ENTIRE s-HARMONIC FUNCTIONS ARE AFFINE

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ABSTRACT. In this paper, we prove that solutions to the equation $(-\Delta)^s u = 0$ in \mathbb{R}^N , for $s \in (0, 1)$, are affine. This allows us to prove the uniqueness of the Riesz potential $|x|^{2s-N}$ in Lebesgue spaces.

1. INTRODUCTION

The classical Liouville theorem for harmonic functions states that *a bounded har*monic function in \mathbb{R}^N is constant, see for instance the particularly short proof by E. Nelson in [12]. The stronger version of it states that a nonnegative harmonic function on \mathbb{R}^N is constant. In the case of the fractional Laplacian $-(-\Delta)^s$, for $s \in (0, 1)$, (see Section 2), the strong form of the Liouville theorem holds as well and was proved by K. Bogdan et al. in [4]. Applications of Liouville theorems for nonlocal operators in the study of nonlocal elliptic systems of equations can be found in [1,8,13,14].

The aim of this paper is to classify all s-harmonic functions in \mathbb{R}^N , thereby obtaining the Liouville theorem for the fractional Laplacian as a particular case.

Theorem 1.1. Every s-harmonic function in \mathbb{R}^N is affine, and constant if $s \in (0, 1/2]$.

The proof of this theorem is mainly based on a Cauchy-type estimate for the derivatives of an s-harmonic function. More precisely, given $s \in (0, 1), \gamma \in \mathbb{N}^N$ and a function u which is s-harmonic in the ball B(0, R), we have the estimate

$$|D^{\gamma}u(0)| \le CR^{2s-|\gamma|} \int_{|y|\ge R/4} |u(y)||y|^{-N-2s} \, dy, \tag{1.1}$$

for some positive constant C depending only on N, γ and s, see Section 3. This estimate is obtained from the Poisson kernel representation formula for s-harmonic functions. We refer to Section 2 for more details.

In the following, we denote by $\mathcal{D}'(\mathbb{R}^N)$ the dual of $C_c^{\infty}(\mathbb{R}^N)$ endowed with the usual topology. An iteration argument based on Theorem 1.1 allows to state the following result.

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Theorem 1.2. Assume that $s \in (0,1)$ and let u be a solution to the equation

$$(-\Delta)^s u = P$$
 in $\mathcal{D}'(\mathbb{R}^N)$,

where P is a polynomial. Then u is affine and P = 0.

Another consequence of the main theorem which is of independent interest is the following result.

Corollary 1.3. Let $p \in [1, \infty)$ and $u \in L^p(\mathbb{R}^N)$ be such that $(-\Delta)^s u = 0 \quad in \mathcal{D}'(\mathbb{R}^N).$

Then $u \equiv 0$.

Combining Corollary 1.3 and the Hardy-Littlewood-Sobolev inequality, we have a uniqueness result.

Corollary 1.4 (Uniqueness of Riesz potential). Let $s \in (0,1)$, $1 and <math>f \in L^p(\mathbb{R}^N)$. Then there exists a unique $u \in L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$ such that

$$(-\Delta)^s u = f \quad in \ \mathcal{D}'(\mathbb{R}^N),$$

and u is given by

$$u(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} \, dy,$$

where

$$\alpha_{N,s} = \pi^{N/2} 2^{2s} \frac{\Gamma(s)}{\Gamma((N-2s)/2)}$$

The paper is organized as follows. In Section 2 we collect some basic facts concerning the fractional Laplacian $-(-\Delta)^s$ and s-harmonic functions. Finally, in Section 3 we prove the Cauchy-type estimate (1.1), the main result and its corollaries.

Note added in proof: We mention that after this paper was submitted, Liouvilletype results for a class of nonlocal operators were proved in [7] and [9] using Fourier transform.

2. Preliminaries

This section is devoted to recall some basic notions about *s*-harmonic functions. We refer the reader to [6, Section 3]. Let \mathcal{L}_s^1 denote the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}} dx < \infty.$$

For functions $\varphi \in C^2(\mathbb{R}^N) \cap \mathcal{L}^1_s$, the fractional Laplacian $-(-\Delta)^s$ is defined by

$$-(-\Delta)^{s}\varphi(x) = C_{N,s}\lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{\varphi(y) - \varphi(x)}{|y-x|^{N+2s}} \, dy \qquad \text{for all } x \in \mathbb{R}^{N}, \tag{2.1}$$

where $C_{N,s} = s(1-s)\pi^{-N/2}4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(2-s)}$.

