GRADIENT ESTIMATES IN FRACTIONAL DIRICHLET PROBLEMS

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ABSTRACT. We obtain some fine gradient estimates near the boundary for solutions to fractional elliptic problems subject to exterior Dirichlet boundary conditions. Our results provide, in particular, the sign of the normal derivative of such solutions near the boundary of the underlying domain.

1. INTRODUCTION

Let Ω be an open bounded subset of \mathbb{R}^N with $C^{1,1}$ boundary and let $s \in (0, 1)$. In this paper we analyze the boundary behavior of distributional solutions $u \in C^s(\mathbb{R}^N)$ to the equation

$$(-\Delta)^{s} u = f(x, u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^{N} \setminus \Omega, \tag{1.1}$$

where $f \in L^{\infty}_{loc}(\mathbb{R}^N \times \mathbb{R})$ and $(-\Delta)^s$ denotes the fractional Laplacian and is defined for $\varphi \in C^{\infty}_{c}(\mathbb{R}^N)$ by

$$(-\Delta)^s \varphi(x) := c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{\varphi(x) - \varphi(x+y)}{|y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N$$

with a normalization constant $c_{N,s} = \frac{s4^s \Gamma(\frac{N}{2}+s)}{\pi^{N/2} \Gamma(1-s)}$. Here and in the following, we assume, in the case $s \in (0, 1/2]$ that, for some $\sigma \in (1-2s, 1)$ and for all M > 0, there exists a constant A_M such that

$$\sup_{t \in (-M,M)} [f(\cdot,t)]_{C^{0,\sigma}(\overline{\Omega})} + \sup_{x \in \Omega} [f(x,\cdot)]_{C^{0,1}[-M,M]} \le A_M.$$
(1.2)

We note that, under the assumptions on f, the solution u to (1.1) belongs to $C^1_{loc}(\Omega)$ by the interior regularity theory, see e.g. [6].

To study the boundary behavior of u, we consider a function δ , which coincides with the distance function dist $(\cdot, \mathbb{R}^N \setminus \Omega)$ in a neighborhood of $\partial\Omega$ and in $\mathbb{R}^N \setminus \Omega$. Moreover, we suppose that δ is positive in Ω and $\delta \in C^{1,1}(\overline{\Omega})$.

Letting $\psi = u/\delta^s$, the known boundary regularity theory for fractional elliptic equations (see e.g. Ros-Oton and Serra [8] followed by [2,5–7]) states that, for any $\alpha \in (0, s)$,

$$\|\psi\|_{C^{\alpha}(\overline{\Omega})} \le C \sup_{x \in \Omega} |f(x, u(x))|, \tag{1.3}$$

where $C = C(N, s, \alpha, \Omega)$ is a positive constant. Moreover, since $u = \delta^s \psi$, we have that

$$\delta^{1-s}(x)\nabla u(x) = s\psi(x)\nabla\delta(x) + \delta(x)\nabla\psi(x) \quad \text{for all } x \in \Omega.$$
(1.4)

However, the identity (1.4) and the estimate in (1.3) do not provide a fine asymptotic of $\nabla u(x)$ near $\partial \Omega$, since one cannot deduce from (1.3) a pointwise estimate of $\nabla \psi$ near $\partial \Omega$. In

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particular, the monotonicity of u in the normal direction near the boundary is in general not known and cannot be deduced from the fractional Hopf lemma, which provides only the sign of ψ on $\partial\Omega$, see e.g. [3]. The purpose of the present paper is to investigate these questions and we show that, for some $\beta > 0$,

$$\delta^{1-s}\nabla u \in C^{\beta}(\overline{\Omega})$$
 and $\delta^{1-s}(x)\nabla u(x) \cdot \nabla \delta(x) = s\psi(x)$ for all $x \in \partial\Omega$. (1.5)

We emphasize that under the assumptions on f, (1.5) does not follow from the known boundary regularity theory for fractional elliptic equations even if Ω is of class C^{∞} . Indeed, by the results of Grubb [4], we have that $\psi \in C^{\alpha}(\overline{\Omega})$ for all $\alpha \in (0, s)$ and also if $2s \leq 1$ then by (1.2), $\psi \in C^{s+\min(s,\sigma)}(\overline{\Omega})$, provided $s + \min(s,\sigma) \notin \mathbb{N}$. Clearly, each of these Hölder regularity on ψ does not imply a pointwise estimate of $\nabla \psi$ and cannot imply (1.5).

Our first main result is the following.

