}\left(B_{R}\right)}=\frac{r_{n}^{1+\beta}}{r_{n}^{2 s+\alpha} \Theta_{n}\left(r_{n}\right)}\left[\nabla g_{n}\right]_{C^{\beta}\left(B_{r_{n} R}\right)} \leq R^{2 s-1+\alpha-\beta} \frac{\Theta_{n}\left(R r_{n}\right)}{\Theta_{n}\left(r_{n}\right)}
$$

Hence by the monotonicity of $\Theta_{n}$, we have

$$
\begin{equation*}
\left[\nabla u_{n}\right]_{C^{\beta}\left(B_{R}\right)} \leq R^{2 s-1+\alpha-\beta} \quad \text { for all } R \geq 1 \tag{5.8}
\end{equation*}
$$

In addition, by (5.7), we get

$$
\left[\nabla u_{n}\right]_{C^{\beta}\left(B_{1}\right)} \geq \frac{1}{2}
$$

This then implies that there exists $x_{n}, h_{n} \in B_{1}$, with $h_{n} \neq 0$, such that

$$
\begin{equation*}
\left|h_{n}\right|^{-\beta}\left|\nabla u_{n}\left(x_{n}+h_{n}\right)-\nabla u_{n}\left(x_{n}\right)\right| \geq \frac{1}{4} \tag{5.9}
\end{equation*}
$$

We define the new sequence

$$
v_{n}(x):=\frac{u_{n}\left(x_{n}+\left|h_{n}\right| x\right)-u_{n}\left(x_{n}\right)-\left|h_{n}\right| \nabla u_{n}\left(x_{n}\right) \cdot x}{\left|h_{n}\right|^{1+\beta}} .
$$

By construction, we have that

$$
\begin{equation*}
v_{n}(0)=\left|\nabla v_{n}(0)\right|=0 \tag{5.10}
\end{equation*}
$$

and by (5.9),

$$
\begin{equation*}
\left|\nabla v_{n}\left(h_{n} /\left|h_{n}\right|\right)\right| \geq \frac{1}{4} \tag{5.11}
\end{equation*}
$$

Moreover by (5.8), for $R \geq 1$ and $x, y \in B_{R}$,

$$
\begin{aligned}
\left|\nabla v_{n}(x)-\nabla v_{n}(y)\right| & =\left|h_{n}\right|^{-\beta}\left|\nabla u_{n}\left(x_{n}+\left|h_{n}\right| x\right)-\nabla u_{n}\left(x_{n}+\left|h_{n}\right| y\right)\right| \\
& \leq\left|h_{n}\right|^{-\beta}\left|h_{n}\right|^{\beta}|x-y|^{\beta}(1+R)^{2 s-1+\alpha-\beta} \leq 2^{2 s-1+\alpha-\beta}|x-y|^{\beta} R^{2 s-1+\alpha-\beta}
\end{aligned}
$$

Combining this with (5.10), we get, for all $R \geq 1$,

$$
\begin{equation*}
\left\|\nabla v_{n}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{2 s-1+\alpha} \tag{5.12}
\end{equation*}
$$

and

$$
\left\|v_{n}\right\|_{C^{1, \beta}\left(B_{R}\right)} \leq C R^{2 s+\alpha}
$$

for some $C=C(s, \alpha, \beta)$. This latter estimate implies that there exists $v \in C_{l o c}^{1, \beta}\left(\mathbb{R}^{N}\right)$ such that, up to a subsequence,

$$
\begin{equation*}
v_{n} \rightarrow v \quad \text { in } C_{l o c}^{1}\left(\mathbb{R}^{N}\right) \tag{5.13}
\end{equation*}
$$

Moreover by (5.11) and (5.10), there exists $e \in S^{N-1}$ such that

$$
\begin{equation*}
|\nabla v(e)| \geq \frac{1}{4} \quad \text { and } \quad \nabla v(0)=0 \tag{5.14}
\end{equation*}
$$

We shall show that $\nabla v \equiv 0$ on $\mathbb{R}^{N}$, which leads to a contradiction. Indeed, given $h \in \mathbb{R}^{N}$, we define $w_{n}(x)=\left(v_{n}\right)_{h, 0}(x)=v_{n}(x+h)-v_{n}(x)$. It follows from (5.12) and the fundamental theorem of calculus that

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{2 s-1+\alpha} \quad \text { for every } R \geq 1 \tag{5.15}
\end{equation*}
$$

where here and in the following of the proof, the letter $C$ is a positive constant only depending on $h, \kappa, N, s, \beta$ and $\alpha$. We put $\rho_{n}:=r_{n}\left|h_{n}\right|, z_{n}:=r_{n} x_{n}$ and we define

$$
\begin{gathered}
\bar{K}_{n}(x, y)=\rho_{n}^{N+2 s} K_{n}\left(z_{n}+\rho_{n} x, z_{n}+\rho_{n} y\right) \\
\bar{K}_{n}^{\prime}(x, y):=r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta}\left[\bar{K}_{n}(x+h, y+h)-\bar{K}_{n}(x, y)\right] \\
U_{n}(x):=r_{n}^{-2 s}\left|h_{n}\right|^{-1} g_{n}\left(z_{n}+\rho_{n} x+\rho_{n} h\right)
\end{gathered}
$$

and

$$
\bar{f}_{n}(x):=r_{n}^{-\alpha-2 s}\left|h_{n}\right|^{-\beta-1} \rho_{n}^{2 s}\left[f_{n}\left(z_{n}+\rho_{n} x+\rho_{n} h\right)-f_{n}\left(z_{n}+\rho_{n} x\right)\right]
$$

For $h \in \mathbb{R}^{N} \backslash\{0\}$, we let $n$ large so that $z_{n}+h \in B_{1 / 2 r_{n}}$, by changing variables and using (5.5), we then have that

$$
\mathcal{L}_{\bar{K}_{n}} w_{n}+\frac{1}{\Theta_{n}\left(r_{n}\right)} \mathcal{L}_{\bar{K}_{n}^{\prime}} U_{n}=\frac{1}{\Theta_{n}\left(r_{n}\right)} \bar{f}_{n} \quad \text { in } B_{1 / 2 r_{n}}
$$

Therefore by (5.4),

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{n}} w_{n}=F_{n} \quad \text { in } B_{1 / 2 r_{n}} \tag{5.16}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{n}(x):=\frac{1}{\Theta_{n}\left(r_{n}\right)} \bar{f}_{n}-\frac{1}{\Theta_{n}\left(r_{n}\right)} \mathcal{E}_{\mathcal{A}_{e, K_{n}^{\prime}}, U_{n}}^{s}-\frac{1}{\Theta_{n}\left(r_{n}\right)} \mathcal{O}_{\mathcal{A}_{o, K_{n}^{\prime}}, U_{n}}^{s} \tag{5.17}
\end{equation*}
$$

By a change of variable, we get

$$
\mathcal{E}_{\mathcal{A}_{e, \bar{K}_{n}^{\prime}}, U_{n}}^{s}(x)=\left|h_{n}\right|^{2 s-1} \int_{S^{N-1}} \int_{0}^{\infty} \delta^{e} g_{n}\left(z_{n}+\rho_{n} x+\rho_{n} h, t, \theta\right) \mathcal{A}_{e, \bar{K}_{n}^{\prime}}\left(x, t / \rho_{n}, \theta\right) t^{-1-2 s} d t d \theta
$$

and

$$
\mathcal{O}_{\mathcal{A}_{o, K_{n}^{\prime}}^{s}, U_{n}}(x)=\left|h_{n}\right|^{2 s-1} \int_{S^{N-1}} \int_{0}^{\infty} \delta^{o} g_{n}\left(z_{n}+\rho_{n} x+\rho_{n} h, t, \theta\right) \mathcal{A}_{o, \bar{K}_{n}^{\prime}}\left(x, t / \rho_{n}, \theta\right) t^{-1-2 s} d t d \theta
$$

We recall that

$$
\mathcal{A}_{\bar{K}_{n}}(x, r, \theta):=\mathcal{A}_{K_{n}}\left(z_{n}+\rho_{n} x, \rho_{n} r, \theta\right)
$$

and

$$
\begin{equation*}
\mathcal{A}_{\bar{K}_{n}^{\prime}}(x, r, \theta):=r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta}\left\{\mathcal{A}_{K_{n}}\left(z_{n}+\rho_{n} x+\rho_{n} h, \rho_{n} r, \theta\right)-\mathcal{A}_{K_{n}}\left(z_{n}+\rho_{n} x, \rho_{n} r, \theta\right)\right\} \tag{5.18}
\end{equation*}
$$

Since $K_{n} \in \widetilde{\mathbb{K}_{\alpha+2 s-1}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$ (recall Definition 1.8), for all $x, y \in \mathbb{R}^{N}, \theta \in S^{N-1}$ and $r>0$,

$$
\begin{align*}
& \left|\mathcal{A}_{o, K_{n}}(x, r, \theta)\right| \leq \frac{1}{\kappa} \min (r, 1)^{2 s-1+\alpha} \\
& \left|\mathcal{A}_{o, K_{n}}(x, r, \theta)-\mathcal{A}_{o, K_{n}}(y, r, \theta)\right| \leq \frac{1}{\kappa} \min (r,|x-y|)^{2 s-1+\alpha} \tag{5.19}
\end{align*}
$$

and

$$
\left|\mathcal{A}_{K_{n}}(x, r, \theta)-\mathcal{A}_{K_{n}}(y, r, \theta)\right| \leq \frac{1}{\kappa}|x-y|^{\alpha}
$$

Therefore,

$$
\left|\mathcal{A}_{e, \bar{K}_{n}^{\prime}}(x, r, \theta)\right| \leq C\left|h_{n}\right|^{\beta} \leq C \quad \text { for all } x \in \mathbb{R}^{N}, r>0 \text { and } \theta \in S^{N-1}
$$

Consequently, since $\left\|g_{n}\right\|_{C^{2 s+\beta}} \leq 1$ and recalling (5.1), we have that

$$
\left|\delta^{e} g_{n}\left(z_{n}+\rho_{n} x+\rho_{n} h, t, \theta\right)\right| \leq C \min \left(1, t^{2 s+\beta}\right)
$$

and thus

$$
\begin{equation*}
\left\|\mathcal{E}_{\mathcal{A}_{e, \bar{K}_{n}^{\prime}}, U_{n}}^{s}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C \tag{5.20}
\end{equation*}
$$

Moreover by (5.19), for all $x \in \mathbb{R}^{N}, r>0$ and $\theta \in S^{N-1}$, we have

$$
\left|\mathcal{A}_{o, \bar{K}_{n}^{\prime}}(x, r, \theta)\right| \leq C r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \min \left(\rho_{n}, \rho_{n} r\right)^{2 s-1+\alpha}
$$

Since $\left|h_{n}\right| \leq 1$ and $\left\|g_{n}\right\|_{C^{0,1}\left(\mathbb{R}^{N}\right)} \leq 1$ (recalling (5.1)), the above estimate implies that

$$
\begin{aligned}
\left|\mathcal{O}_{\mathcal{A}_{o, K_{n}^{\prime}}, U_{n}}^{s}(x)\right| & \leq C r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \int_{0}^{\infty} \min (1, t) \min \left(\rho_{n}, t\right)^{2 s-1+\alpha} t^{-1-2 s} d t \\
& \leq C r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \int_{0}^{\rho_{n}} t^{2 s+\alpha} t^{-1-2 s} d t \\
& +C r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \rho_{n}^{2 s-1+\alpha} \int_{\rho_{n}}^{1} t^{-2 s} d t+C r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \rho_{n}^{2 s-1+\alpha} \int_{1}^{\infty} t^{-1-2 s} d t
\end{aligned}
$$

Using that $2 s>1$ and recalling that $\rho_{n}=r_{n}\left|h_{n}\right|$, we then conclude that

$$
\begin{equation*}
\left\|\mathcal{O}_{\mathcal{A}_{o, \bar{K}_{n}^{\prime}}^{s}, U_{n}}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C \tag{5.21}
\end{equation*}
$$

Because $\left[f_{n}\right]_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq 1$, it is plain that

$$
\begin{equation*}
\left\|\bar{f}_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C \tag{5.22}
\end{equation*}
$$

Recalling (5.17), it follow from (5.20), (5.21) and (5.22) that

$$
\begin{equation*}
\left\|F_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq \frac{C}{\Theta_{n}\left(r_{n}\right)} \tag{5.23}
\end{equation*}
$$

In view of (5.13) and (5.15), for every $h \in \mathbb{R}^{N}$, we have that

$$
\begin{equation*}
w_{n}=\left(v_{n}\right)_{h, 0} \rightarrow v_{h, 0} \quad \text { in } L_{s}\left(\mathbb{R}^{N}\right) \cap H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \tag{5.24}
\end{equation*}
$$

Letting $a_{n}(\theta):=\mathcal{A}_{\bar{K}_{n}}\left(z_{n}, 0, \theta\right)$, we have that

$$
\left|\mathcal{A}_{\bar{K}_{n}}(x, r, \theta)-a_{n}(\theta)\right| \leq C \min \left(\rho_{n}^{\alpha}(|x|+r)^{\alpha}, 1\right) \quad \text { for all } x \in \mathbb{R}^{N}, r>0 \text { and } \theta \in S^{N-1}
$$

so that

$$
\begin{equation*}
\left|\bar{K}_{n}(x, y)-\mu_{a_{n}}(x, y)\right| \leq C \min \left(\rho_{n}^{\alpha}(|x|+|x-y|)^{\alpha}, 1\right) \mu_{1}(x, y) \quad \text { for all } x \neq y \in \mathbb{R}^{N} \tag{5.25}
\end{equation*}
$$

Moreover $a_{n}$ satisfies (3.1) for all $n$. Therefore in view of Lemma 3.1, Lemma 2.2, (5.24), (5.25) and (5.23), passing to the limit in (5.16), we deduce that

$$
\begin{equation*}
\mathcal{L}_{\mu_{b}} v_{h, 0}=0 \quad \text { in } \mathbb{R}^{N} \tag{5.26}
\end{equation*}
$$

were $b$ is the weak-star limit of $a_{n}$ in $L^{\infty}\left(S^{N-1}\right)$. Furthermore by (5.12),

$$
\left\|v_{h, 0}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{2 s-1+\alpha}
$$

Thanks to (5.26) and since $2 s-1+\alpha<1$, we can apply Lemma 3.2 to get a constant $c=$ $c(h, \alpha, \beta, N, s, \kappa)$ such that $v_{h, 0}(x)=v(x+h)-v(x)=c$ for all $x, h \in \mathbb{R}^{N}$. Hence, since $\nabla v(0)=0$, we find that $\nabla v(h)=0$ for all $h \in \mathbb{R}^{N}$. This contradicts the first inequality in (5.14). The proof of $(i)$ is thus finished.