For $u \in \mathcal{L}^1_s$, the expression $(-\Delta)^s u$ defines a distribution on every open set $\Omega \subset \mathbb{R}^N$ by

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u(x)(-\Delta)^s \varphi(x) \, dx \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

In the case where $(-\Delta)^s u = 0$ in $\mathcal{D}'(\Omega)$, we will say that u is s-harmonic in Ω . We note that affine functions u belong to \mathcal{L}^1_s if s > 1/2 and constant functions u belong to \mathcal{L}^1_s if $s \in (0, 1/2]$. Moreover, by using (2.1), in both cases, we can see that $(-\Delta)^s u(x) = 0$ for every $x \in \mathbb{R}^N$. Furthermore, thanks to [6, Lemma 3.3], we have $(-\Delta)^s u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

The fractional Laplacian has an explicit Poisson kernel with respect to the ball B(x, r) (see [3]). It is given by

$$P_r(x,y) = \begin{cases} \beta_{N,s} \frac{(r^2 - |x|^2)^s}{(|y|^2 - r^2)^s} |y - x|^{-N} & \text{for } |x| < r, \ |y| > r, \\ 0 & \text{otherwise,} \end{cases}$$
(2.2)

where $\beta_{N,s} = \Gamma(N/2)\pi^{-N/2-1}\sin(s\pi)$. Therefore (see also [4]), if u is s-harmonic in Ω then for every ball $B(a,r) \subset \subset \Omega$ we have

$$u(x) = \int_{\mathbb{R}^N} P_r(x - a, y - a)u(y) \, dy \qquad \text{for all } x \in B(a, r).$$

We now consider the regularization of P_r as in [6]. To this end, we pick a function $\phi \in C_c^{\infty}(1,4)$ such that $\int_{\mathbb{R}} \phi(r) dr = 1$ and define $\Psi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(y) = \int_{1}^{4} P_{r}(0, y)\phi(r) \, dr = \beta_{N,s} |y|^{-N} \int_{\min(1, |y|)}^{\min(4, |y|)} r^{2s} (|y|^{2} - r^{2})^{-s} \phi(r) \, dr.$$

Observe that, if $|y| \leq 1$ then $(0, |y|) \cap (1, 4) = \emptyset$ and thus

$$\Psi(y) = 0 \qquad \text{for every } y \in B(0,1). \tag{2.3}$$

Furthermore, as shown e.g. in [6, Lemma 3.11], we have $\Psi \in C^{\infty}(\mathbb{R}^N)$. Moreover, for every $\gamma \in \mathbb{N}^N$ there holds

$$|D^{\gamma}\Psi(y)| \le C |y|^{-N-2s-|\gamma|} \quad \text{for every } y \in \mathbb{R}^N \setminus \{0\}, \qquad (2.4)$$

where $C = C(N, \gamma, s)$ denotes, here and in the following, a positive constant depending only on N, γ and s.

We define $\Psi_{r_0}(y) = r_0^{-N} \Psi(y/r_0)$, for $y \in \mathbb{R}^N$ and $r_0 > 0$. Then, for any *s*-harmonic function u in an open set $\Omega \subset \mathbb{R}^N$, we have

$$u(x) = u \star \Psi_{r_0}(x)$$
 for all almost every $x \in \Omega_{4r_0}$, (2.5)

where $\Omega_{4r_0} = \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega) > 4r_0\}$, see [10, Lemma 2.6] or [6, Page 65]. We will therefore assume, in the sequel, that *s*-harmonic functions in some open set are smooth in that set.

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3. Proof of the main result and its consequences

The following result (from which we will derive our main result) can be seen as a nonlocal version of the Cauchy estimate for bounded harmonic functions, see e.g. [2, Chapter 2].