Theorem 1.1. Let $N \ge 1$, $s \in (1/2, 1)$, $\alpha \in (0, 1)$ and $\Omega \subset \mathbb{R}^N$ be an open bounded set of class $C^{1,\gamma}$, with $\gamma > s$. Let $u \in C^s(\mathbb{R}^N)$ and $g \in L^{\infty}(\mathbb{R}^N)$ be such that

$$(-\Delta)^s u = g \quad in \ \Omega, \qquad u = 0 \quad in \ \mathbb{R}^N \setminus \Omega.$$
 (1.6)

Let $\psi = u/\delta^s$ satisfy

$$\|\psi\|_{C^{\alpha}(\overline{\Omega})} \le C_0 \left(\|g\|_{L^{\infty}(\mathbb{R}^N)} + \|u\|_{L^{\infty}(\Omega)} \right), \tag{1.7}$$

for some constant $C_0 > 0$. Then provided $\alpha \neq s$, we have

$$|\nabla\psi(x)| \le C\delta^{\min(\alpha,s)-1}(x) \left(\|g\|_{L^{\infty}(\mathbb{R}^{N})} + \|u\|_{L^{\infty}(\Omega)} \right) \qquad \text{for almost all } x \in \Omega.$$
(1.8)

If moreover Ω is of class $C^{1,1}$, then for all $\beta \in (0, \min(\alpha, 2s - 1))$,

$$\|\delta^{1-s}\nabla u\|_{C^{\beta}(\overline{\Omega})} \le C\left(\|g\|_{L^{\infty}(\mathbb{R}^{N})} + \|u\|_{L^{\infty}(\Omega)}\right)$$
(1.9)

and

$$\delta^{1-s}(x)\nabla u(x) \cdot \nabla \delta(x) = s\psi(x) \qquad \text{for all } x \in \partial\Omega. \tag{1.10}$$

Here, $C = C(\Omega, N, s, \alpha, \gamma, \beta, C_0).$

We recall that by [1,6], if Ω is of class $C^{2,\varepsilon}$ and $g \in C^{\varepsilon}(\mathbb{R}^N)$, for some $\varepsilon > 0$, then (1.7) holds for some $\alpha > s$. In this case, $\delta^{\min(\alpha,s)-1}(x)$ in (1.8) can be replaced with $\delta^{s-1}(x)$.

To state our next results, we will consider a function $U \in C^{s}(\mathbb{R}^{N}) \cap C^{1}_{loc}(\Omega)$ satisfying

$$(-\Delta)^s U \in C^{\sigma}(\overline{\Omega}) \tag{1.11}$$

and

$$c\delta^s(x) \le U(x) \le \frac{1}{c}\delta^s(x)$$
 for all $x \in \mathbb{R}^N$, (1.12)

for some positive constant c.

Our second main result is the following.

Theorem 1.2. Let $N \ge 1$, $s \in (0, 1/2]$, and $\Omega \subset \mathbb{R}^N$ be an open bounded set of class $C^{1,1}$. Let $u \in C^s(\mathbb{R}^N)$ be a solution to (1.1), where f satisfies (1.2) and let U satisfy (1.11) and (1.12). Suppose that u satisfies (1.3), for some $\alpha \in (0, 1)$ with $\alpha \neq s$. Let $\Psi := \frac{U}{\delta^s}$ and suppose that $\Psi \in C^{\alpha}(\overline{\Omega})$. Then the following statements holds.

(i) We have

$$|\nabla \psi(x)| \le C \left(\delta^{\min(s,\alpha)-1}(x) + |\nabla \Psi(x)| \right) \qquad \text{for all } x \in \Omega.$$
(1.13)

(ii) If
$$\delta^{1-s}\nabla U \in C^{\gamma}(\overline{\Omega})$$
, for some $\gamma > 0$, then for all $\beta \in (0, \min\{\gamma, \alpha, s, \sigma - 1 + 2s\}]$,
 $\|\delta^{1-s}\nabla u\|_{C^{\beta}(\overline{\Omega})} \leq C.$ (1.14)

For $M := ||u||_{L^{\infty}(\mathbb{R}^N)}$, the constant C above depends only on N, s, β , Ω , γ , σ , α , A_M , U and $||f||_{L^{\infty}(\Omega \times [-M,M])}$.