The proof of $(i i)$ is similar to the one of $(i)$, we therefore give a sketch below, emphasizing the main differences. Indeed, following the proof, we put

$$
\bar{u}_{n}(x)=\frac{1}{r_{n}^{2 s+\alpha} \Theta_{n}\left(r_{n}\right)}\left\{g_{n}\left(r_{n} x\right)-\mathbf{Q}_{r_{n}, g_{n}}\left(r_{n} x\right)\right\}
$$

with $\Theta_{n}(r)$ is a nonincreasing function as above, with $g_{n}$ replaced with $g_{n}-\mathbf{Q}_{r, g_{n}}$. From the definition of $\mathbf{Q}_{r, g_{n}}$, the monotonicity of $\Theta_{n}$ and the fact that $2 s-1+\alpha>1$, we then get $\left\|\nabla \bar{u}_{n}\right\|_{C^{\beta}\left(B_{R}\right)} \leq$ $C R^{2 s-1+\alpha}$ for all $R \geq 1$. On the other hand, there are $x_{n} \in B_{1}$ and $h_{n} \in B_{1} \backslash\{0\}$ such that $\left|\nabla \bar{u}_{n}\left(x_{n}+h_{n}\right)-\nabla \bar{u}_{n}\left(x_{n}\right)\right|\left|h_{n}\right|^{-\beta} \geq \frac{1}{4}$. Similarly as above, we define

$$
\bar{v}_{n}(x):=\frac{\bar{u}_{n}\left(x_{n}+\left|h_{n}\right| x\right)-\bar{u}_{n}\left(x_{n}\right)-\left|h_{n}\right| \nabla \bar{u}_{n}\left(x_{n}\right) \cdot x}{\left|h_{n}\right|^{1+\beta}}
$$

so that $\left\|\nabla \bar{v}_{n}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{2 s-1+\alpha}$ for all $R \geq 1$. Moreover, $\bar{v}_{n}$ is bounded in $C_{l o c}^{1, \beta}\left(\mathbb{R}^{N}\right)$, so that $\bar{v}_{n} \rightarrow \bar{v}$ in $C_{l o c}^{1}\left(\mathbb{R}^{N}\right)$. In addition,

$$
\begin{equation*}
\nabla \bar{v}(0)=0 \quad \text { and } \quad|\nabla \bar{v}(e)| \geq \frac{1}{4}, \quad \text { for some } e \in S^{N-1} \tag{5.27}
\end{equation*}
$$

and $\|\bar{v}(\cdot+h)-\bar{v}\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{2 s-1+\alpha}$ for all $R \geq 1$. Next, letting, $\bar{w}_{n}(x)=\left(\bar{v}_{n}\right)_{h, 0}(x)=\bar{v}_{n}(x+h)-$ $\bar{v}_{n}(x)$ we find that

$$
\mathcal{L}_{\bar{K}_{n}} \bar{w}_{n}-\frac{2 \rho_{n}^{2}}{r_{n}^{2 s+\alpha}\left|h_{n}\right|^{1+\beta} \Theta_{n}\left(r_{n}\right)} \sum_{i, j=1}^{N} T_{i j}^{r_{n}, g_{n}} h_{j} \mathcal{L}_{\bar{K}_{n}} x_{i}=F_{n} \quad \text { in } B_{1 / 2 r_{n}}
$$

where $F_{n}$ is given by (5.17). By (5.4) and the fact that $\mathcal{E}_{\mathcal{A}_{e, \bar{K}_{n}}, x_{i}} \equiv 0$ on $\mathbb{R}^{N}$, we then get

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{n}} \bar{w}_{n}=F_{n}+\frac{2 \rho_{n}^{2}}{r_{n}^{2 s+\alpha}\left|h_{n}\right|^{1+\beta} \Theta_{n}\left(r_{n}\right)} \sum_{i, j=1}^{N} T_{i j}^{r_{n}, g_{n}} h_{j} \mathcal{O}_{\mathcal{A}_{o, \bar{K}_{n}}^{s}, x_{i}} \quad \text { in } B_{1 / 2 r_{n}} \tag{5.28}
\end{equation*}
$$

We start by estimating $F_{n}$ defined in (5.17). Thanks to (5.18), we have

$$
\begin{aligned}
& \mathcal{A}_{o, \bar{K}_{n}^{\prime}}(x, r, \theta)=r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \rho_{n} \int_{0}^{1} D_{x} \mathcal{A}_{o, K_{n}}\left(z_{n}+\rho_{n} x+\varrho \rho_{n} h, \rho_{n} r, \theta\right) h d \varrho \\
& =r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \rho_{n} r \int_{0}^{1}\left[D_{r} \mathcal{A}_{o, K_{n}}\left(z_{n}+\rho_{n} x+\rho_{n} h, \varrho \rho_{n} r, \theta\right)-D_{r} \mathcal{A}_{o, K_{n}}\left(z_{n}+\rho_{n} x, \varrho \rho_{n} r, \theta\right)\right] d \varrho
\end{aligned}
$$

Recall that $K_{n} \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, 1+2 s-2+\alpha, Q_{\infty}\right)$ with $1>2 s-2+\alpha>0$, and so by definition,

$$
\left\|D_{r} \mathcal{A}_{K_{n}}\right\|_{C^{2 s+\alpha-2}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}+\left\|D_{x} \mathcal{A}_{o, K_{n}}\right\|_{L_{2 s+\alpha-2}^{\infty}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)} \leq C
$$

It follows that

$$
\left|\mathcal{A}_{o, \bar{K}_{n}^{\prime}}(x, r, \theta)\right| \leq C r_{n}^{-\alpha}\left|h_{n}\right|^{-\beta} \min \left(\rho_{n} r \rho_{n}^{2 s+\alpha-2}, \rho_{n}\left(\rho_{n} r\right)^{2 s+\alpha-2}\right)
$$

From this and the fact that $\left|\delta^{o} g_{n}(y, t, \theta)\right| \leq t$, we deduce that

$$
\begin{aligned}
\left|\mathcal{O}_{\mathcal{A}_{o, \bar{K}_{n}^{\prime}}, U_{n}}(x)\right| & \leq C r_{n}^{-2 s-\alpha}\left|h_{n}\right|^{-1-\beta} \int_{0}^{\infty} \rho_{n} r \min \left(\rho_{n} r \rho_{n}^{2 s+\alpha-2}, \rho_{n}\left(\rho_{n} r\right)^{2 s+\alpha-2}\right) r^{-1-2 s} d r \\
& =C r_{n}^{-2 s-\alpha}\left|h_{n}\right|^{-1-\beta} \rho_{n}^{2 s} \int_{0}^{\infty} \min \left(t \rho_{n}^{2 s+\alpha-2}, \rho_{n} t^{2 s+\alpha-2}\right) t^{-2 s} d t \\
& \leq C r_{n}^{-2 s-\alpha}\left|h_{n}\right|^{-1-\beta} \rho_{n}^{2 s} \int_{0}^{\rho_{n}} \rho_{n}^{2 s+\alpha-2} t^{1-2 s} d t+C r_{n}^{-2 s-\alpha}\left|h_{n}\right|^{-1-\beta} \rho_{n}^{2 s} \int_{\rho_{n}}^{\infty} \rho_{n} t^{\alpha-2} d t .
\end{aligned}
$$

Therefore, $\left\|\mathcal{O}_{\mathcal{A}_{o, K_{n}^{\prime}}^{\prime}, U_{n}}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C$. By combining this with (5.20) and (5.22), we get

$$
\left\|F_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq \frac{C}{\Theta_{n}\left(r_{n}\right)}
$$

Next, we estimate the last term in the right hand in (5.28). From the first inequality in (5.19), we get

$$
\begin{equation*}
\left|\mathcal{O}_{\mathcal{A}_{o, K_{n}}, x_{i}}(x)\right| \leq C \int_{0}^{\infty} \min \left(\rho_{n} r, 1\right) r^{-2 s} d r \leq C \rho_{n}^{2 s-1} \tag{5.29}
\end{equation*}
$$

Since $g_{n}(0)=\left|\nabla g_{n}(0)\right|=0$ and $\left\|g_{n}\right\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)} \leq 1$, we then have that $\left|T_{i j}^{r_{n}, g_{n}}\right| \leq C r_{n}^{-2+2 s+\beta}$. Hence, since $2 s+\beta \geq 1+\alpha$, by (5.29), the last term in the right hand in (5.28) is bounded by $\frac{C}{\Theta_{n}\left(r_{n}\right)}$. Since $\frac{1}{\Theta_{n}\left(r_{n}\right)}$ tends to zero as $n \rightarrow \infty$, thanks to Lemma [2.2, we can pass to the limit in (5.28), to get $\underline{\mathcal{L}}_{\mu_{b}} \bar{v}_{h, 0}=0$ in $\mathbb{R}^{N}$. Hence, since $2 s-1+\alpha<2 s$, Lemma 3.2 implies that, there exist $\bar{c} \in \mathbb{R}$ and $\bar{d} \in \mathbb{R}^{N}$, only depending on $h, \alpha, \beta, N, s$ and $\kappa$, such that $\bar{v}_{h, 0}(x)=\bar{v}(x+h)-\bar{v}(x)=\bar{d} \cdot x+\bar{c}$ for all $x, h \in \mathbb{R}^{N}$. Now, since $\nabla \bar{v}(0)=0$, we find $\nabla \bar{v} \equiv 0$ on $\mathbb{R}^{N}$. This contradicts (5.27) and the proof of (ii) is finished.

The proof of (iii) follows (verbatim) the same argument as the one of (i). The fact that $K_{n} \in$ $\widetilde{K}_{\alpha+\varepsilon}^{1 / 2}\left(\kappa, \alpha, Q_{\infty}\right)$ is only needed to deduce the uniform bound $\left\|\mathcal{O}_{\mathcal{A}_{o, K_{n}^{\prime}}^{1 / 2}, U_{n}}^{1}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C$ from the uniform estimate $\left|\mathcal{A}_{o, \bar{K}_{n}^{\prime}}(x, r, \theta)\right| \leq C r_{n}^{-\alpha}\left|h_{n}\right|^{-\alpha / 2} \min \left(\rho_{n}, \rho_{n} r\right)^{\alpha+\varepsilon}$.

The proof of $(i v)$ does not differ much from the one of $(i)$. We skip the details.
As a consequence of the previous result, we have the following
Theorem 5.3. Let $s \in(0,1), N \geq 1, \kappa>0$ and $\alpha \in(0,1)$. Let $K \in \widetilde{\mathscr{K}_{\tau}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$ with $\tau=$ $\min \left(\alpha+(2 s-1)_{+}, 1\right)$. Let $u \in H^{s}\left(B_{2}\right) \cap C^{\alpha}\left(\mathbb{R}^{N}\right)$ and $f \in C^{\alpha}\left(B_{2}\right)$ satisfy

$$
\mathcal{L}_{K} u=f \quad \text { in } B_{2}
$$

(i) If $2>2 s+\alpha>1,2 s+\alpha \neq 2$ and $2 s \neq 1$, then there exists $C=C(N, s, \kappa, \alpha)>0$ such that

$$
\|u\|_{C^{2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

(ii) If $2 s+\alpha>2$ and $K \in{\widetilde{\mathcal{K}_{0}^{s}}}^{s}\left(\kappa, 2 s-1+\alpha, Q_{\infty}\right)$ then there exists $C=C(N, s, \kappa, \alpha)>0$ such that

$$
\|u\|_{C^{2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

(iii) If $2 s=1$ and $K \in \widetilde{\mathscr{K}_{\alpha+\varepsilon}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$, for some $\varepsilon>0$, then there exists $C=C(N, s, \kappa, \alpha, \varepsilon)>0$ such that

$$
\|u\|_{C^{1+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

(iv) If $2 s+\alpha<1$, then there exists $C=C(N, s, \kappa, \alpha)>0$ such that

$$
\|u\|_{C^{2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) .
$$

Proof. From [30], there exists $\beta \in(0, \alpha)$, only depending on $N, s, \kappa$ and $\alpha$ such that, if $2 s+\beta \notin \mathbb{N}$,

$$
\begin{equation*}
\|u\|_{C^{2 s+\beta}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right), \tag{5.30}
\end{equation*}
$$

for some $C=C(N, s, \alpha, \beta, \kappa)$.
Case 1: $1<2 s+\alpha<2$. For $z \in B_{1}$, we define

$$
\begin{equation*}
g_{z}(x)=u(x+z)-u(z)-\varphi_{4}(x) \nabla u(z) \cdot x \tag{5.31}
\end{equation*}
$$

which satisfies $g_{z}(0)=\left|\nabla g_{z}(0)\right|=0$. We introduce the cut-off function $\varphi_{4}$ only because the functions $x \mapsto x_{i}$, for $i=1, \ldots, N$, do not belong to $L_{s}\left(\mathbb{R}^{N}\right)$ when $2 s=1$. By construction and (5.30), we have

$$
\begin{equation*}
\left\|g_{z}\right\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)} \leq C\|u\|_{C^{2 s+\beta}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) . \tag{5.32}
\end{equation*}
$$

In addition, by Lemma 2.3 (iii),

$$
\mathcal{L}_{K_{z}} g_{z}=f(\cdot+z)-\mathcal{L}_{K_{z}} U \quad \text { in } B_{1},
$$

where $K_{z}(x, y)=K(x+z, y+z), U(x):=\varphi_{4}(x) \nabla u(z) \cdot x$. From Lemma 2.3(iii), we get

$$
\mathcal{L}_{K_{z}}\left(\varphi_{1 / 2} g_{z}\right)=\tilde{f}-\mathcal{L}_{K_{z}} U \quad \text { in } B_{1 / 8}
$$

for some function $\widetilde{f}$, satisfying

$$
\begin{equation*}
\|\widetilde{f}\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) . \tag{5.33}
\end{equation*}
$$

Using (5.4), we then have that

$$
\begin{equation*}
\mathcal{L}_{K_{z}}\left(\varphi_{1 / 2} g_{z}\right)=F \quad \text { in } B_{1 / 8} \tag{5.34}
\end{equation*}
$$

where $F:=\widetilde{f}-\mathcal{E}_{\mathcal{A}_{e, K}, U}^{s}-\mathcal{O}_{\mathcal{A}_{o, K}, U}^{s}$. Because $K_{z} \in \widetilde{\mathscr{K}_{\alpha+2 s-1}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$ for $2 s>1$ and $K_{z} \in$ $\widetilde{\mathcal{K}_{\alpha+\varepsilon}^{s}}\left(\kappa, \alpha, Q_{\infty}\right)$ for $2 s=1$ and since $U \in C_{c}^{2}\left(\mathbb{R}^{N}\right)$, we deduce from Lemma 5.1 that

$$
\left\|\mathcal{E}_{\mathcal{A}_{e, K_{z}}, U}^{s}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|\mathcal{O}_{\mathcal{A}_{0, K_{z}}, U}^{s}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C(N, s, \alpha)\|\nabla u\|_{L^{\infty}\left(B_{1}\right)} .
$$

This with (5.33) and (5.30) imply that

$$
\begin{equation*}
\|F\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) . \tag{5.35}
\end{equation*}
$$

Thanks to (5.34), applying Proposition [5.2( $i$ ) and (iii) and using (5.35), provided $1<2 s+\alpha<2$, we get

$$
\left\|\nabla\left(\varphi_{1 / 2} g_{z}\right)\right\|_{L^{\infty}\left(B_{r}\right)} \leq C r^{2 s-1+\alpha}\left(\left\|\varphi_{1 / 2} g_{z}\right\|_{C^{2 s+\beta}\left(\mathbb{R}^{N}\right)}+\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