Lemma 3.1. For every $\gamma \in \mathbb{N}^N$, there exists a constant C > 0 only depending on N, γ and s such that for every function u which is s-harmonic in B(0, R),

$$|D^{\gamma}u(0)| \le C R^{2s-|\gamma|} \int_{|y|\ge R/4} |u(y)||y|^{-N-2s} \, dy.$$

Proof. Let u be an s-harmonic function in B(0, R) and $r_0 \in (0, R/4)$. Then by (2.5) we have

$$u(x) = u \star \Psi_{r_0}(x) \qquad \text{for all } x \in B(0, R - 4r_0).$$

By (2.3), (2.4) and the dominated convergence theorem, we deduce that

 $D^{\gamma}u(0) = u \star D^{\gamma}\Psi_{r_0}(0) \qquad \text{for all } \gamma \in \mathbb{N}^N.$

Using once more (2.3) and (2.4), we get

$$|D^{\gamma}u(0)| = \left|\int_{|y| \ge r_0} u(y)D^{\gamma}\Psi_{r_0}(-y)\,dy\right| \le C\,r_0^{2s}\int_{|y| \ge r_0} |u(y)|\,|y|^{-N-2s-|\gamma|}\,dy.$$

It follows that

$$|D^{\gamma}u(0)| \le C r_0^{2s-|\gamma|} \int_{|y|\ge r_0} |u(y)| \, |y|^{-N-2s} \, dy.$$

Letting $r_0 \to R/4$, we get the desired estimate.

As a consequence of Lemma 3.1, we have the following result.

Corollary 3.2. Let Ω be a nonempty open set of \mathbb{R}^N such that $\Omega \neq \mathbb{R}^N$. Then for every $\gamma \in \mathbb{N}^N$, there exists a constant C > 0 only depending on N, γ and s such that for every function u which is s-harmonic in Ω ,

$$|D^{\gamma}u(x)| \le C\,\delta_{\Omega}^{2s-|\gamma|}(x)\,\int_{|y|\ge\frac{\delta_{\Omega}(x)}{4}}|u(y)||y|^{-N-2s}\,dy\qquad \text{for all }x\in\Omega,$$

where $\delta_{\Omega}(x) = dist(x, \mathbb{R}^N \setminus \Omega).$

Proof. By assumption, $\delta_{\Omega}(x) < \infty$ for every $x \in \Omega$. Since $B(x, \delta_{\Omega}(x)) \subset \Omega$, applying Lemma 3.1 to the function $y \mapsto u(y+x)$ and $R = \delta_{\Omega}(x)$, we get the desired result.

Proof of Theorem 1.1.

Let $x \in \mathbb{R}^N$ and r > 0. We apply Corollary 3.2 with $\Omega = B(x, r)$ and $|\gamma| \ge 2s$. Then, letting $r \to \infty$, we get $|D^{\gamma}u(x)| = 0$ for every $|\gamma| \ge 2s$ and $x \in \mathbb{R}^N$. The proof of the theorem is thus completed. \Box

Remark 3.3. It is well known that there are smooth functions u — hence in $L^1_{loc}(\mathbb{R}^N)$ — satisfying $\Delta u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ for $N \ge 2$ which are not polynomials. Therefore a natural question arises: does there exist a larger space of distributions, strictly containing \mathcal{L}^1_s , where the fractional laplacian is appropriately defined and where there are nontrivial entire s-harmonic functions which are not affine?

Proof of Theorem 1.2. The proof will be done by induction. Suppose ℓ is the degree of P. Assume that $\ell = 0$ so that P is a constant. Let $h \in \mathbb{R}^N$ and $u_h(x) = u(x+h) - u(x)$. It is clear that $u_h \in \mathcal{L}^1_s$. In addition $(-\Delta)^s u_h = 0$. It follows from Theorem 1.1 that $\partial_{i,j}u_h(0) = 0$ and therefore $\partial_{i,j}u(h) = \partial_{i,j}u(0)$ for every $h \in \mathbb{R}^N$. This implies that u is a second order polynomial and since it belongs to \mathcal{L}^1_s , it is affine.