As an example of a function $U \in C^{s}(\mathbb{R}^{N}) \cap C^{1}_{loc}(\Omega)$ satisfying (1.11) and (1.12), we can consider the solution to

$$(-\Delta)^s U = 1$$
 in Ω and $U = 0$ in $\mathbb{R}^N \setminus \Omega$. (1.15)

Here, by [4], if Ω is of class C^{∞} then $\Psi = U/\delta^s \in C^{\infty}(\overline{\Omega})$. Therefore, by combining (1.4), (1.14) and (1.13), we get (1.5).

Remark 1.3. We notice that the results in Theorem 1.1 and Theorem 1.2 remain valid if we replace $(-\Delta)^s$ with the anisotropic fractional Laplacian $(-\Delta)^s_a$ with $a \in L^{\infty}(S^{N-1})$ if $2s \leq 1$ and $a \in C^{\sigma}(S^{N-1})$ for some $\sigma > 1-2s$ if $2s \leq 1$. Here, the anisotropic fractional Laplacian is defined for $\varphi \in C_c^{\infty}(\mathbb{R}^N)$ as

$$(-\Delta)_a^s \varphi(x) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \left(\varphi(x) - \varphi(x+y) \right) \frac{a(y/|y|)}{|y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N$$

In these cases, by [5], the interior and boundary regularity that is needed in the proofs in Section 2 below remains valid.

Our next result is a consequence of Theorem 1.2 and the recent results in [1] where the authors show the existence of a function satisfying (1.11) and (1.12) in $C^{1,1}$ domains.

Corollary 1.4. Let $N \ge 1$, $s \in (0, 1/2]$, and $\Omega \subset \mathbb{R}^N$ be an open bounded set of class $C^{1,1}$. Let $u \in C^s(\mathbb{R}^N)$ be a solution to (1.1) satisfying (1.3), where f satisfies (1.2). Provided $\alpha \neq s$, the following statements hold.

(i) We have

$$|\nabla \psi(x)| \le C\delta^{\min(s,\alpha)-1}(x) \qquad \text{for all } x \in \Omega.$$
(1.16)

(ii) For all $\beta \in (0, \min\{\alpha, s, \sigma - 1 + 2s\}],$

$$\|\delta^{1-s}\nabla u\|_{C^{\beta}(\overline{\Omega})} \le C \tag{1.17}$$

and

$$\delta^{1-s}(x)\nabla u(x)\cdot\nabla\delta(x) = s\psi(x) \qquad \text{for all } x \in \partial\Omega.$$
(1.18)

Here, for $M := ||u||_{L^{\infty}(\mathbb{R}^N)}$, the constant C above depends only on N, s, β , Ω , σ , α , A_M and $||f||_{L^{\infty}(\Omega \times [-M,M])}$.

Remark 1.5. In view of the above results, the following question remain open. Our arguments yield a bound of $|\nabla \psi|$ in terms of $\delta^{\min(s,\alpha)}$. Does the estimate $|\nabla \psi| \leq C \delta^{\alpha-1}$ hold for some $\alpha > s$?

To prove Theorem 1.1, we consider the function $y \mapsto v_x(y) := \delta^s(y)(\psi(y) - \psi(x))$, with $y \in B(x, \delta(x))$. Note that $v_x(y) = u(y) - \delta^s(y)\psi(x)$ and its order of vanishing near $\partial\Omega$ is $\delta^{s+\min(\alpha,s)}(x)$. We then apply interior regularity theory to the translated and rescaled equation for v_x to deduce the estimate $\|v_x\|_{C^{1,\beta}(B(x,\delta(x)/2))} \leq C\delta^{s+\min(\alpha,s)}(x)$, from which we conclude the proof. In the case of Theorem 1.2, we adopt the same strategy as in the proof of Theorem 1.1. However, since we do not know a sharp result for the Hölder continuity of $(-\Delta)^s \delta^s$ in $C^{1,1}$ domains, we replace δ^s with U in the definition of v_x . We then apply interior regularity theory and a bootstrap argument to the translated and rescaled problem.

2. Proof of the main results

We recall the interior regularity for the fractional Laplacian for equation to $(-\Delta)^s v = g$ in B_1 with $v \in L^{\infty}(\mathbb{R}^N)$. Then, see e.g. [5], we have the following estimates with a constant C depending only on N, s and τ .