As a consequence, by (5.32), for all $r \in(0,1 / 2)$ and all $z \in B_{1}$,

$$
\|\nabla u-\nabla u(z)\|_{L^{\infty}\left(B_{r}(z)\right)} \leq C r^{2 s-1+\alpha}\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right),
$$

for some $C=C(N, s, \alpha, \kappa)$. We then conclude that

$$
\|\nabla u\|_{C^{2 s-1+\alpha}\left(B_{1 / 8}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) .
$$

Therefore ( $i$ ) and (iii) follow from a covering and scaling argument.
Case 2: $2 s+\alpha>2$. We know from Case 1 that for all $\beta \in(0,2-2 s)$,

$$
\begin{equation*}
\|u\|_{C^{2 s+\beta}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) \tag{5.36}
\end{equation*}
$$

We then consider the function $g_{z}$ defined in (5.31). Hence, thanks to (5.34), by Proposition $5.2(\mathrm{ii})$, (5.36) and (5.35) we get

$$
\begin{equation*}
\left\|\nabla\left(\varphi_{1 / 2} g_{z}\right)-\nabla \mathbf{Q}_{r, \varphi_{1 / 2} g_{z}}\right\|_{L^{\infty}\left(B_{r}\right)} \leq C r^{2 s-1+\alpha}\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) \tag{5.37}
\end{equation*}
$$

provided $2 s+\beta \geq 1+\alpha$. In view of (5.37), we can use an iteration argument to obtain, for all $r \in(0,1 / 2)$,

$$
\left\|\nabla u-\nabla u(z)-M_{z}(\cdot-z)\right\|_{L^{\infty}\left(B_{r}(z)\right)} \leq C r^{2 s-1+\alpha}\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right) .
$$

for some $(N \times N)$-matrix $M_{z}$ satisfying $\left|M_{z}\right| \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)$. Since $2 s-1+\alpha>1$, we deduce that $M_{z}=D^{2} u(z)$. Using now an extension theorem, see e.g. 60] [Page 177], we conclude that

$$
\|\nabla u\|_{C^{1,2 s-2+\alpha}\left(B_{1 / 8}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

We thus get (iii) after a covering and scaling argument.
Case 3: $2 s+\alpha<1$. Here, we argue as in Case 1, by applying Proposition 5.2 (iii) to the function $g_{z}(x)=\varphi_{1 / 2}(x)\{u(x+z)-u(z)\}$. We skip the details.

By an induction argument, we have the following result.
Theorem 5.4. Let $N \geq 1, s \in(0,1)$ and $\alpha \in(0,1)$. Let $\kappa>0, k \in \mathbb{N}$ and $K \in \widetilde{K_{\tau}^{s}}\left(\kappa, k+\alpha, Q_{\infty}\right)$, with $\tau=\min \left(1, \alpha+(2 s-1)_{+}\right)$. Let $u \in H^{s}\left(B_{2}\right) \cap C^{k+\alpha}\left(\mathbb{R}^{N}\right)$ and $f \in C^{k+\alpha}\left(\mathbb{R}^{N}\right)$ such that

$$
\mathcal{L}_{K} u=f \quad \text { in } B_{2}
$$

(i) If $2 s \neq 1$ and $2 s+\alpha<2$, then there exists $C=C(N, s, k, \kappa, \alpha)$ such that

$$
\|u\|_{C^{k+2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

(ii) If $2 s=1$ and $K \in \widetilde{K}_{\alpha+\varepsilon}^{1 / 2}\left(\kappa, k+\alpha, Q_{\infty}\right)$, for some $\varepsilon>0$, then there exists $C=C(N, k, \kappa, \alpha, \varepsilon)$ such that

$$
\|u\|_{C^{k+1+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

(iii) If $2 s+\alpha>2$ and $K \in \widetilde{\mathscr{K}_{0}^{s}}\left(\kappa, k+2 s-1+\alpha, Q_{\infty}\right)$, then there exists $C=C(N, s, \kappa, \alpha)>0$ such that

$$
\|u\|_{C^{k+2 s+\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

Proof. We will prove $(i)$ and (ii), since the one of (iii) will follow the same arguments. The case $k=0$, that $u \in C^{2 s+\alpha}\left(B_{1}\right)$, is proved in Thoerem 5.3. We prove the statement first for $k=1$. By Lemma 2.3( $i i$, we have that $u^{1}:=\varphi_{1} u \in C^{2 s+\alpha}\left(\mathbb{R}^{N}\right) \cap C^{1, \alpha}\left(\mathbb{R}^{N}\right)$ and

$$
\mathcal{L}_{K} u^{1}=f^{1} \quad \text { in } B_{2^{-2}}
$$

for some function $f^{1}$ satisfying

$$
\begin{equation*}
\left\|f^{1}\right\|_{C^{1+\alpha}\left(B_{2}-2\right)} \leq C\left(\|u\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}\right) \tag{5.38}
\end{equation*}
$$

Let $h \in B_{2^{-4}}$, with $h \neq 0$. Then (recalling (2.1))

$$
\mathcal{L}_{K} u_{h, 1}^{1}-\mathcal{L}_{K_{h, 1}} u^{1}(\cdot+h)=f_{h, 1}^{1} \quad \text { in } B_{2^{-4}}
$$

We note that $K_{h, 1}$ satisfies (2.3), with $\alpha^{\prime}=0$. By Corollary 3.5. we obtain $u_{h, 1}^{1}$ is uniformly bounded in $C^{\sigma}\left(B_{2^{-5}}\right)$, for some $\sigma \in(s, 2 s)$. Therefore $\nabla u^{1} \in C^{0, \sigma}\left(B_{2^{-5}}\right) \subset H^{s}\left(B_{2^{-5}}\right)$. Using (5.4), we then have that

$$
\mathcal{L}_{K} u_{h, 1}^{1}-\mathcal{E}_{\mathcal{A}_{e, K_{h, 1}}, u_{h}^{1}}^{s}-\mathcal{O}_{\mathcal{A}_{o, K_{h, 1}}, u_{h}^{1}}^{s}=f_{h, 1}^{1} \quad \text { in } B_{2^{-4}} .
$$

Letting $h \rightarrow 0$, we see that, for all $i_{1} \in\{1, \ldots, N\}$,

$$
\mathcal{L}_{K} \partial_{i_{1}} u^{1}=\mathcal{E}_{\partial_{i_{1}} \mathcal{A}_{e, K}, u^{1}}^{s}+\mathcal{O}_{\partial_{i_{1}} \mathcal{A}_{o, K}, u^{1}}^{s}+\partial_{i_{1}} f^{1}=: f_{i_{1}}^{1} \quad \text { in } B_{2^{-5}}
$$

By Lemma 5.1 and (5.38), if $2 s \neq 1$, the right hand-side in the above display belongs to $C^{\alpha}\left(\mathbb{R}^{N}\right)$ and satisfies

$$
\left\|f_{i_{1}}^{1}\right\|_{C^{\alpha}\left(B_{2}-2\right)} \leq C\left(\|u\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

On the other hand if $2 s=1$, then Lemma 5.1 and (5.38) yield

$$
\left\|f_{i_{1}}^{1}\right\|_{C^{\alpha-\delta}\left(B_{2}-2\right)} \leq C\left(\|u\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

for all $\delta \in(0,1-\alpha)$. It follows from Theorem 5.3 that if $2 s \neq 1$, then

$$
\left\|\partial_{i_{1}} u^{1}\right\|_{C^{2 s+\alpha}\left(B_{2}-6\right)} \leq C\left(\|u\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

and for $2 s=1$,

$$
\left\|\partial_{i_{1}} u^{1}\right\|_{C^{2 s+\alpha-\delta}\left(B_{2}-6\right)} \leq C\left(\|u\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

We now remove the $\delta$ in the above estimate (for $2 s=1$ ). Indeed, we define $\bar{u}^{1}:=\varphi_{2-7} u^{1} \in$ $C^{2 s+\alpha+(1-\delta)}\left(\mathbb{R}^{N}\right)$ which, by Lemma 2.3) (iii), satisfies

$$
\mathcal{L}_{K} \bar{u}^{1}=\bar{f}^{1} \quad \text { in } B_{2^{-8}}
$$

with

$$
\left\|\bar{f}^{1}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C\left(\|u\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

Therefore proceeding as above, we have

$$
\begin{equation*}
\mathcal{L}_{K} \partial_{i_{1}} \bar{u}^{1}=\mathcal{E}_{\partial_{i_{1}} \mathcal{A}_{e, K}, \bar{u}^{1}}^{s}+\mathcal{O}_{\partial_{i_{1}} \mathcal{A}_{o, K}, \bar{u}^{1}}^{s}+\partial_{i_{1}} \bar{f}^{1}=: \bar{f}_{i_{1}}^{1} \quad \text { in } B_{2^{-9}} . \tag{5.39}
\end{equation*}
$$

Since $K \in \widetilde{\mathscr{K}}_{\alpha+\varepsilon}^{1 / 2}\left(\kappa, \alpha, Q_{\infty}\right)$ and $\bar{u}^{1}:=\varphi_{2-7} u^{1} \in C^{2 s+\alpha+1-\delta}\left(\mathbb{R}^{N}\right)$, Lemma 5.1 yields

$$
\left\|\mathcal{E}_{\partial_{i_{1}} \mathcal{A}_{e, K}, \bar{u}^{1}}^{s}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|\mathcal{O}_{\partial_{i_{1}} \mathcal{A}_{o, K}, \bar{u}^{1}}^{s}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C\left\|\bar{u}^{1}\right\|_{C^{2 s+\alpha+(1-\delta)}\left(\mathbb{R}^{N}\right)}
$$

Applying Theorem 5.3 to the equation (5.39), we then get

$$
\left\|\partial_{i_{1}} \bar{u}^{1}\right\|_{C^{1+\alpha}\left(B_{2-10}\right)} \leq C\left(\left\|\bar{u}^{1}\right\|_{C^{1+\alpha}\left(\mathbb{R}^{N}\right)}+\left\|\bar{f}_{i_{1}}^{1}\right\|_{C^{\alpha}\left(B_{2}\right)}\right)
$$

The theorem is thus proved for $k=2$.
Let $k>2$. We now prove by induction that for every $\left(i_{1}, i_{2}, \ldots, i_{k}\right) \in\{1, \ldots, N\}^{k}$ there exist a constant $r_{k}$, only depending on $k$, and a constant $C_{k}>0$, only depending on $N, s, \kappa, \alpha$ and $k$, such that

$$
\begin{equation*}
\left\|\partial_{i_{1} i_{2} \ldots i_{k}}^{k} u\right\|_{C^{2 s+\alpha+k}\left(B_{r_{k}}\right)} \leq C_{k}\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(B_{2}\right)}\right) \tag{5.40}
\end{equation*}
$$

We assume, as induction hypothesis that, the result is true up order $k-1$. That is, there exist $r_{k-1}, C_{k-1}>0$, as above, such that

$$
\begin{equation*}
\|u\|_{C^{2 s+k-1+\alpha}\left(B_{r_{k-1}}\right)} \leq C_{k-1}\left(\|u\|_{C^{k-1+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k-1+\alpha}\left(B_{2}\right)}\right) \tag{5.41}
\end{equation*}
$$

We then consider

$$
u^{k}:=\varphi_{r_{k-1} / 2} u \in C^{2 s+\alpha+k-1}\left(\mathbb{R}^{N}\right) \cap C^{k, \alpha}\left(\mathbb{R}^{N}\right)
$$

By Lemma 2.3(ii), we then have that

$$
\mathcal{L}_{K} u^{k}=f^{k} \quad \text { in } B_{r_{k-1} / 4}
$$

for some function

$$
\begin{equation*}
\left\|f^{k}\right\|_{C^{k+\alpha}\left(B_{r_{k-1} / 4}\right)} \leq C_{k}^{\prime}\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(B_{2}\right)}\right) \tag{5.42}
\end{equation*}
$$

where, unless otherwise stated, $C_{k}^{\prime}$ denotes a positive constant, only depending on $N, s, \kappa, \alpha$ and $k$. Proceeding as above, we can differentiate the equation $k$ times to deduce that for all $\left(i_{1}, i_{2}, \ldots, i_{k}\right) \in$ $\{1, \ldots, N\}^{k}$,

$$
\begin{equation*}
\mathcal{L}_{K} \partial_{i_{1} i_{2} \ldots i_{k}}^{k} u^{k}=g^{k}+\partial_{i_{1} i_{2} \ldots i_{k}}^{k} f^{k} \quad \text { in } B_{r_{k}^{\prime}} \tag{5.43}
\end{equation*}
$$

for constant $r_{k}^{\prime}<r_{k}$, only depending on $r_{k}$ and $k$, and for some function $g^{k}:=\sum_{j=1}^{m} c_{j}^{e} \mathcal{E}_{a_{j}^{e}, v_{j}}^{s}+$ $\sum_{j=1}^{m} c_{j}^{o} \mathcal{O}_{a_{j}^{o}, w_{j}}^{s}$ where $c_{j}^{e}, c_{j}^{o}$ are real numbers, $a_{j}^{e}, a_{j}^{o}, v_{j}$ and $w_{j}$ are respectively given by the partial derivatives in $x$ of $\mathcal{A}_{e, K}, \mathcal{A}_{o, K}$ up to order $k$ and $v_{j}$ together with $w_{j}$ are given by partial derivatives of $u^{k}$ up to order $k-1$. Therefore, provided $2 s \neq 1$, by Lemma 5.1.

$$
\left\|g^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C_{k}^{\prime}\left\|u^{k}\right\|_{C^{2 s+\alpha+k-1}\left(\mathbb{R}^{N}\right)} \leq C\left(\|u\|_{C^{k-1+\alpha}\left(\mathbb{R}^{N}\right)}+\left\|f^{k}\right\|_{C^{k, \alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

Now Theorem 5.3 implies that, for $2 s \neq 1$,

$$
\left\|\partial_{i_{1} i_{2} \ldots i_{k}}^{k} u^{k}\right\|_{C^{2 s+\alpha}\left(B_{r_{k}^{\prime} / 2}\right)} \leq C_{k}^{\prime}\left(\left\|\partial_{i_{1} i_{2} \ldots i_{k-1}}^{k} u^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|g^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|f^{k}\right\|_{C^{k, \alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

By (5.42), we get (5.40) in the case $2 s \neq 1$. Therefore $(i)$ follows by a covering and scaling argument.