Now assume that the result holds true for a polynomial of degree up to $\ell \geq 0$ and suppose that $(-\Delta)^s u = P_{\ell+1}$, a polynomial of degree $\ell+1$. Then, for $h \in \mathbb{R}^N$, using the binomial formula we can see that $(-\Delta)^s u_h = P_{\ell,h}$, where $P_{\ell,h}$ is a polynomial of degree ℓ . It follows from our assumption that u_h is affine for any $h \in \mathbb{R}^N$. This again implies that u is a second order polynomial and thus an affine function, since it belongs to \mathcal{L}^1_s . \Box

Proof of Corollary 1.3. We just note that $L^q(\mathbb{R}^N) \subset \mathcal{L}^1_s$ for every $q \in [1, \infty]$ by Hölder's inequality. \Box

Proof of Corollary 1.4. We define the function $\tilde{u}(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy$. Let $f_n \in C_c^{\infty}(\mathbb{R}^N)$ be such that $f_n \to f$ in $L^p(\mathbb{R}^N)$. Define $u_n(x) = \alpha_{N,s} \int_{\mathbb{R}^N} \frac{f_n(y)}{|x-y|^{N-2s}} dy$. By the Hardy-Littlewood-Sobolev inequality (see [11, Theorem 4.3]), we have $u_n \to \tilde{u}$ in $L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$. In particular, $u_n \to \tilde{u}$ in \mathcal{L}_s^1 by Hölder's inequality. Thanks to [5, Lemma 5.3], we have $(-\Delta)^s u_n = f_n$ in $\mathcal{D}'(\mathbb{R}^N)$. Passing to the limit as $n \to \infty$, we deduce that $(-\Delta)^s \tilde{u} = f$ in $\mathcal{D}'(\mathbb{R}^N)$. Finally, if $u \in L^{\frac{Np}{N-2sp}}(\mathbb{R}^N)$ is an arbitrary solution to $(-\Delta)^s u = f$ in $\mathcal{D}'(\mathbb{R}^N)$ then $(-\Delta)^s(u - \tilde{u}) = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. We thus conclude, from Corollary 1.3, that $u = \tilde{u}$. \Box

References

- N. Abatangelo, Large s-harmonic functions and boundary blow-up solutions for the fractional laplacian. Discrete Contin. Dyn. Syst. A, 35; (2015), no. 12, 5555-5607.
- [2] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory. Second edition.* Graduate Texts in Mathematics, 137. Springer-Verlag, New York, 2001.
- [3] R. M. Blumenthal, R. K. Getoor and D. B. Ray, On the distribution of first hits for the symmetric stable processes. Trans. Amer. Math. Soc. 99 (1961) 540-554.
- [4] K. Bogdan, T. Kulczycki, A. Nowak, Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes. Illinois J. Math. 46 (2002), no. 2, 541-556.
- [5] K. Bogdan and T. Byczkowski, Potential theory of Schrödinger operator based on fractional Laplacian. Probab. Math. Statist. 20 (2000), no. 2, Acta Univ. Wratislav. No. 2256, 293-335.
- [6] K. Bogdan, T. Byczkowski, Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains. Studia Math. 133 (1999), no. 1, 53-92.

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- [7] W. Chen, L. D'Ambrosio and Y. Li, Some Liouville theorem for the fractional Laplacian. Nonlin. Anal. 121, (2015) 370-381.
- [8] F. Ferrari and I. Verbitsky, Radial fractional Laplace operators and Hessian inequalities. J. Differential Equations 253 (2012), no. 1, 244-272.
- [9] M. M. Fall and T. Weth, *Liouville theorems for a general class of nonlocal operators*. http://arxiv.org/abs/1504.00419.
- [10] M. M. Fall and T. Weth, Monotonicity and nonexistence results for some fractional elliptic problems in the half space. http://arxiv.org/abs/1309.7230 (To appear in Com. Contemp. Math. DOI: 10.1142/S0219199715500121)
- [11] E. H. Lieb, M. Loss, Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
- [12] E. Nelson, A proof of Liouville's theorem. Proc. Amer. Math. Soc. 12 (1961), 995.
- [13] X. Ros-Oton and J. Serra, *Regularity theory for general stable operators*. http://arxiv.org/abs/1412.3892.
- [14] R. Zhuo, W. Chen, X. Cui and Z. Yuan, A Liouville theorem for the fractional Laplacian. http://arxiv.org/pdf/1401.7402.

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