(i) If 2s > 1 and $\tau \in (0, 2s - 1)$,

$$\|v\|_{C^{1,\tau}(B_{1/2})} \le C(\|g\|_{L^{\infty}(B_1)} + \|v\|_{L^{\infty}(B_1)} + \|v\|_{L^1_s}).$$
(2.1)

(ii) If $2s + \tau \notin \mathbb{N}$, then

$$\|v\|_{C^{2s+\tau}(B_{1/2})} \le C(\|g\|_{C^{\tau}(B_1)} + \|v\|_{L^{\infty}(B_1)} + \|v\|_{L^1_s}).$$
(2.2)

Here and in the following $B_t := B(0,t)$ denotes the centered ball of radius t > 0 in \mathbb{R}^N and

$$\|v\|_{L^1_s} = \int_{\mathbb{R}^N} \frac{|v(x)|}{1+|x|^{N+2s}} dx.$$

2.1. Proof of Theorem 1.1. For simplicity, we will assume that

$$||u||_{L^{\infty}(\mathbb{R}^N)} + ||g||_{L^{\infty}(\mathbb{R}^N)} \le 1.$$

From now on, C always denotes a positive constant depending on N, Ω , s, σ , α , β and γ , which may change from line to line.

Since Ω is of class $C^{1,\gamma}$, we can assume that δ , defined Section 1, is Lipschitz continuous in \mathbb{R}^N . Fix $x \in \Omega$ and for $z \in \mathbb{R}^N$ we define

$$u_x(z) := u(x + z\delta(x)), \quad \delta_x(z) := \delta(x + z\delta(x)), \text{ and } v_x(z) := u_x(z) - \delta_x^s(z)\psi(x).$$

Since δ is Lipschitz continuous, for $z \in B_{1/2}$ we have that

$$\frac{1}{2}\delta(x) \le \delta_x(z) \le 2\delta(x) \quad \text{and} \quad |\nabla\delta_x(z)| \le C\delta(x).$$
(2.3)

Since $\psi \in C^{\alpha}(\overline{\Omega})$ and $\delta^{s} \in C^{s}(\mathbb{R}^{N})$, we have for $z \in \mathbb{R}^{N}$

$$|v_x(z)| = \delta_x^s(z)|\psi(x) - \psi(x + \delta(x)z)| \le C\delta^{s+\alpha}(x)|z|^{\alpha}(1+|z|^s)\mathbf{1}_{\frac{\Omega-x}{\delta(x)}}(z),$$

where we used that δ is zero on $\mathbb{R}^N \setminus \Omega$. As a consequence, for $z \in B_1$ we have

$$|\psi_x(z)| = \delta_x^s(z)|\psi(x) - \psi(x + \delta(x)z)| \le C\delta^{s+\alpha}(x).$$

$$(2.4)$$

We observe that for some $R = R(\Omega)$, we have $\frac{\Omega - x}{\delta(x)} \subset B_{R/\delta(x)}$. In particular, if $\alpha \neq s$,

$$\begin{aligned} \|v_x\|_{L^1_s} &\leq C\delta^{s+\alpha}(x) + C\delta^s(x) \int_{B_{R/\delta(x)} \setminus B_1} \frac{\delta^{\alpha}(x)|z|^{\alpha}}{|z|^{N+s}} dz \\ &\leq C\delta^{s+\alpha}(x) + C\delta^{s+\alpha}(x) \int_1^{R/\delta(x)} t^{\alpha-1-s} dt \\ &\leq C\delta^{s+\alpha}(x)(1+\delta^{s-\alpha}(x)). \end{aligned}$$

We then conclude that, for $\alpha \neq s$,

$$||v_x||_{L^1_s} \le C\delta^{s+\min(s,\alpha)}(x).$$
 (2.5)

Next, we note that by the scaling properties of the fractional Laplacian, we have for $z \in B_1$

$$(-\Delta)^{s} v_{x}(z) = \delta^{2s}(x)g(x+\delta(x)z) - \psi(x)\delta^{2s}(x)[(-\Delta)^{s}\delta^{s}](x+\delta(x)z).$$
(2.6)
mplete the proof of the theorem

We now complete the proof of the theorem.