Now when $2 s=1$, then we can argue similarly as above, noticing that, under the induction hypothesis (5.41), by Lemma 5.1, the function $g^{k}$ in the right hand side of (5.46) belongs to $C^{\alpha-\delta}\left(\mathbb{R}^{N}\right)$, for all $\delta \in(0, \alpha)$. Hence Theorem 5.3 implies that, for all $\delta \in(0, \alpha)$,

$$
\left\|\partial_{i_{1} i_{2} \ldots i_{k}}^{k} u^{k}\right\|_{C^{2 s+\alpha-\delta}\left(B_{r_{k}^{\prime} / 2}\right)} \leq C_{k}^{\prime \prime}\left(\left\|\partial_{i_{1} i_{2} \ldots i_{k-1}}^{k} u^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|g^{k}\right\|_{C^{\alpha-\delta}\left(\mathbb{R}^{N}\right)}+\left\|f^{k}\right\|_{C^{k, \alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

where, unless otherwise stated, $C_{k}^{\prime \prime}$ denotes a positive constant, only depending on $N, \kappa, \alpha, \delta, \varepsilon$ and $k$. To remove the parameter $\delta$, we consider

$$
\bar{u}^{k}:=\varphi_{r_{k}^{\prime} / 4} u \in C^{k+1+\alpha-\delta}\left(\mathbb{R}^{N}\right)
$$

which satisfies

$$
\begin{equation*}
\left\|\bar{u}^{k}\right\|_{C^{k+\alpha+(1-\delta)}\left(\mathbb{R}^{N}\right)} \leq C_{k}^{\prime \prime}\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}\right) \tag{5.44}
\end{equation*}
$$

By Lemma 2.3(ii), we then have that

$$
\mathcal{L}_{K} \bar{u}^{k}=\bar{f}^{k} \quad \text { in } B_{r_{k}^{\prime} / 8}
$$

for some function

$$
\begin{equation*}
\left\|\bar{f}^{k}\right\|_{C^{k+\alpha}\left(B_{r_{k}^{\prime} / 8}\right)} \leq C_{k}^{\prime \prime}\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}\right) \tag{5.45}
\end{equation*}
$$

As above, we then differentiate the equation $k$ times to deduce that for all $\left(i_{1}, i_{2}, \ldots, i_{k}\right) \in\{1, \ldots, N\}^{k}$,

$$
\begin{equation*}
\mathcal{L}_{K} \partial_{i_{1} i_{2} \ldots i_{k}}^{k} \bar{u}^{k}=\bar{g}^{k}+\partial_{i_{1} i_{2} \ldots i_{k}}^{k} \bar{f}^{k} \quad \text { in } B_{r_{k}^{\prime \prime}} \tag{5.46}
\end{equation*}
$$

for constant $r_{k}^{\prime \prime}$, only depending on $k$, and for some function $\bar{g}^{k}(x):=\sum_{j=1}^{m} c_{j}^{e} \mathcal{E}_{a_{j}^{e}, v_{j}}^{1 / 2}+\sum_{j=1}^{m} c_{j}^{o} \mathcal{O}_{a_{j}^{o}, w_{j}}^{1 / 2}$ where $c_{j}^{e}, c_{j}^{o}$ are real numbers and $a_{j}^{e}, a_{j}^{o}$ (resp. $v_{j}$ and $w_{j}$ ) are respectiveley given by the partial derivatives in $x$-variable of $\mathcal{A}_{e, K}, \mathcal{A}_{o, K}$ up to order $k$ (resp. $v_{j}$ together with $w_{j}$ are given by partial derivatives of $\bar{u}^{k}$ up to order $k-1$ ). Therefore by Lemma 5.1. (5.44), and since $K \in \widetilde{K}_{\alpha+\varepsilon}^{1 / 2}\left(\kappa, k+\alpha, \mathbb{R}^{N}\right)$, we obtain

$$
\begin{equation*}
\left\|\bar{g}^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C_{k}^{\prime \prime}\left\|\bar{u}^{k}\right\|_{C^{1+\alpha+k-\delta}\left(\mathbb{R}^{N}\right)} \leq C_{k}^{\prime \prime}\left(\|u\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{k+\alpha}\left(\mathbb{R}^{N}\right)}\right) \tag{5.47}
\end{equation*}
$$

Applying Theorem 5.3] we conclude that

$$
\left\|\partial_{i_{1} i_{2} \ldots i_{k}}^{k} \bar{u}^{k}\right\|_{C^{1+\alpha}\left(B_{r_{k}^{\prime \prime} / 2}\right)} \leq C_{k}^{\prime \prime}\left(\left\|\partial_{i_{1} i_{2} \ldots i_{k-1}}^{k-1} \bar{u}^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|\bar{g}^{k}\right\|_{C^{\alpha}\left(\mathbb{R}^{N}\right)}+\left\|\bar{f}^{k}\right\|_{C^{k, \alpha}\left(\mathbb{R}^{N}\right)}\right)
$$

Hence, since $\bar{u}^{k}=u$ on $B_{r_{k}^{\prime \prime} / 2}$, by (5.44), (5.45) and (5.47) with then obtain (5.40). Now (ii) follows by scaling and covering.

## 6. Proof of the main Results

We start this section with the following result which shows how to pass from a nonlocal equation with kernels in $\widetilde{\mathscr{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{\delta}\right)$ to a nonlocal equation driven by kernels in $\widetilde{\mathscr{K}_{\tau}^{s}}\left(\kappa, m+\alpha, Q_{\infty}\right)$.
Lemma 6.1. Let $K \in \widetilde{\mathscr{K}}_{\tau}^{s}\left(\kappa, m+\alpha, Q_{4 R}\right)$, for some $\alpha \in[0,1), \tau \in[0,1], m \in \mathbb{N}$ and $R>0$. Let $v \in H^{s}\left(B_{4 R}\right) \cap L_{s}\left(\mathbb{R}^{N}\right)$ and $f \in L_{l o c}^{1}\left(B_{4 R}\right)$ satisfy

$$
\mathcal{L}_{K} v=f \quad \text { in } B_{4 R} .
$$

Let

$$
\bar{K}(x, y)=\varphi_{2 R}(x) \varphi_{2 R}(y) K(x, y)+\left(2-\varphi_{R}(x)-\varphi_{R}(y)\right) \mu_{1}(x, y)
$$

Then

$$
\begin{equation*}
\mathcal{L}_{\bar{K}} v+\bar{V} v=f+\bar{f} \quad \text { in } B_{R / 4} \tag{6.1}
\end{equation*}
$$

where, for $x \in B_{R / 4}$,

$$
\bar{V}(x)=G_{1, K, 2 R}(x)-G_{1, \mu_{1}, R}(x), \quad \bar{f}(x)=G_{v, K, 2 R}(x)-G_{v, \mu_{1}, R}(x),
$$

and $G_{v, K, \rho}$ is given by (2.14). In particular, $\bar{K} \in \widetilde{\mathscr{K}_{\tau}^{s}}\left(\bar{\kappa}, m+\alpha, Q_{\infty}\right)$, for some constant $\bar{\kappa}=$ $\bar{\kappa}(\kappa, \alpha, m, R, \tau, s, N)$.

Proof. The proof of (6.1) is elementary, and we skip it. Next, we observe that

$$
\mathcal{A}_{\bar{K}}(x, r, \theta)=\varphi_{2 R}(x) \varphi_{2 R}(x+r \theta) \mathcal{A}_{K}(x, r, \theta)+\left(2-\varphi_{R}(x)-\varphi_{R}(x+r \theta)\right)
$$

Recalling the definition of the cut-off function $\varphi_{R}$ in the beginning of Section 2, we easily deduce that

$$
\min (\kappa, 1) \leq \mathcal{A}_{\bar{K}}(x, r, \theta) \leq \max (1 / k, 4) \quad \text { for all } x \in \mathbb{R}^{N}, r \geq 0, \theta \in S^{N-1}
$$

This in particular implies that $\bar{K}$ satisfies (2.2). Moreover, it is also not difficult to check that

$$
\left\|\mathcal{A}_{\bar{K}}\right\|_{C^{m, \alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}+\left\|\mathcal{A}_{o, \bar{K}}\right\|_{\mathcal{C}_{\tau}^{m}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)} \leq C(\kappa, m, \alpha, \tau, R)
$$

Proof of Theorem 1.9. As mentioned in the first section, the case $2 s \leq 1$ was already proven in [30]. Now the case $2 s>1$ follows from Theorem 4.3, Lemma 6.1, Lemma 2.3 and the fact that $L^{p}\left(\mathbb{R}^{N}\right) \hookrightarrow \mathcal{M}_{N / p}$.

Proof of Theorem 1.10. First applying Theorem 5.3 and using Lemma 6.1 together with Lemma 2.3 we get the estimates.

Proof of Theorem 1.3 . It follows from Theorem 1.9 ,

Proof of Theorem 1.4. By Lemma 6.1 we have that

$$
\begin{equation*}
\mathcal{L}_{\bar{K}} u=f+\bar{f}-\bar{V} u \quad \text { in } B_{1 / 2} \tag{6.2}
\end{equation*}
$$

with $\bar{K} \in \widetilde{\mathscr{K}_{\tau_{s}}^{s}}\left(\bar{\kappa}, m+\alpha+(2 s-1)_{+}, Q_{\infty}\right)$ with $\tau_{s}:=\alpha+(2 s-1)_{+}$for $2 s+\alpha<2$ and $\tau_{s}=0$ for $2 s+\alpha>2$. In addition, by Lemma 2.3(iii), we have

$$
\begin{equation*}
\|\bar{V}\|_{C^{m+\alpha}\left(B_{1 / 2}\right)} \leq C \tag{6.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\bar{f}\|_{C^{m+\alpha}\left(B_{1 / 2}\right)} \leq C\|u\|_{L_{s}\left(\mathbb{R}^{N}\right)} \tag{6.4}
\end{equation*}
$$

We consider first the case $2 s \neq 1$. Since $u$ satisfies (6.2), applying Theorem 5.4 and using Lemma 2.3 (iii), we get

$$
\left\|\varphi_{1 / 2} u\right\|_{C^{2 s+m+\alpha}\left(B_{2-4}\right)} \leq C\left(\left\|\varphi_{1 / 2} u\right\|_{C^{m+\alpha}\left(\mathbb{R}^{N}\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|F\|_{C^{m+\alpha}\left(B_{1 / 2}\right)}\right)
$$

where $F:=f+\bar{f}-\bar{V} u$. Consequently, by (6.3) and (6.4)

$$
\|u\|_{C^{2 s+m+\alpha}\left(B_{2}-4\right.} \leq C\left(\|u\|_{C^{m+\alpha}\left(B_{1}\right)}+\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m+\alpha}\left(B_{2}\right)}\right)
$$

Using now adimentional Hölder norms and interpolation (see e.g. 5.38), we can absorb the $C^{m+\alpha}\left(B_{1}\right)-$ norm of $u$ to deduce that

$$
\|u\|_{C^{2 s+m+\alpha}\left(B_{2}-5\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m+\alpha}\left(B_{2}\right)}\right)
$$

If now $2 s=1$, then since $\bar{K} \in \widetilde{\mathscr{K}}_{\alpha}^{1 / 2}\left(\bar{\kappa}, m+\alpha, Q_{\infty}\right)$ and in view of Theorem 5.4, the same arguments as above yield

$$
\|u\|_{C^{1+m+\alpha-\varepsilon}\left(B_{2}-5\right)} \leq C\left(\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{m+\alpha-\varepsilon}\left(B_{2}\right)}\right)
$$

for all $\varepsilon \in(0, \alpha)$. Now by scaling and covering, we get the result.
6.1. Proof of Theorem 1.5. The following fundamental lemma allows, in particular, to consider truncation of the nonlocal mean curvature kernel $1_{B_{r}}(x) 1_{B_{r}}(y) \mathcal{K}_{u}(x, y)$ without any assumption on $u$ in the exterior of $B_{r}$.
Lemma 6.2. Let $u: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be a measurable function and $\Gamma^{u, R}: B_{R / 2} \rightarrow \mathbb{R}$ be given by

$$
\begin{equation*}
\Gamma^{u, R}(x):=\int_{\mathbb{R}^{N}}\left(1-1_{B_{R}}(y)\right) \frac{\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(-p_{u}(x, y)\right)}{|x-y|^{N+2 s-1}} d y \tag{6.5}
\end{equation*}
$$

If $u \in C^{k, \alpha}\left(B_{R / 2}\right)$, for $k \geq 1$ and $\alpha \in[0,1]$, then, there exists a constant $C=C(N, s, k, \alpha, R)$ such that

$$
\begin{equation*}
\left\|\Gamma^{u, R}\right\|_{C^{k, \alpha}\left(B_{R / 2}\right)} \leq C\left(1+\|u\|_{C^{k, \alpha}\left(B_{R / 2}\right)}\right)^{2 k} \tag{6.6}
\end{equation*}
$$

If $u \in C^{0,1}\left(B_{R / 2}\right)$ then, there exists a constant $C=C(N, s, R)$ such that

$$
\begin{equation*}
\left\|\Gamma^{u, R}\right\|_{C^{0,1}\left(B_{R / 2}\right)} \leq C\left(1+\|u\|_{C^{0,1}\left(B_{R / 2}\right)}\right) \tag{6.7}
\end{equation*}
$$

Proof. For simplicity, we assume that $R=2$, and to alleviate the notations, we put $\Gamma^{u}:=\Gamma^{u, R}$. We first observe, from (1.15), that $\mathcal{F}_{s}^{\prime}(p)=-\left(1+p^{2}\right)^{(-N-2 s) / 2}$, so that for all $j \in \mathbb{N}$,

$$
\begin{equation*}
|p|^{j}\left|\mathcal{F}_{s}^{(j)}(p)\right| \leq C(N, s, j) \quad \text { for all } p \in \mathbb{R} \tag{6.8}
\end{equation*}
$$

In particular, since $2 s>1$,

$$
\begin{equation*}
\left\|\Gamma^{u}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C(N, s) \tag{6.9}
\end{equation*}
$$

Next, for all $(x, y) \in B_{1} \times \mathbb{R}^{N} \backslash B_{2}$, we have

$$
\begin{equation*}
\left|\partial_{x}^{\mu} p_{u}(x, y)\right| \leq C(k)\left(|u(y)||y|^{-1}+\|u\|_{C^{k-1,1}\left(B_{1}\right)}\right) \quad \text { for } \mu \in \mathbb{N}^{N} \text { with }|\mu| \leq k \tag{6.10}
\end{equation*}
$$

On the other hand, by writing $u(y)=(u(y)-u(z))+u(z)$, we easily deduce that

$$
\begin{equation*}
|u(y)||y|^{-1} \leq C\left(\left|p_{u}(z, y)\right|+\|u\|_{L^{\infty}\left(B_{1}\right)}\right) \quad \text { for all } z \in B_{1} \text { and } y \in \mathbb{R}^{N} \backslash B_{2} \tag{6.11}
\end{equation*}
$$

Using this in (6.10), we see that, for $\mu \in \mathbb{N}^{N}$ with $|\mu| \leq k$,

$$
\begin{equation*}
\left|\partial_{x}^{\mu} p_{u}(x, y)\right| \leq C(k)\left(\left|p_{u}(z, y)\right|+\|u\|_{C^{k-1,1}\left(B_{1}\right)}\right) \quad \text { for all } x, z \in B_{1} \text { and } y \in \mathbb{R}^{N} \backslash B_{2} \tag{6.12}
\end{equation*}
$$