Proof of Theorem 1.1 completed. We start by recalling that by [7, Proposition 2.6], if Ω is of class $C^{1,\gamma}$ with $\gamma > s$, then

$$|(-\Delta)^s \delta^s(x)| \le C \qquad \text{for all } x \in \Omega.$$

From the assumptions on g, (2.4), and (2.5) the estimate (2.1) applied to the equation (2.6) gives, for $\beta \in (0, 2s - 1)$ and $\alpha \neq s$,

$$\|v_x\|_{C^{1,\beta}(B_{1/4})} \le C\left(\delta^{2s}(x) + \delta^{2s}(x) \sup_{z \in B_{1/2}} |(-\Delta)^s \delta^s(x + \delta(x)z)| + \|v_x\|_{L^{\infty}(B_{1/2})} + \|v_x\|_{L^1_s}\right) \le C\left(\delta^{2s}(x) + \delta^{s + \min(s,\alpha)}(x)\right) \le C\delta^{s + \min(s,\alpha)}(x).$$

$$(2.7)$$

We then deduce from this that, for all $x \in \Omega$

$$|\nabla u(x) - \psi(x)\nabla\delta^s(x)| = \delta^{-1}(x)|\nabla v_x(0)| \le C\delta^{s+\min(s,\alpha)-1}(x).$$
(2.8)

Since $\delta^s(x)\nabla\psi(x) = \nabla u(x) - \psi(x)\nabla\delta^s(x)$, we get (1.8).

We now prove (1.9) and we recall our assumption that Ω is of class $C^{1,1}$. By (2.7), for $\alpha \neq s$, we get

$$\|\nabla u - \psi(x)\nabla\delta^s\|_{L^{\infty}(B(x,\delta(x)/4))} \le C\delta^{s+\min(s,\alpha)-1}(x)$$
(2.9)

and

$$\begin{aligned} [\nabla u - \psi(x) \nabla \delta^{s}]_{C^{0,\beta}(B(x,\delta(x)/4))} &= \delta^{-1-\beta}(x) [\nabla u - \psi(x) \nabla \delta^{s}]_{C^{0,\beta}(B(0,1/4))} \\ &\leq C \delta^{s + \min(s,\alpha) - 1 - \beta}(x). \end{aligned}$$
(2.10)

Define $w_x(y) := \delta^{1-s}(y) (\nabla u(y) - \psi(x) \nabla \delta^s(y))$. Then, for $y_1, y_2 \in B(x, \delta(x)/4)$ and by (2.9) and (2.10), we have

$$|w_{x}(y_{1}) - w_{x}(y_{2})| \leq C\left(\delta^{-s}(x)|y_{1} - y_{2}|\delta^{s+\min(s,\alpha)-1}(x) + \delta^{1-s}(x)\delta^{s+\min(s,\alpha)-1-\beta}(x)|y_{1} - y_{2}|^{\beta}\right) \leq C\left(\delta^{1-\beta-s}(x)|y_{1} - y_{2}|^{\beta}\delta^{s+\min(s,\alpha)-1}(x) + \delta^{\min(s,\alpha)-\beta}(x)|y_{1} - y_{2}|^{\beta}\right) \leq C\delta^{\min(s,\alpha)-\beta}(x)|y_{1} - y_{2}|^{\beta},$$

where we used that $\delta \in C^1(B(x, \delta(x)/4)))$ and (2.3). Hence form this and (2.9), provided $\beta \in (0, \min(\alpha, s, 2s - 1))$, we get

$$[w_x]_{C^{\beta}(B(x,\delta(x)/4)))} \le C\delta^{\min(s,\alpha)-\beta}(x) \le C.$$

Therefore, noticing that $\delta^{1-s}(y)\nabla u(y) = w_x(y) - \psi(x)\nabla\delta(y), \ \nabla\delta \in C^{0,1}(B(x,\delta(x)/4))$ and $|\psi(x)| \leq C$, we find, for all $\beta \in (0, \min(\alpha, s, 2s - 1))$, that

$$[\delta^{1-s}\nabla u]_{C^{\beta}(B(x,\delta(x)/4)))} \le C.$$

It then follows from a very similar argument as in the proof of [8, Proposition 1.1] that $[\delta^{1-s}\nabla u]_{C^{\beta}(\overline{\Omega})} \leq C$. Since $\|\delta^{1-s}\nabla u\|_{L^{\infty}(\Omega)} \leq C$ by (2.8), we get (1.9).

Finally, since Ω is of class $C^{1,1}$, then $\nabla \delta \in C(\overline{\Omega})$ and $\nabla \psi \in C(\Omega)$. Therefore (1.10) follows from (1.4), (1.8) and (1.9). The proof is thus complete.

2.2. **Proof of Theorem 1.2.** To simplify the write up, we assume for the following that with $M = ||u||_{L^{\infty}(\Omega)}$ we have $||f||_{L^{\infty}(\Omega \times [-M,M])} \leq 1$ and $A_M \leq 1$ (recall (1.2)).

From now on, the letter C always denotes a positive constant depending on N, Ω , s, σ , α , β , γ , U and M, which may change from line to line.

By (1.12) and (1.3), we see that $\overline{\psi} = \frac{u}{U} = \frac{\psi}{\Psi} \in C^{\alpha}(\overline{\Omega})$. In the following, we fix $r_0 \leq c^{\frac{1}{s}}$, with c being the constant appearing in (1.12). Recalling (1.12), for $x \in \Omega$ and $z \in \mathbb{R}^N$, we define

 $\overline{u}_x(z) := u(x + zU^{\frac{1}{s}}(x)), \quad \overline{U}_x(z) := U(x + zU^{\frac{1}{s}}(x)), \quad \text{and} \quad \overline{v}_x(z) := \overline{u}_x(z) - \overline{U}_x(z)\overline{\psi}(x).$ We observe that, since $\overline{\psi} = u/U$,

$$\overline{v}_x(z) = \overline{U}_x(z) \left(\overline{\psi}_x(z) - \overline{\psi}(x) \right).$$

Therefore in view of (1.12) and the fact that $\overline{\psi} \in C^{\alpha}(\overline{\Omega})$, by using similar argument as in the beginning of Section 2.1, we find that, provided $\alpha \neq s$,

$$\|\overline{v}_x\|_{L^1_s} \le C\delta^{s+\min(s,\alpha)}(x) \tag{2.11}$$

and

$$\|\overline{v}_x\|_{L^{\infty}(B_{r_0/2})} \le C\delta^{s+\min(s,\alpha)}(x).$$
 (2.12)

Now direct computations, based on the scaling property of the fractional Laplacian and (1.11), yield for all $r_0 \leq c^{\frac{1}{s}}$ and $z \in B_{r_0/2}$

$$(-\Delta)^s \overline{v}_x(z) = U^2(x) f\left(x + U^{\frac{1}{s}}(x)z, \overline{u}_x(z)\right) - \overline{\psi}(x) U^2(x) (-\Delta)^s U(x + U^{\frac{1}{s}}(x)z).$$
(2.13)

We now complete the proof of the theorem.

Proof of Theorem 1.2 completed. We start by recalling that $2s \leq 1$. Let

$$\overline{g}_x(z) = f\left(x + U^{\frac{1}{s}}(x)z, \overline{u}_x(z)\right) = f\left(x + U^{\frac{1}{s}}(x)z, \overline{v}_x(z) + \overline{\psi}(x)\overline{U}_x(z)\right).$$
(2.14)

Then from the assumptions on f and (1.12), we get

$$\|\overline{g}_x\|_{C^{\min(s,\sigma)}(B_{r_0/2})} \le C\left(1 + \|\overline{v}_x\|_{C^s(B_{r_0/2})}\right).$$
(2.15)

By (1.11), (2.2) and (2.13), provided $\tau_0 = 2s + \min(s, \sigma) \notin \mathbb{N}$, we have

$$\|\overline{v}_x\|_{C^{\tau_0}(B_{r_0/4})} \le C\left(U^2(x)\|\overline{g}_x\|_{C^{\min(s,\sigma)}(B_{r_0/2})} + U^2(x) + \|\overline{v}_x\|_{L^{\infty}(B_{r_0/2})} + \|\overline{v}_x\|_{L^1_s}\right),$$

and if $\tau_0 \in \mathbb{N}$, we can replace $\|\overline{v}_x\|_{C^{\tau_0}(B_{r_0/4})}$ above with $\|\overline{v}_x\|_{C^{\tau_0-\varepsilon}(B_{r_0/4})}$ for an arbitrary small $\varepsilon > 0$. Hence using (2.15), (2.12), and (2.11) we obtain

$$\|\overline{v}_x\|_{C^{\tau_0}(B_{r_0/4})} \le C\left(\|\overline{v}_x\|_{C^{\min(s,\sigma)}(B_{r_0/2})} + 1\right)\delta^{s+\min(s,\alpha)}(x) \le C\delta^{s+\min(s,\alpha)}(x).$$
(2.16)

We consider a sequence of numbers $r_i = c^{\frac{1}{s}} 2^{-i-2}$ and $\tau_{i+1} = \min(2s + \tau_i, \sigma)$ for $i \in \mathbb{N}$. Then by (2.13) and (2.14), for all $z \in B_{r_i}$, $i \in \mathbb{N}$ we have

$$(-\Delta)^s \overline{v}_x(z) = U^2(x)\overline{g}_x(z) - \overline{\psi}(x)U^2(x)(-\Delta)^s U(x + U^{\frac{1}{s}}(x)z).$$