By the Faà di Bruno formula (see e.g. [43]), for $|\gamma|=k$ and $(x, y) \in B_{1} \times \mathbb{R}^{N} \backslash B_{2}$, we get

$$
\begin{equation*}
\partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}(x, y)\right)=\sum_{\Pi \in \mathscr{P}_{k}} \mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}(x, y)\right) \prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}(x, y) \tag{6.13}
\end{equation*}
$$

where $\mathscr{P}_{k}$ denotes the set of all partitions of $\{1, \ldots, k\}$. Hence, for $x \in B_{1}$ and $y \in \mathbb{R}^{N} \backslash B_{2}$, by (6.12), we have that

$$
\begin{aligned}
\left|\partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}(x, y)\right)\right| & \leq C \sum_{\Pi \in \mathscr{P}_{k}} 2^{\Pi-1}\left(\left|p_{u}(x, y)\right|^{|\Pi|}\left|\mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}(x, y)\right)\right|+\|u\|_{C^{k-1,1}\left(B_{1}\right)}^{|\Pi|}\left|\mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}(x, y)\right)\right|\right) \\
& \leq C\left(1+\|u\|_{C^{k-1,1}\left(B_{1}\right)}^{k}+\sum_{\Pi \in \mathscr{P}_{k}} 2^{\Pi-1}\left|p_{u}(x, y)\right|^{|\Pi|}\left|\mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}(x, y)\right)\right|\right)
\end{aligned}
$$

From this and (6.8), we deduce that, for all $\gamma \in \mathbb{N}^{N}$ with $|\gamma|=k$,

$$
\begin{equation*}
\sup _{(x, y) \in B_{1} \times \mathbb{R}^{N} \backslash B_{2}}\left|\partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}(x, y)\right)\right|+\sup _{(x, y) \in B_{1} \times \mathbb{R}^{N} \backslash B_{2}}\left|\partial_{x}^{\gamma} \mathcal{F}_{s}\left(-p_{u}(x, y)\right)\right| \leq C\left(1+\|u\|_{C^{k-1,1}\left(B_{1}\right)}\right)^{k} \tag{6.14}
\end{equation*}
$$

with $C=C(s, N, k)$. Since $2 s>1$, from the above estimate, (6.9) and the dominated convergence theorem, we can differentiate under the integral sign in (6.5) to deduce that

$$
\begin{equation*}
\left\|\Gamma^{u}\right\|_{C^{k-1,1}\left(B_{1}\right)} \leq C\left(1+\|u\|_{C^{k-1,1}\left(B_{1}\right)}\right)^{k} \tag{6.15}
\end{equation*}
$$

Moreover, to see (6.7), we note that if $u \in C^{0,1}\left(B_{1}\right)$, then Rademarcher's theorem implies that $u$ is equivalent to a differentiable function. Therefore (6.14) holds (with $k=1$ ) and replacing "sup" with "essup". Now by the dominated convergence theorem, we get (6.7).
Let us now fix $x_{1}, x_{2} \in B_{1}$ and $y \in \mathbb{R}^{N} \backslash B_{2}$. Direct computations yield

$$
\left|\partial_{x}^{\mu} p_{u}\left(x_{1}, y\right)-\partial_{x}^{\mu} p_{u}\left(x_{2}, y\right)\right| \leq C\left|x_{1}-x_{2}\right|^{\alpha}\left(\left|u(y)\left\|\left.y\right|^{-1}+\right\| u \|_{C^{k, \alpha}\left(B_{1}\right)}\right) \quad \text { for } \mu \in \mathbb{N}^{N} \text { with }|\mu| \leq k\right.
$$

Note that, (6.11) implies that

$$
|u(y) \| y|^{-1} \leq C\left\{\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{L^{\infty}\left(B_{1}\right)}\right\}
$$

Therefore, for all $\mu \in \mathbb{N}^{N}$ with $|\mu| \leq k$, we get

$$
\begin{equation*}
\left|\partial_{x}^{\mu} p_{u}\left(x_{1}, y\right)-\partial_{x}^{\mu} p_{u}\left(x_{2}, y\right)\right| \leq C\left|x_{1}-x_{2}\right|^{\alpha}\left\{\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{C^{k, \alpha}\left(B_{1}\right)}\right\} \tag{6.16}
\end{equation*}
$$

and, by (6.12),

$$
\begin{equation*}
\left|\partial_{x}^{\mu} p_{u}\left(x_{1}, y\right)\right| \leq C\left\{\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{C^{k}\left(B_{1}\right)}\right\} \tag{6.17}
\end{equation*}
$$

Next, we define

$$
g_{s} \in C^{\infty}\left(\mathbb{R}_{+}, \mathbb{R}\right), \quad g_{s}(r)=-r^{-(N+2 s-1) / 2}
$$

so that $\mathcal{F}_{s}^{\prime}(p)=g_{s}\left(1+p^{2}\right)$ for all $p \in \mathbb{R}$. Moreover, for $r>0$,

$$
\begin{equation*}
g_{s}^{(j)}(r)=(-1)^{j+1} 2^{-j} \prod_{i=0}^{j-1}(N+2 s-1+2 i) r^{-\frac{N+2 s-1+2 j}{2}} \tag{6.18}
\end{equation*}
$$

From this and the generalized chain rule for higher derivatives, we get

$$
\begin{equation*}
\mathcal{F}_{s}^{(j+1)}(p)=\sum_{\left(m_{1}, m_{2}\right) \in \mathcal{N}_{j}} \tau_{j}\left(m_{1}, m_{2}\right) p^{m_{1}} g_{s}^{\left(m_{1}+m_{2}\right)}\left(1+p^{2}\right), \tag{6.19}
\end{equation*}
$$

where $\tau_{j}\left(m_{1}, m_{2}\right)=\frac{j!!^{m_{1}}}{m_{1}!m_{2}!}$ and $\mathcal{N}_{j}:=\left\{\left(m_{1}, m_{2}\right) \in \mathbb{N}^{2}: m_{1}+2^{m_{2}} m_{2}=j\right\}$. Hence, for all $p_{1}, p_{2} \in \mathbb{R}$,

$$
\begin{aligned}
&\left|\mathcal{F}_{s}^{(j+1)}\left(p_{1}\right)-\mathcal{F}_{s}^{(j+1)}\left(p_{2}\right)\right| \leq \sum_{\left(m_{1}, m_{2}\right) \in \mathcal{N}_{j}} \tau_{j}\left(m_{1}, m_{2}\right)\left|p_{1}^{m_{1}}-p_{2}^{m_{1}}\right|\left|g_{s}^{\left(m_{1}+m_{2}\right)}\left(1+p_{1}^{2}\right)\right| \\
&+\sum_{\left(m_{1}, m_{2}\right) \in \mathcal{N}_{j}} \tau_{j}\left(m_{1}, m_{2}\right)\left|p_{2}^{m_{1}}\right|\left|p_{1}^{2}-p_{2}^{2}\right| \int_{0}^{1}\left|g_{s}^{\left(m_{1}+m_{2}+1\right)}\left(1+t p_{1}^{2}+(1-t) p_{2}^{2}\right)\right| d t
\end{aligned}
$$

It then follows from, (6.16) and (6.17), that

$$
\begin{align*}
& \left|\mathcal{F}_{s}^{(j+1)}\left(p_{u}\left(x_{1}, y\right)\right)-\mathcal{F}_{s}^{(j+1)}\left(p_{u}\left(x_{2}, y\right)\right)\right| \\
& \leq C\left|x_{1}-x_{2}\right|^{\alpha} \sum_{\left(m_{1}, m_{2}\right) \in \mathcal{N}_{j}} \tau_{j}\left(m_{1}, m_{2}\right) \frac{\left(\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{C^{k, \alpha}\left(B_{1}\right)}\right)^{m_{1}}}{\left(1+\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)^{2}\right)^{\frac{N+2 s-1+2\left(m_{1}+m_{2}\right)}{2}}} \\
& +C\left|x_{1}-x_{2}\right|^{\alpha} \sum_{\left(m_{1}, m_{2}\right) \in \mathcal{N}_{j}} \tau_{j}\left(m_{1}, m_{2}\right) \frac{\left(\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{\left.C^{k, \alpha}\left(B_{1}\right)\right)^{m_{1}+2}}\right.}{\left(1+\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)^{2}\right)^{\frac{N+2 s-1+2\left(m_{1}+m_{2}+1\right)}{2}}} . \tag{6.20}
\end{align*}
$$

On the other hand, it is immediate, from (6.18) and (6.19), that

$$
\begin{equation*}
\left|\mathcal{F}_{s}^{(j+1)}\left(p_{u}\left(x_{2}, y\right)\right)\right| \leq \frac{C}{\left(1+\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)^{2}\right)^{\frac{N+2 s-1+2(j+1)}{2}}} \tag{6.21}
\end{equation*}
$$

Using (6.17), (6.16) and an induction argument, we get

$$
\begin{equation*}
\left|\prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}\left(x_{1}, y\right)-\prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}\left(x_{1}, y\right)\right| \leq C\left|x_{1}-x_{2}\right|^{\alpha}\left\{\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{C^{k, \alpha}\left(B_{1}\right)}\right\}^{|\Pi|} \tag{6.22}
\end{equation*}
$$

Moreover (6.17) yields

$$
\begin{equation*}
\left|\prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}\left(x_{2}, y\right)\right| \leq C\left(\min \left(\left|p_{u}\left(x_{1}, y\right)\right|,\left|p_{u}\left(x_{2}, y\right)\right|\right)+\|u\|_{C^{k}\left(B_{1}\right)}\right)^{|\Pi|} \tag{6.23}
\end{equation*}
$$

We have, from (6.13), that

$$
\begin{align*}
\mid \partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}\left(x_{1}, y\right)\right)- & \partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}\left(x_{2}, y\right)\right)\left|\leq \sum_{\Pi \in \mathscr{P}_{k}}\right| \mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}\left(x_{1}, y\right)\right)-\mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}\left(x_{2}, y\right)\right)| | \prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}\left(x_{1}, y\right) \mid \\
& +\sum_{\Pi \in \mathscr{P}_{k}}\left|\mathcal{F}_{s}^{(|\Pi|)}\left(p_{u}\left(x_{2}, y\right)\right)\right|\left|\prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}\left(x_{1}, y\right)-\prod_{\mu \in \Pi} \partial_{x}^{\mu} p_{u}\left(x_{2}, y\right)\right| \tag{6.24}
\end{align*}
$$

Next, we observe that for $\left(m_{1}, m_{2}\right) \in \mathcal{N}_{|\Pi|-1}$, then

$$
|\Pi|+m_{1}-2\left(m_{1}+m_{2}\right)-(N+2 s-1)<0
$$

Now from this, (6.20), (6.21), (6.22), (6.23) and (6.24), we deduce that, for all $x_{1}, x_{2} \in B_{1}$ and $y \in \mathbb{R}^{N} \backslash B_{2}$,

$$
\left|\partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}\left(x_{1}, y\right)\right)-\partial_{x}^{\gamma} \mathcal{F}_{s}\left(p_{u}\left(x_{2}, y\right)\right)\right| \leq C\left|x_{1}-x_{2}\right|^{\alpha}\left(1+\|u\|_{C^{k, \alpha}\left(B_{1}\right)}\right)^{2 k}
$$

Combining this with (6.14), we get $\sup _{y \in \mathbb{R}^{N} \backslash B_{2}}\left\|\mathcal{F}_{s}\left(p_{u}(\cdot, y)\right)\right\|_{C^{k, \alpha}\left(B_{1}\right)} \leq C\left(1+\|u\|_{C^{k, \alpha}\left(B_{1}\right)}\right)^{2 k}$. Since the same estimates remains valid when $p_{u}$ is replaced with $-p_{u}$, then (6.6) follows.

We will need the following elementary result which follows from the fact that $\mathcal{F}_{s}^{\prime}$ is even on $\mathbb{R}$ and the fundamental theorem of calculus.

Lemma 6.3. For all $a, b \in \mathbb{R}$, we have

$$
\left[\mathcal{F}_{s}(a)-\mathcal{F}_{s}(b)\right]-\left[\mathcal{F}_{s}(-a)-\mathcal{F}_{s}(-b)\right]=2(a-b) \int_{0}^{1} \mathcal{F}_{s}^{\prime}(b+\rho(a-b)) d \rho
$$

We now complete the
Proof of Theorem 1.5. In view of (1.19), we have

$$
\begin{equation*}
\mathcal{L}_{\tilde{\mathcal{K}}_{u}} u=f-\Gamma^{u} \quad \text { in } B_{1 / 2}, \tag{6.25}
\end{equation*}
$$

where

$$
\widetilde{\mathcal{K}}_{u}(x, y):=1_{B_{1}}(x) 1_{B_{1}}(y) \mathcal{K}_{u}(x, y) \quad \text { for all } x \neq y \in \mathbb{R}^{N}
$$

and, for $x \in B_{1 / 2}$,

$$
\Gamma^{u}(x):=\int_{\mathbb{R}^{N}}\left(1-1_{B_{1}}(y)\right) \frac{\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(-p_{u}(x, y)\right)}{|x-y|^{N+2 s-1}} d y
$$

We recall from the fundamental theorem of calculus that

$$
\begin{equation*}
(u(x)-u(y)) \mathcal{K}_{u}(x, y)=\left[\mathcal{F}_{s}\left(p_{u}(x, y)\right)-\mathcal{F}_{s}\left(-p_{u}(x, y)\right)\right]|x-y|^{-(N+2 s-1)} \tag{6.26}
\end{equation*}
$$

Let $h \in B_{1 / 4}$ with $h \neq 0$. Then, recalling the notation in (2.1), by Lemma 6.3 (6.26) and (6.25),

$$
\mathcal{L}_{K_{h}^{u}} u_{h, 1}=f_{h, 1}+\Gamma_{h, 1}^{u} \quad \text { in } B_{1 / 4}
$$

where

$$
\begin{equation*}
K_{h}^{u}(x, y):=1_{B_{1}}(x) 1_{B_{1}}(y) \frac{1}{|x-y|^{N+2 s}} q_{h}^{u}(x, y) \tag{6.27}
\end{equation*}
$$

and

$$
q_{h}^{u}(x, y):=-2 \int_{0}^{1} \mathcal{F}_{s}^{\prime}\left(p_{u(\cdot+h)}(x, y)+\rho p_{u-u(\cdot+h)}(x, y)\right) d \rho
$$

Since $\mathcal{F}_{s}^{\prime}$ is even and $p_{w}(x, y)=-p_{w}(y, x)$, we see that $K_{h}^{u}(x, y)=K_{h}^{u}(y, x)$. Moreover

$$
\begin{equation*}
K_{h}^{u}(x, y) \geq C|x-y|^{-N-2 s} \quad x \neq y \in B_{1} \tag{6.28}
\end{equation*}
$$

for some constant $C>0$, only depending on $N, s$ and $\|u\|_{C^{0,1}\left(B_{1}\right)}$. Letting $v:=\varphi_{1 / 8} u_{h, 1}$ and using Lemma 2.3 (i), we have that

$$
\begin{equation*}
\mathcal{L}_{K_{h}^{u}} v=f_{h, 1}+\Gamma_{h, 1}^{u}+G_{h}, \quad \text { in } B_{2^{-4}} \tag{6.29}
\end{equation*}
$$

with $G_{h}:=G_{K_{h}^{u}, u_{h, 1}, 1 / 4}$ satisfying (note that $K_{h}^{u}$ is supported in $B_{1} \times B_{1}$ and $\left|q_{h}^{u}\right| \leq 2$ )

$$
\begin{equation*}
\left\|G_{h}\right\|_{L^{\infty}\left(B_{2}-5\right)} \leq C(N, s)\left\|u_{h, 1}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C\|\nabla u\|_{L^{\infty}\left(B_{2}\right)} \tag{6.30}
\end{equation*}
$$