Hence iterating the above argument, provided $\tau_{i+1} \notin \mathbb{N}$ (or else we replace τ_{i+1} with $\tau_{i+1} - \varepsilon$ for an arbitrary small $\varepsilon > 0$), we get

$$\|\overline{v}_x\|_{C^{\tau_{i+1}}(B_{\tau_{i+1}})} \le C\left(\|\overline{v}_x\|_{C^{\tau_i}(B_{\tau_i})} + 1\right)\delta^{s+\min(s,\alpha)}(x).$$

Therefore by (2.16) and the fact that $1 > \sigma > 1 - 2s$, we must have that for some $i_0 \in \mathbb{N}$,

$$\|\overline{v}_x\|_{C^{1,\sigma-1+2s}(B_{r_{i_0}})} \le C\delta^{s+\min(s,\alpha)}(x).$$
(2.17)

This implies, in particular that

$$|\nabla \overline{v}_x(0)| \le C\delta^{s+\min(s,\alpha)}(x).$$
 for all $x \in \Omega$,

Therefore, since by (1.12),

$$\nabla u(x) - \overline{\psi}(x)\nabla U(x)| = U^{-\frac{1}{s}}(x)|\nabla \overline{v}_x(0)| \le C\delta^{s+\min(s,\alpha)-1}(x).$$

Using that $|U\nabla\overline{\psi}| = |\nabla u(x) - \overline{\psi}(x)\nabla U(x)|$ and (1.12), we then get

$$|\nabla \overline{\psi}(x)| \le C\delta^{\min(s,\alpha)-1}(x)$$
 for all $x \in \Omega$

and thus, since $\overline{\psi} = \frac{\psi}{\Psi}$,

$$|\nabla \psi(x)| \le C\left(\delta^{\min(s,\alpha)-1}(x) + |\nabla \Psi(x)|\right)$$
 for all $x \in \Omega$,

which is (1.13).

To see (1.14), we argue as in the proof of the case 2s > 1 in Section 2.1. Indeed, we start by noting that, using (1.12) and (2.17), we can find a constant $c_0 = c_0(\Omega, N, s) > 0$ such that

$$\|\overline{v}_x\|_{C^{1,\sigma-1+2s}(B(x,c_0\delta(x)))} \le C\delta^{s+\min(s,\alpha)}(x),$$

provided $\alpha \neq s$. This implies that

$$\|\nabla u - \overline{\psi}(x)\nabla U\|_{L^{\infty}(B(x,c_0\delta(x)))} \le C\delta^{s+\min(s,\alpha)-1}(x)$$
(2.18)

and for all $\beta \in (0, \sigma - 1 + 2s]$

$$[\nabla u - \overline{\psi}(x)\nabla U]_{C^{\beta}(B(x,c_0\delta(x)))}C\delta^{s+\min(s,\alpha)-1-\beta}(x).$$
(2.19)

To finish the proof, we proceed as in Section 2.1. Define $\overline{w}_x = \delta^{1-s} \left(\nabla u - \overline{\psi}(x) \nabla U \right)$ and then, for $y_1, y_2 \in B(x, c_0 \delta(x))$ and $\beta \in (0, \min(\sigma - 1 + 2s)]$ we have

$$\left|\overline{w}_x(y_1) - \overline{w}_x(y_2)\right| \le C\delta^{\min(s,\alpha) - \beta}(x)|y_1 - y_2|^{\beta},\tag{2.20}$$

with the same calculation as in the proof of Theorem 1.1 based on (2.18), (2.19) and the regularity of δ . Therefore, provided $\alpha \neq s$, we have for $\beta \in (0, \min(\alpha, s, \sigma - 1 + 2s)]$

$$[\overline{w}_x]_{C^\beta(B(x,c_0\delta(x)))} \le C.$$

Next, we define $V(y) := \delta^{1-s}(y) \nabla u(y)$, and we note that

$$V(y) = \overline{w}_x(y) + \delta^{1-s}(y)\nabla U(y)\overline{\psi}(x)$$

By assumption $\delta^{1-s}\nabla U \in C^{\gamma}(\overline{\Omega})$ and thus $\delta \nabla \Psi \in L^{\infty}(\Omega)$, so that by (1.13) and (1.4)

$$\|V\|_{L^{\infty}(\Omega)} \le C. \tag{2.21}$$

On the other hand by (2.20) we have

$$[V]_{C^{\beta}(B(x,c_0\delta(x)))} \leq C \quad \text{for all } \beta \in (0,\min\{\gamma,\alpha,s,\sigma-1+2s\}],$$