We would like to apply [21, Theorem 2.4] to get the $C^{\alpha_{0}}$ bound of $v$, but our kernel $K_{h}^{u}$, which is compactly supported might vanish at some diagonal points $\{x=y\}$. A way out to such difficulty, is to use the argument in [30, Remark 2.1] (see also Lemma 6.1) by considering

$$
\bar{K}_{h}^{u}(x, y)=K_{h}^{u}(x, y)+\left(2-1_{B_{1 / 2}}(x)-1_{B_{1 / 2}}(y)\right)|x-y|^{-N-2 s} .
$$

We then deduce, from (6.29), that

$$
\begin{equation*}
\mathcal{L}_{\bar{K}_{h}^{u} v+\bar{V} v=f_{h, 1}+\Gamma_{h, 1}^{u}+G_{h}+\bar{f}, \quad \text { in } B_{2^{-4}}, ., ~}^{\text {, }} \tag{6.31}
\end{equation*}
$$

for some functions $\|\bar{V}\|_{L^{\infty}\left(B_{2-4}\right)} \leq C(N, s)$ and $\|\bar{f}\|_{L^{\infty}\left(B_{2^{-4}}\right)} \leq C(N, s)\|v\|_{L^{\infty}\left(B_{1}\right)}$. Since $K_{h}^{u}(x, y)$ satisfies (6.28), we find that $\bar{K}_{h}^{u}$ satisfies (2.2). Therefore, since $v \in H^{s}\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$, by [21, Theorem 2.4], we have that

$$
\|v\|_{C^{0, \alpha_{0}}\left(B_{2^{-5}}\right)} \leq C\left(\|v\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left\|f_{h, 1}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left\|\Gamma_{h, 1}^{u}\right\|_{L^{\infty}\left(B_{2^{-4}}\right)}+\left\|G_{h}\right\|_{L^{\infty}\left(B_{2^{-4}}\right)}\right)
$$

for some $\alpha_{0}>0$ and $C>0$, only depending on $N, s$ and $\|u\|_{C^{0,1}\left(B_{1}\right)}$. From (6.30) and the fact that $v=u_{h, 1}$ on $B_{1 / 8}$, we get

$$
\left\|u_{h, 1}\right\|_{C^{0, \alpha_{0}}\left(B_{2}-5\right)} \leq C\left(\left\|u_{h, 1}\right\|_{L^{\infty}\left(B_{1}\right)}+\left\|f_{h, 1}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left\|\Gamma_{h, 1}^{u}\right\|_{L^{\infty}\left(B_{2}-4\right)}\right)
$$

This and Lemma 6.2 imply that

$$
\begin{equation*}
\|u\|_{C^{1, \alpha_{0}}\left(B_{2}-6\right)} \leq C\left(1+\|u\|_{C^{0,1}\left(B_{1}\right)}+\|f\|_{C^{0,1}\left(\mathbb{R}^{N}\right)}\right), \tag{6.32}
\end{equation*}
$$

which proves (1.20).
To obtain the gradient estimate of $v$ from Theorem 1.3 we check that $\mathcal{L}_{K_{h}^{u}}$ is a $C^{0, \alpha_{0}}$-nonlocal operator. To this scope, for every $w \in C^{0,1}\left(B_{1}\right)$, we define $Z_{w}: B_{1 / 2} \times[0,1 / 2) \times S^{N-1} \rightarrow \mathbb{R}$ by

$$
Z_{w}(x, r, \theta):=-\int_{0}^{1} \nabla w(x+r t \theta) \cdot \theta d t \quad \text { for } r \in[0,1 / 2), x \in B_{1 / 2} \text { and } \theta \in S^{N-1}
$$

Clearly $Z_{w}$ is as smooth as $\nabla w$ and $Z_{w}(x, r, \theta):=p_{w}(x, x+r \theta)$ for $r>0$. We then define $\mathcal{A}_{K_{h}^{u}}$ : $B_{1 / 4} \times[0,1 / 4) \times S^{N-1} \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
\mathcal{A}_{K_{h}^{u}}(x, r, \theta):=1_{B_{2}}(x) 1_{B_{2}}(x+r \theta) \int_{0}^{1} \frac{2 d \rho}{\left(1+\left(Z_{u(\cdot+h)}(x, r, \theta)+\rho Z_{u-u(\cdot+h)}(x, r, \theta)\right)^{2}\right)^{(N+2 s) / 2}}, \tag{6.33}
\end{equation*}
$$

which by (6.27), satisfies $\mathcal{A}_{K_{h}^{u}}(x, r, \theta)=r^{N+2 s} K_{h}^{u}(x, x+r \theta)$ for all $(x, r, \theta) \in B_{1 / 4} \times(0,1 / 4) \times S^{N-1}$. Moreover,

$$
\mathcal{A}_{K_{h}^{u}}(x, 0, \theta)-\mathcal{A}_{K_{h}^{u}}(x, 0,-\theta)=0 \quad \text { for all }(x, \theta) \in B_{1 / 4} \times S^{N-1}
$$

In addition from, (6.32) together with (6.33), we have that

$$
\left\|\mathcal{A}_{K_{h}^{u}}\right\|_{C^{\alpha_{0}}\left(Q_{2-7} \times S^{N-1}\right)} \leq C
$$

with $C$, only depending on $N, s,\|u\|_{C^{0,1}\left(B_{2}\right)}, \alpha_{0}$ and $\|f\|_{C^{0,1}\left(B_{2}\right)}$. We then conclude that $K_{h}^{u} \in$ $\mathscr{K}^{s}\left(\kappa, \alpha_{0}, Q_{2^{-7}}\right)$, for some $\kappa$, only depending on $N, s,\|u\|_{C^{0,1}\left(B_{2}\right)}, \alpha_{0}$ and $\|f\|_{C^{0,1}\left(B_{2}\right)}$. Therefore applying Theorem 1.3 (ii) to (6.29), we deduce that

$$
\|\nabla v\|_{C^{\min \left(2 s-1-\varepsilon, \alpha_{0}\right)}\left(B_{2}-8\right)} \leq C\left(\|v\|_{L^{\infty}\left(B_{1}\right)}+\left\|f_{h, 1}\right\|_{L^{\infty}\left(B_{2}\right)}\right)
$$

for all $\varepsilon \in(0,2 s-1)$ and $C$ a constant, only depending on $N, s,\|u\|_{C^{0,1}\left(B_{2}\right)}, \alpha_{0}, \varepsilon$ and $\|f\|_{C^{0,1}\left(B_{2}\right)}$. Hence, recalling that $v=u_{h, 1}$ in $B_{1 / 8}$, we get

$$
\|\nabla u\|_{C^{1, \alpha_{1}}\left(B_{2}-9\right)} \leq C
$$

with $\alpha_{1}:=\min \left(2 s-1-\varepsilon, \alpha_{0}\right)$. Hence, for all $h \in B_{2^{-10}}$, we have $K_{h}^{u} \in \mathscr{K}^{s}\left(\kappa, 1, Q_{2^{-10}}\right)$, for some $\kappa$, only depending on $N, s,\|u\|_{C^{0,1}\left(B_{2}\right)}, \alpha_{1}$ and $\|f\|_{C^{0,1}\left(B_{2}\right)}$. We apply once more Theorem 1.3 (ii) to (6.29), to get $\|v\|_{C^{1,2 s-1-\varepsilon}\left(B_{2-11}\right)} \leq C$, so that

$$
\begin{equation*}
\|u\|_{C^{2,2 s-1-\varepsilon}\left(B_{2}-12\right)} \leq C \tag{6.34}
\end{equation*}
$$

This finishes the proof of $(i)$ after a scaling and covering.
For (ii), we consider first the case $m=1$. Clearly (6.34) and (6.33) imply that $K_{h}^{u} \in \mathscr{K}^{s}(\kappa, 2 s-1+$ $\left.\alpha, Q_{2^{-13}}\right)$, for all $h \in B_{2^{-13}}$ and $\alpha \in(0,1)$. In particular, by Lemma 2.3 ( iii ), we have $\left\|G_{h}\right\|_{C^{0, \alpha}\left(B_{2}-13\right)} \leq$ $C\left\|u_{h, 1}\right\|_{L^{\infty}\left(B_{1}\right)}$. Now by (6.34) and Lemma 6.2 for all $h \in B_{2^{-13}}$, we have $\left\|\Gamma_{h, 1}^{u}\right\|_{C^{1,2 s-1-\varepsilon}\left(B_{2}-13\right)} \leq C$. Therefore, applying Theorem 1.4 to the equation (6.29), we get $\|v\|_{C^{2 s+\alpha}\left(B_{2}-15\right)} \leq C$, provided $2 s+\alpha \notin \mathbb{N}$. Hence

$$
\|u\|_{C^{1+2 s+\alpha}\left(B_{2-16}\right)} \leq C .
$$

If now $m \geq 2$, then the above estimate implies that $K_{h}^{u} \in \mathscr{K}^{s}\left(\kappa, 2 s+\alpha, Q_{2^{-18}}\right)$ for all $h \in B_{2-18}$. Hence, Lemma 2.3 (iii) implies that $\left\|G_{h}\right\|_{C^{1, \alpha}\left(B_{2-18}\right)} \leq C\left\|u_{h, 1}\right\|_{L^{\infty}\left(B_{1}\right)}$. On the other hand, by Lemma 6.2, $\Gamma_{h, 1}^{u} \in C^{2 s+\alpha}\left(B_{2-16}\right) \subset C^{1, \alpha}\left(B_{2^{-16}}\right)$, because $2 s>1$. It then follows, from (6.29) and Theorem 1.4. that $\left\|\partial_{x_{i}} v\right\|_{C^{2 s+\alpha}\left(B_{2}-18\right)} \leq C$. This yields $\left\|\partial_{x_{i}} u\right\|_{C^{1+2 s+\alpha}\left(B_{2}-19\right)} \leq C$, because $v=\varphi_{1 / 8} u_{h, 1}$. Now iterating the above argument, then for all $k \in\{1, \ldots, m\}$ and $i=\{1, \ldots, N\}$, we can find two constants $r_{k}$, only depending on $k$, and a constant $C_{k}>0$, only depending on $N, s,\|u\|_{C^{0,1}\left(B_{2}\right)}, k, \alpha$ and $\|f\|_{C^{m, \alpha}\left(B_{2}\right)}$, such that

$$
\left\|\partial_{x_{i}}^{k} u_{h, 1}\right\|_{C^{2 s+\alpha}\left(B_{r_{k}}\right)} \leq C_{k}
$$

for all $h \in B_{r_{k} / 2}$. A covering and scaling arguments yield (iii).
6.2. Proof of Theorem 1.6 and Theorem 1.7. Up to a change of coordinates and a scaling, we may assume that a neighborhood of $0 \in \Sigma$ is parameterized by a $C^{1, \gamma}$-diffeomorphism $\Phi: B_{2} \rightarrow \Sigma$, for some $\gamma \in(0,1)$, satisfying $\Phi(0)=0$ and

$$
\begin{equation*}
|D \Phi(x)-i d| \leq \frac{1}{2} \quad \text { for all } x \in B_{2} \tag{6.35}
\end{equation*}
$$

We consider the following open sets in $\Sigma$ given by

$$
\mathcal{B}_{r}:=\Phi\left(B_{r}\right) \quad \text { for } r \in(0,2]
$$

and we define $\eta_{r}(\bar{x})=\varphi_{r}\left(\Phi^{-1}(\bar{x})\right)$. For $\Psi \in C_{c}^{\infty}\left(\mathcal{B}_{1 / 2}\right)$, we then we have

$$
\begin{align*}
\int_{\mathcal{B}_{2}} \int_{\mathcal{B}_{2}} \frac{(u(\bar{x})-u(\bar{y}))(\Psi(\bar{x})-\Psi(\bar{y}))}{|\bar{x}-\bar{y}|^{N+2 s}} \eta_{2}(\bar{x}) \eta_{2}(\bar{y}) d \sigma(\bar{x}) d \sigma(\bar{y}) & +\int_{\Sigma} V_{1}(\bar{x}) u(\bar{x}) \Psi(\bar{x}) d \sigma(\bar{x}) \\
& =\int_{\Sigma} f_{1}(\bar{x}) \Psi(\bar{x}) d \sigma(\bar{x}) \tag{6.36}
\end{align*}
$$

where

$$
\begin{equation*}
V_{1}(\bar{x}):=V(\bar{x})+\int_{\Sigma}\left(1-\eta_{2}(\bar{y})\right)|\bar{x}-\bar{y}|^{-N-2 s} d \sigma(\bar{y}) \tag{6.37}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{1}(\bar{x}):=f(\bar{x})+\int_{\Sigma}\left(1-\eta_{2}(\bar{y})\right) u(\bar{y})|\bar{x}-\bar{y}|^{-N-2 s} d \sigma(\bar{y}) \tag{6.38}
\end{equation*}
$$

We denote by $J a c_{\Phi}$ the Jacobian determinant of $\Phi$. Let $\psi(x)=\Psi(\Phi(x)), v(x)=u(\Phi(x)), \tilde{V}(x)=$ $V_{1}(x) J a c_{\Phi}(x)$ and $\widetilde{f}(x)=f_{1}(x) J a c_{\Phi}(x)$. Then by the changes of variables $\bar{x}=\Phi(x)$ and $\bar{y}=\Phi(y)$, in (6.36), we get

$$
\frac{1}{2} \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}}(v(x)-v(y))(\psi(x)-\psi(y)) K(x, y) d x d y+\int_{B_{1}} \widetilde{V}(x) u(x) \psi(x) d x=\int_{B_{1}} \widetilde{f}(x) \psi(x) d x
$$

where

$$
\begin{equation*}
K(x, y)=\varphi_{2}(x) \varphi_{2}(y) J a c_{\Phi}(x) J a c_{\Phi}(y)|\Phi(x)-\Phi(y)|^{-N-2 s} \tag{6.39}
\end{equation*}
$$

We further consider $w=\varphi_{1 / 4} v \in H^{s}\left(\mathbb{R}^{N}\right)$, so that by Lemma 2.3,

$$
\begin{equation*}
\mathcal{L}_{K} w+\widetilde{V} w=\widetilde{f}+G \quad \text { in } B_{1 / 16} \tag{6.40}
\end{equation*}
$$

where

$$
\begin{equation*}
G(x)=\int_{B_{2}}\left(1-\varphi_{1 / 4}(y)\right) v(y) K(x, y) d y \tag{6.41}
\end{equation*}
$$