provided $\alpha \neq s$. Arguing similarly as in the proof of [8, Proposition 1.1] we have $[V]_{C^{\beta}(\overline{\Omega})} \leq C$. This together with (2.21) yield (*ii*). Proof of Corollary 1.4. First, we observe that the results hold in $\Omega_{\beta} := \{x \in \Omega : \delta(x) \ge \beta\}$ for all $\beta > 0$. From now on, we fix $\beta > 0$ small such that any point $x \in \Omega \setminus \Omega_{\beta}$ has a unique projection $\sigma(x) \in \partial\Omega$ and that the map $x \mapsto \sigma(x)$ is $C^1(\overline{\Omega} \setminus \Omega_{\beta})$.

By [1], there exists a function $U \in C^s(\mathbb{R}^N) \cap C^2_{loc}(\Omega)$ satisfying (1.11), (1.12) and, for all $\varepsilon \in (0, 1)$, there exists a constant $C_0 = C_0(s, \Omega, N, \varepsilon)$ such that

$$U^{\frac{1}{s}} \in C^{1,1-\varepsilon}(\overline{\Omega})$$
 and $|D^2 U^{\frac{1}{s}}(x)| \le C_0 \delta^{-\varepsilon}(x)$ for all $x \in \Omega$. (2.22)

From now on, we fix $\varepsilon = 1 - \alpha \in (0, 1)$. In the following the constant C is as in the statement of the corollary.

Since U = 0 on $\partial\Omega$, for $x \in \Omega \setminus \Omega_{\beta}$, by the mean value theorem, we have

$$U^{\frac{1}{s}}(x) = \delta(x) \int_0^1 \nabla U^{\frac{1}{s}}(x - t\delta(x)\nabla\delta(\sigma(x))) \cdot \nabla\delta(\sigma(x)) dt.$$
(2.23)

From this, (2.22) and the fact that δ is smooth on $\partial \Omega$ (in fact it vanishes there), we deduce that for every $x \in \Omega \setminus \Omega_{\beta}$,

$$|\nabla (U^{\frac{1}{s}}/\delta)| \le C_0 \int_0^1 \delta \left(x - t\delta(x)\nabla \delta(\sigma(x))\right)^{\alpha - 1} dt + C \le C\delta^{\alpha - 1}(x).$$

It then follows that, for every $x \in \Omega \setminus \Omega_{\beta}$,

$$|\nabla \Psi(x)| = |\nabla (U^{\frac{1}{s}}/\delta)^{s}(x)| = s(U^{\frac{1}{s}}/\delta)^{s-1}(x)|\nabla (U^{\frac{1}{s}}/\delta)(x)| \le C\delta^{\alpha-1}(x),$$

where we used (1.12). Hence by the regularity of U and δ , we obtain

$$|\nabla \Psi(x)| \le C\delta^{\alpha-1}(x)$$
 for all $x \in \Omega$.

Therefore by applying Theorem 1.2 (i), we obtain (i).

We now prove (ii). We start by noting that, by (2.22), (2.23) and (1.12) we get

$$\frac{U^{\frac{1}{s}}}{\delta} \in C^{\alpha}(\overline{\Omega} \setminus \Omega_{\beta}) \quad \text{and} \quad \frac{U^{\frac{1}{s}}}{\delta} \ge C \quad \text{in } \overline{\Omega} \setminus \Omega_{\beta}.$$
 (2.24)

Direct computations yield

$$\delta^{1-s}\nabla U = \delta^{1-s}\nabla (U^{\frac{1}{s}})^s = s\delta^{1-s} (U^{\frac{1}{s}})^{s-1}\nabla U^{\frac{1}{s}} = s(U^{\frac{1}{s}}/\delta)^{s-1}\nabla U^{\frac{1}{s}}.$$

From this, (2.24), (2.22) and (2.23) we obtain $\delta^{1-s}\nabla U \in C^{\alpha}(\overline{\Omega} \setminus \Omega_{\beta})$. We then conclude, from the regularity of U and δ , that $\delta^{1-s}\nabla U \in C^{\alpha}(\overline{\Omega})$. Now by Theorem 1.2 (*ii*) we have $\|\delta^{1-s}\nabla u\|_{C^{\beta}(\overline{\Omega})} \leq C$ and the proof of (1.14) is complete.

Finally (1.18) follows from (1.4), (1.16) and (1.14).

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