Next, we observe that the function $\mathcal{A}_{K}: B_{1} \times[0,1] \times S^{N-1} \rightarrow \mathbb{R}^{N}$, given by

$$
\mathcal{A}_{K}(x, r, \theta)=\varphi_{2}(x) \varphi_{2}(x+r \theta) J a c_{\Phi}(x) J a c_{\Phi}(x+r \theta)\left|\int_{0}^{1} D \Phi(x+r \theta) \theta d t\right|^{-N-2 s}
$$

is an extension of $(x, r, \theta) \mapsto r^{N+2 s} K(x, x+r \theta)$ on $B_{1} \times[0,1] \times S^{N-1}$. Moreover, since $\Phi \in C^{1, \gamma}\left(B_{2}\right)$, we see that

$$
\begin{align*}
& \left\|\mathcal{A}_{K}\right\|_{C^{\gamma}\left(B_{1 / 2} \times[0,1 / 2] \times S^{N-1}\right)} \leq \frac{1}{\kappa}  \tag{6.42}\\
& \mathcal{A}_{K}(x, 0, \theta)=\mathcal{A}_{K}(x, 0,-\theta) \quad \text { for all }(x, \theta) \in \mathbb{R}^{N} \times S^{N-1}
\end{align*}
$$

for some $\kappa>0$, only depending on $N, s, \gamma$ and $\|\Phi\|_{C^{1, \gamma}\left(B_{2}\right)}$. Consequently by (6.42), (6.35) and (6.39), decreasing $\kappa$ if necessary, we see that $K \in \mathscr{K}^{s}\left(\kappa, \gamma, Q_{1 / 2}\right)$. In addition, from (6.37) and (6.38), we easily deduce that for $p>1$,

$$
\begin{equation*}
\|\widetilde{f}\|_{L^{p}\left(B_{1 / 16}\right)}+\|G\|_{L^{p}\left(B_{1 / 16}\right)} \leq C\left(\|u\|_{L_{s}(\Sigma)}+\|f\|_{L^{p}\left(\mathcal{B}_{2}\right)}\right) \quad \text { and } \quad\|\widetilde{V}\|_{L^{p}\left(B_{1 / 2}\right)} \leq C \tag{6.43}
\end{equation*}
$$

where $C$ is a constant only depending on $N, s, p, \gamma,\|\nabla \Phi\|_{C^{1, \gamma}\left(B_{2}\right)},\|V\|_{L^{p}\left(\mathcal{B}_{2}\right)}$ and $\|1\|_{L_{s}(\Sigma)}$.
Proof of Theorem 1.6 (completed). From the computations above, we have that $w=\varphi_{1 / 2} u \circ \Phi \in$ $H^{s}\left(\mathbb{R}^{N}\right)$ satisfies (6.40) with $K \in \mathscr{K}^{s}\left(\kappa, \gamma, Q_{1 / 2}\right)$. Thanks to (6.43), we can apply Theorem (1.3) to get the result.

Proof of Theorem 1.7 (completed). We know from Theorem 1.6 and the above argument that $w=$ $\varphi_{1 / 2} u \circ \Phi \in H^{s}\left(\mathbb{R}^{N}\right) \cap C^{\min (2 s-\varepsilon, 1)}\left(B_{1 / 4}\right)$, for all $\varepsilon \in(0,2 s)$ and solves (6.40) with $K \in \mathscr{K}^{s}\left(\kappa, \gamma, Q_{1 / 2}\right)$. However, in view of (6.37) and (6.38), we can use similar arguments as in the proof of Lemma 2.3 (iv) to deduce that

$$
\begin{equation*}
\|\widetilde{f}\|_{C^{\alpha}\left(B_{1 / 16}\right)} \leq C\left(\|u\|_{L_{s}(\Sigma)}+\|f\|_{C^{\alpha}\left(\mathcal{B}_{2}\right)}\right) \tag{6.44}
\end{equation*}
$$

and, using also (1.22), we get

$$
\begin{equation*}
\|\widetilde{V}\|_{C^{\alpha}\left(B_{1 / 16}\right)} \leq C\left(\|V\|_{C^{\alpha}\left(\mathcal{B}_{2}\right)}+\|1\|_{L_{s}(\Sigma)}\right) \leq C \tag{6.45}
\end{equation*}
$$

where here and below, the letter $C$ denotes a positive constant which may vary from line to line but only depends on $N, s, \alpha, \gamma,\|V\|_{C^{\alpha}\left(\mathcal{B}_{2}\right)},\|\nabla \Phi\|_{C^{1, \gamma}\left(B_{2}\right)}$ and $\|1\|_{L_{s}(\Sigma)}$. Moreover, recalling (6.41), applying Lemma 2.3(iii), we have that

$$
\begin{equation*}
\|G\|_{C^{\alpha}\left(B_{1 / 16}\right)} \leq C\|w\|_{L^{1}\left(B_{2}\right)} \leq C\|u\|_{L_{s}(\Sigma)} \tag{6.46}
\end{equation*}
$$

In view of (6.40), (6.44), (6.45) and (6.46), we can apply Theorem 1.4 (i) and use a bootstrap argument, to deduce that

$$
\begin{aligned}
\|w\|_{C^{2 s+\alpha}\left(B_{r_{0}}\right)} & \leq C\left(\|w\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|\widetilde{f}\|_{C^{\alpha}\left(B_{2}\right)}+\|G\|_{C^{\alpha}\left(B_{1 / 16}\right)}\right) \\
& \leq C\left(\|u\|_{L^{2}\left(\mathcal{B}_{2}\right)}+\|u\|_{L_{s}(\Sigma)}+\|f\|_{C^{\alpha}\left(\mathcal{B}_{2}\right)}\right)
\end{aligned}
$$

for some $r_{0}, C>0$, depends only on $N, s, \alpha, \gamma, c_{1}, c_{0},\|V\|_{C^{\alpha}\left(\mathcal{B}_{2}\right)},\|\nabla \Phi\|_{C^{1, \gamma}\left(B_{2}\right)}$ and $I_{s, \Sigma}$. The proof is thus completed by scaling, covering and a change of variables.

## 7. Appendix

Proof of Lemma 5.1. Case $2 s+\alpha<2$. For simplicity, recalling (1.25) and (1.26), we assume that

$$
\|A\|_{C^{k+2 s+\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}+\|B\|_{\mathcal{C}_{\tau_{s}}^{k}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)} \leq 1
$$

where $\tau_{s}:=\alpha+(2 s-1)_{+}$if $2 s \neq 1$ and $\tau_{1 / 2}:=\alpha+\varepsilon$ if $2 s=1$. We also assume that

$$
\|u\|_{C^{k+2 s+\alpha+\varepsilon_{s}\left(\mathbb{R}^{N}\right)}} \leq 1
$$

where $\varepsilon_{s}:=0$ if $2 s \neq 1$ and $\varepsilon_{s}:=\varepsilon$ if $2 s=1$.
We consider the case $m=0$. Since $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq 1$, we have

$$
\begin{equation*}
\left|\delta^{e} u(x, r, \theta)\right| \leq C \min \left(1, r^{2 s+\alpha}\right) \tag{7.1}
\end{equation*}
$$

Here, for $2 s \geq 1$, we use the fact that $2 \delta^{e} u(x, r)=r \int_{0}^{1}(\nabla u(x+\operatorname{tr} \theta)-\nabla u(x-\operatorname{tr} \theta)) \cdot \theta d t$. Moreover for $x_{1}, x_{2} \in \mathbb{R}^{N}$ and $r>0$, then for $2 s+\alpha<1$, we have

$$
\begin{equation*}
\left|\delta^{e} u\left(x_{1}, r, \theta\right)-\delta^{e} u\left(x_{2}, r, \theta\right)\right| \leq C \min \left(r^{2 s+\alpha},\left|x_{1}-x_{2}\right|^{2 s+\alpha}\right) \tag{7.2}
\end{equation*}
$$

and if $2 s \geq 1$, we have

$$
\begin{equation*}
\left|\delta^{e} u\left(x_{1}, r, \theta\right)-\delta^{e} u\left(x_{2}, r, \theta\right)\right| \leq C \min \left(r^{2 s+\alpha}, r\left|x_{1}-x_{2}\right|^{\tau_{s}}\right) \tag{7.3}
\end{equation*}
$$

On the other hand, for all $s \in(0,1)$,

$$
\begin{equation*}
\left|\delta^{o} u(x, r, \theta)\right| \leq C \min (1, r)^{\min (2 s+\alpha, 1)} \tag{7.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\delta^{o} u\left(x_{1}, r, \theta\right)-\delta^{o} u\left(x_{2}, r, \theta\right)\right| \leq C \min \left(r,\left|x_{1}-x_{2}\right|\right)^{\min (2 s+\alpha, 1)} \tag{7.5}
\end{equation*}
$$

Using (17.4), for $s \in(0,1)$, we estimate

$$
\begin{aligned}
\left|\mathcal{O}_{B, u}^{s}(x)\right| & \leq C \int_{0}^{\infty} \min (r, 1)^{\min (2 s+\alpha, 1)} \min (r, 1)^{(2 s-1)_{+}+\alpha} r^{-1-2 s} d r \\
& \leq C \int_{0}^{1} r^{\min (2 s+\alpha, 1)} r^{(2 s-1)_{+}+\alpha} r^{-1-2 s} d r+C \int_{1}^{\infty} r^{-1-2 s} d r
\end{aligned}
$$

so that,

$$
\begin{equation*}
\left\|\mathcal{O}_{B, u}^{s}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C \tag{7.6}
\end{equation*}
$$

We consider next $\mathcal{E}_{A, u}^{s}$. For all $x \in \mathbb{R}^{N}$ and for all $s \in(0,1)$, by (7.1), we have

$$
\left|\mathcal{E}_{A, u}^{s}(x)\right| \leq C \int_{0}^{\infty} \min \left(r^{2 s+\alpha}, 1\right) r^{-1-2 s} d r \leq C \int_{0}^{1} r^{\alpha-1} d r+C \int_{1}^{\infty} r^{-1-2 s} d r
$$

yielding

$$
\begin{equation*}
\left\|\mathcal{E}_{A, u}^{s}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C \tag{7.7}
\end{equation*}
$$

Let $x_{1}, x_{2} \in \mathbb{R}^{N}$ with $\left|x_{1}-x_{2}\right| \leq 1$. Using (7.5), for $s \in(0,1)$ we have

$$
\begin{aligned}
\mid \mathcal{O}_{B, u}^{s}\left(x_{1}\right) & -\mathcal{O}_{B, u}^{s}\left(x_{2}\right) \mid \leq C \int_{0}^{\infty} \min \left(r,\left|x_{1}-x_{2}\right|\right)^{\min (2 s+\alpha, 1)} \min (r, 1)^{\tau_{s}} r^{-1-2 s} d r \\
& +C \int_{0}^{\infty} \min (r, 1)^{\min (2 s+\alpha, 1)} \min \left(r,\left|x_{1}-x_{2}\right|\right)^{\tau_{s}} r^{-1-2 s} d r \\
& \leq C \int_{0}^{\left|x_{1}-x_{2}\right|} r^{\min (2 s+\alpha, 1)+\tau_{s}} r^{-1-2 s} d r+C\left|x_{1}-x_{2}\right|^{\min (2 s+\alpha, 1)} \int_{\left|x_{1}-x_{2}\right|}^{1} r^{\tau_{s}-1-2 s} d r \\
& +C\left|x_{1}-x_{2}\right|^{\tau_{s}} \int_{\left|x_{1}-x_{2}\right|}^{1} r^{\min (2 s+\alpha, 1)-1-2 s} d r \\
& +C\left|x_{1}-x_{2}\right|^{\min (2 s+\alpha, 1)} \int_{1}^{\infty} r^{-1-2 s} d r+C\left|x_{1}-x_{2}\right|^{\tau_{s}} \int_{1}^{\infty} r^{-1-2 s} d r \\
& \leq C\left|x_{1}-x_{2}\right|^{\alpha}
\end{aligned}
$$

In the above estimate, it is used that $\tau_{s}=\alpha+\varepsilon$, for $s=1 / 2$. This together with (7.6) imply that $\left\|\mathcal{O}_{B, u}^{s}\right\|_{C^{0, \alpha}\left(\mathbb{R}^{N}\right)} \leq C$, for all $s \in(0,1)$.
Now for $2 s \geq 1$, let $x_{1} \neq x_{2} \in \mathbb{R}^{N}$ with $\left|x_{1}-x_{2}\right| \leq 1$. Using (7.3) and (7.1) we have

$$
\begin{aligned}
& \left|\mathcal{E}_{A, u}^{s}\left(x_{1}\right)-\mathcal{E}_{A, u}^{s}\left(x_{2}\right)\right| \\
& \leq C \int_{0}^{\infty} \min \left(r^{2 s+\alpha}, r\left|x_{1}-x_{2}\right|^{\tau_{s}}\right) r^{-1-2 s} d r+C\left|x_{1}-x_{2}\right|^{\alpha} \int_{0}^{\infty} \min \left(r^{2 s+\alpha}, 1\right) r^{-1-2 s} d r \\
& \leq C \int_{0}^{\left|x_{1}-x_{2}\right|} r^{\alpha-1} d r+C\left|x_{1}-x_{2}\right|^{\tau_{s}} \int_{\left|x_{1}-x_{2}\right|}^{\infty} r^{-2 s} d r+C\left|x_{1}-x_{2}\right|^{\alpha} \leq C\left|x_{1}-x_{2}\right|^{\alpha}
\end{aligned}
$$

Hence using (7.7), for $2 s \geq 1$, we get $\left\|\mathcal{E}_{A, u}^{s}\right\|_{C^{0, \alpha}\left(\mathbb{R}^{N}\right)} \leq C$.
We now consider the case $2 s+\alpha<1$. For $x_{1}, x_{2} \in \mathbb{R}^{N},\left|x_{1}-x_{2}\right| \leq 1$, by (7.2), we estimate

$$
\begin{aligned}
& \left|\mathcal{E}_{A, u}^{s}\left(x_{1}\right)-\mathcal{E}_{A, u}^{s}\left(x_{2}\right)\right| \\
& \leq C \int_{0}^{\infty} \min \left(r,\left|x_{1}-x_{2}\right|\right)^{2 s+\alpha} r^{-1-2 s} d r+C\left|x_{1}-x_{2}\right|^{\alpha} \int_{0}^{\infty} \min \left(r^{2 s+\alpha}, 1\right) r^{-1-2 s} d r \\
& \leq C \int_{0}^{\left|x_{1}-x_{2}\right|} r^{-1+\alpha} d r+C\left|x_{1}-x_{2}\right|^{2 s+\alpha} \int_{\left|x_{1}-x_{2}\right|}^{\infty} r^{-1-2 s} d r+C\left|x_{1}-x_{2}\right|^{\alpha} \leq C\left|x_{1}-x_{2}\right|^{\alpha}
\end{aligned}
$$

We then conclude from this and (7.7) that $\left\|\mathcal{E}_{A, u}^{s}\right\|_{C^{0, \alpha}\left(\mathbb{R}^{N}\right)} \leq C$, provided $2 s+\alpha<1$.
If $m>1$, we can use the Leibniz formula for the derivatives of the product of two functions. Note that for all $\gamma \in \mathbb{N}^{N}$ with $|\gamma| \leq m$, we have that $\delta^{e} \partial^{\gamma} u$ (resp. $\delta^{o} \partial^{\gamma} u$ ) satisfies (7.1) and (7.2) (resp. (7.4) and (7.5).

Case $2 s+\alpha>2$. We first observe from the arguments in the previous case that

$$
\begin{align*}
&\left\|\mathcal{E}_{A, u}^{s}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C\|A\|_{C^{0}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha}\left(\mathbb{R}^{N}\right)}, \\
&\left\|\mathcal{O}_{B, u}^{s}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C\|A\|_{C_{1}^{0}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha}\left(\mathbb{R}^{N}\right)} \tag{7.8}
\end{align*}
$$

Since $B(y, 0, \theta)=0$, we have

$$
B\left(x_{1}, r, \theta\right)-B\left(x_{2}, r, \theta\right)=r \int_{0}^{1}\left(D_{r} B\left(x_{1}, \varrho r, \theta\right)-D_{r} B\left(x_{2}, \varrho r, \theta\right)\right) d \varrho
$$

On the other hand

$$
B\left(x_{1}, r, \theta\right)-B\left(x_{2}, r, \theta\right)=\int_{0}^{1} D_{x} B\left(\varrho x_{1}+(1-\varrho) x_{2}, r, \theta\right) \cdot\left(x_{1}-x_{2}\right) d \varrho
$$

The above two estimates yield

$$
\begin{align*}
& \left|B\left(x_{1}, r, \theta\right)-B\left(x_{2}, r, \theta\right)\right| \leq\left(\|B\|_{C^{2 s+\alpha-1}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\right. \\
& \left.\quad+\|B\|_{\mathcal{C}_{2 s+\alpha-2}^{1}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\right) \min \left(r\left|x_{1}-x_{2}\right|^{2 s+\alpha-2}, r^{2 s+\alpha-2}\left|x_{1}-x_{2}\right|\right) . \tag{7.9}
\end{align*}
$$

In addition, we have

$$
\delta^{o} u\left(x_{1}, r, \theta\right)-\delta^{o} u\left(x_{2}, r, \theta\right)=\int_{0}^{1} D_{x} \delta^{o} u\left(\varrho x_{1}+(1-\varrho) x_{2}, r, \theta\right) \cdot\left(x_{1}-x_{2}\right) d \varrho
$$

so that

$$
\left.\mid \delta^{o} u\left(x_{1}, r, \theta\right)-\delta^{o} u\left(x_{2}, r, \theta\right)\right) \mid \leq C\|u\|_{C^{2 s+\alpha}\left(\mathbb{R}^{N}\right)} \min \left(r, r^{2 s+\alpha-2}\left|x_{1}-x_{2}\right|\right)
$$

Using this and (7.9), we find that, for all $x_{1}, x_{2} \in \mathbb{R}^{N}$,

$$
\begin{equation*}
\left|\mathcal{O}_{B, u}^{s}\left(x_{1}\right)-\mathcal{O}_{B, u}^{s}\left(x_{2}\right)\right| \leq C\left(\|B\|_{C^{2 s+\alpha-1}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}+\|B\|_{\mathcal{C}_{2 s+\alpha-2}^{1}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\right)\left|x_{1}-x_{2}\right|^{\alpha} \tag{7.10}
\end{equation*}
$$

Next, we write $2 \delta^{e} u(x, r)=r \int_{0}^{1}(\nabla u(x+\operatorname{tr} \theta)-\nabla u(x-\operatorname{tr} \theta)) \cdot \theta d t$ from which we deduce that

$$
\delta^{e} u(x, r)=r^{2} \int_{0}^{1} t \int_{0}^{1} D_{x}^{2} \delta^{o} u(x, \varrho t r, \theta)[\theta, \theta] d \varrho d t
$$

and

$$
\delta^{e} u\left(x_{1}, r\right)-\delta^{e} u\left(x_{2}, r\right)=r \int_{0}^{1} \int_{0}^{1} D_{x}^{2} \delta^{o} u\left(\varrho x_{1}+(1-\varrho) x_{2}, t r, \theta\right)\left[x_{1}-x_{2}, \theta\right] d t d \varrho
$$

By combining the above two estimates, we get

$$
\left|\delta^{e} u\left(x_{1}, r\right)-\delta^{e} u\left(x_{2}, r\right)\right| \leq C\|u\|_{C^{2 s+\alpha}\left(\mathbb{R}^{N}\right)} \min \left(r^{2 s+\alpha}, r^{2 s+\alpha-1}\left|x_{1}-x_{2}\right|\right)
$$

Using now the above estimate and the fact that $A \in C^{\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)$, we immediately deduce that $\left[\mathcal{E}_{A, u}^{s}\right]_{C^{\alpha}\left(\mathbb{R}^{N}\right)} \leq C\|A\|_{C^{\alpha}\left(Q_{\infty}\right) \times L^{\infty}\left(S^{N-1}\right)}\|u\|_{C^{2 s+\alpha}\left(\mathbb{R}^{N}\right) \text {. From this, (7.8) and (7.10), we get the }}$ statement in the lemma for $m=0$ and $2 s+\alpha>2$. In the general case that $m \geq 1$, we can use the Leibniz formula for the derivatives of the product of two functions and argue as above to get the desired estimates.

## References

[1] N. Abatangelo and L. Dupaigne, Nonhomogeneous boundary conditions for the spectral fractional Laplacian, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), no. 2, 439-467.
[2] N. Abatangelo and E. Valdinoci, Getting acquainted with the fractional Laplacian. Contemporary Research in Elliptic PDEs and Related Topics pp 1-105.
[3] G. Barles, E. Chasseigne, C. Imbert, Hölder continuity of solutions of second-order elliptic integro-differential equations, J. Eur. Math. Soc. 13 (2011), 1-26.
[4] G. Barles, E. Chasseigne, C. Imbert, The Dirichlet problem for second-order elliptic integro-differential equations, Indiana Univ. Math. J. 57 (2008), 213-146.
[5] B. Barrios, A. Figalli, E. Valdinoci, Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 13 (2014), 609-639.
[6] R. Bass, D. Levin, Harnack inequalities for jump processes, Potential Anal. 17 (2002), 375-388.
[7] E. Bombieri, E. De Giorgi, M. Miranda, Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche, Arch. Ration. Mech. Anal. 32 (1969), 255-267.
[8] C. Bucur and M. Squassina, Asymptotic mean value properties for fractional anisotropic operators. J. Math. Anal. Appl., 466, 1, 2018, Pages 107-126
[9] C. Bucur, E. Valdinoci, Nonlocal diffusion and applications, Lecture Notes of the Unione Matematica Italiana 20, Springer International Publishing Switzerland 2016.
[10] X. Cabré and M. Cozzi, A gradient estimate for nonlocal minimal graphs. Duke Math. J. 168 (2019), no. 5, 775-848.
[11] X. Cabré, M. M. Fall, T. Weth, Near-sphere lattices with constant nonlocal mean curvature. Math. Ann. (2018) 370(3), 1513-1569.
[12] L. Caffarelli and X. Cabré, Fully Nonlinear Elliptic Equations, Amer.Math. Soc. Colloq. Publ. 43, Amer. Math. Soc., Providence, 1995.
[13] L. Caffarelli, C. H. Chan and A. Vasseur. Regularity theory for parabolic nonlinear integral operators. J. Am. Math. Soc V. 24, No 3, 2011, 849-869.
[14] L. Caffarelli, J.-M. Roquejoffre, O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math. 63 (2010), 11111144.
[15] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), 597-638.
[16] L. Caffarelli, L. Silvestre, Regularity results for nonlocal equations by approximation, Arch. Rat. Mech. Anal. 200 (2011), 59-88.
[17] L. Caffarelli, L. Silvestre, The Evans-Krylov theorem for nonlocal fully nonlinear equations, Ann. of Math. 174 (2011), 1163-1187.
[18] L. Caffarelli, L. Silvestre Hölder regularity for generalized master equations with rough kernels, Advances in analysis: the legacy of Elias M. Stein, Princeton Math. Ser., vol. 50, Princeton Univ. Press, Princeton, NJ, 2014, pp. 63-83.
[19] L. Caffarelli, P. E. Souganidis, Convergence of nonlocal threshold dynamics approximations to front propagation, Arch. Ration. Mech. Anal. 195 (2010), 1-23.
[20] L. Caffarelli, E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments, Adv. Math. 248 (2011) 25 Pages 843-871.
[21] M. Cozzi, Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes, J. Funct. Anal. 272 (2017), no. 11, 4762-4837.
[22] J. Dávila, M. del Pino, J. Wei, Nonlocal s-minimal surfaces and Lawson cones, to appear in J. Diff. Geom.
[23] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3), 3:2-43, 1957.
[24] A. Di Castro, T. Kuusi and G. Palatucci, Local behavior of fractional p-minimizers. Ann. Inst. H. Poincaré Anal. Non Linéaire. V. 33, 5, 2016, 1279-1299.
[25] S. Dipierro, O. Savin, E. Valdinoci, Definition of fractional Laplacian for functions with polynomial growth, Rev. Mat. Iberoam. 35 (2019), no. 4, 1079-1122.
[26] S. Dipierro, O. Savin, E. Valdinoci, Graph properties for nonlocal minimal surfaces, Calc. Var. Partial Differential Equations 55 (2016), no. 4, Paper No. 86, 25 pp.
[27] B. Dyda and M. Kassman, Regularity estimates for elliptic nonlocal operators. Anal. PDE Volume 13, Number 2 (2020), 317-370.
[28] M. M. Fall, Constant Nonlocal Mean Curvature surfaces and related problems. Proc. Int. Cong. of Math. 2018 Rio de Janeiro, Vol. 1 (21-30) DOI: 10.9999/icm2018-v1-p21.
[29] M. M. Fall, V. Felli, A. Ferrero, and A. Niang, Asymptotic expansions and unique continuation at DirichletNeumann boundary junctions for planar elliptic equations. Mathematics in Engineering, 2018, 1(1): 84-117. doi: 10.3934/Mine.2018.1.84.
[30] M. M. Fall, Regularity estimates for nonlocal Schrödinger equation. Discrete Contin. Dyn. Syst. 39.3 (2019), 1405-1456.
[31] M. M. Fall and T. Weth, Monotonicity and nonexistence results for some fractional elliptic problems in the half space. Commun. Contemp. Math. 18 (2016), no. 1, 1550012, 25 pp.
[32] M. Felsinger, M. Kassmann, P. Voigt, The Dirichlet problem for nonlocal operators. Math. Z. 279 (2015), no. 3-4, 779-809.
[33] X. Fernández-Real, X. Ros-Oton, Regularity theory for general stable operators: parabolic equations. J. Funct. Anal. 272 (2017), no. 10, 4165-4221.
[34] X. Fernández-Real and X. Ros-Oton, Boundary regularity for the fractional heat equation, Rev. Acad. Cienc. Ser. A Math. 110 (2016), 49-64.
[35] A. Figalli, E. Valdinoci, Regularity and Bernstein-type results for nonlocal minimal surfaces, J. Reine Angew. Math. 729 (2017), 263-273.
[36] A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini, Isoperimetry and stability properties of balls with respect to nonlocal energies, Commun. Math. Phys. 336 (2015), 441-507.
[37] R. Finn, New estimates for equations of minimal surface type, Arch. Ration. Mech. Anal. 14 (1963), 337-375.
[38] D. Gilbarg, N. Trudinger, l S. Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.
[39] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. Multiscale Model. Simul., 7(3):1005-1028, 2008.
[40] G. Grubb, Fractional Laplacians on domains, a development of Hormander's theory of $\mu$-transmission pseudodifferential operators, Adv. Math. 268 (2015, 478-528).
[41] G. Grubb, Local and nonlocal boundary conditions for $\mu$-transmission and fractional elliptic pseudodifferential operators, Anal. PDE 7 (2014), 1649-1682.
[42] T. Jin, J. Xiong, Schauder estimates for nonlocal fully nonlinear equations. Ann. Inst. H. Poincaré Anal. Non Linéaire V. 33, (5), 2016, 1375-1407.
[43] W. P. Johnson, The Curious History of Faá di Bruno's Formula, Am. Math. Monthly 109 (2002), 217-227.
[44] M. Kassmann, A priori estimates for integro-differential operators with measurable kernels. Calc. Var. Partial Differential Equations 34 (2009), 1-21.
[45] M. Kassmann, A. Mimica, Intrinsic scaling properties for nonlocal operators. J. Eur. Math. Soc. (JEMS) 19 (2017), no. 4, 983-1011.
[46] M. Kassmann, M. Rang, R. W. Schwab, Integro-differential equations with nonlinear directional dependence. Indiana Univ. Math. J. 63 (2014), no. 5, 1467-1498.
[47] M. Kassmann, R. W. Schwab, Regularity results for nonlocal parabolic equations. Riv. Math. Univ. Parma (N.S.) 5 (2014), no. 1, 183-212.
[48] D. Kriventsov, $C^{1, \alpha}$ interior regularity for nonlinear nonlocal elliptic equations with rough kernels. Comm. Partial Differential Equations 38 (2013), no. 12, 2081-2106.
[49] T. Kuusi, G. Mingione, Y. Sire: nonlocal equations with measure data. Commun. Math. Phys. (2015), v. 337, 3, pp 1317-1368.
[50] G. Mingione, Gradient potential estimates. J. Eur. Math. Soc. 13, 459-486 (2011).
[51] S. Mosconi, Optimal elliptic regularity: a comparison between local and nonlocal equations. Discrete Contin. Dyn. Syst. Volume 11, Number 3, (2018), 547-559.
[52] C. Mou and Y. Yi, Interior Regularity for Regional Fractional Laplacian. Commun. Math. Phys. (2015) 340 : 233.
[53] X. Ros-Oton, J. Serra, Regularity theory for general stable operators, J. Differential Equations 260 (2016), 86758715.
[54] X. Ros-Oton, J. Serra, Boundary regularity for fully nonlinear integro-differential equations, Duke Math. J. 165 (2016), no. 11, 2079-2154.
[55] R. W. Schwab, L. Silvestre, Regularity for parabolic integro-differential equations with very irregular kernels. Anal. PDE 9 (2016), no. 3, 727-772.
[56] J. Serra, $C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels, Calc. Var. Partial Differential Equations 54 (2015), 3571-3601.
[57] J. Serra, Regularity for fully nonlinear nonlocal parabolic equations with rough kernels. Calc. Var. Partial Differential Equations 54 (2015), no. 1, 615-629.
[58] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60 (2007), 67-112.
[59] L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. Indiana Univ. Math. J., 55(3):1155-1174, 2006.
[60] E. Stein, Singular integrals and differentiability properties of functions. Princeton, University Press, 1970.
[61] E. V. Teixeira, Sharp regularity for general Poisson equations with borderline sources. J. Math. Pures Appl. (9) 99 (2013), no. 2, 150-164.
M.M.F.: African Institute for Mathematical Sciences in Senegal, KM 2, Route de Joal, B.P. 1418. Mbour, SÉNÉGAL

E-mail address: mouhamed.m.fall@aims-senegal.org, mouhamed.m.fall@gmail.com


[^0]:    The author's work is supported by the Alexander von Humboldt foundation. He thanks Joaquim Serra, for his interest in this work and with whom he had stimulating discussions that help to improve the first section of this paper. He also thanks the anonymous referee for useful comments.

