UNBOUNDED PERIODIC SOLUTIONS TO SERRIN'S OVERDETERMINED BOUNDARY VALUE PROBLEM

MOUHAMED MOUSTAPHA FALL, IGNACE ARISTIDE MINLEND, AND TOBIAS WETH

ABSTRACT. We study the existence of nontrivial unbounded domains Ω in $\mathbb{R}^N$ such that the overdetermined problem

$$-\Delta u = 1 \quad \text{in } \Omega, \qquad u = 0, \quad \partial_\nu u = \text{const} \qquad \text{on } \partial\Omega$$

admits a solution u. By this, we complement Serrin's classification result from 1971 which yields that every bounded domain admitting a solution of the above problem is a ball in $\mathbb{R}^N$. The domains we construct are periodic in some variables and radial in the other variables, and they bifurcate from a straight (generalized) cylinder or slab. We also show that these domains are uniquely self Cheeger relative to a period cell for the problem.

1. INTRODUCTION AND MAIN RESULT

In 1971, Serrin [23] established a celebrated result on the overdetermined problem of finding a domain $\Omega \subset \mathbb{R}^N$ and a C^2-function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$- \Delta u = 1 \qquad \text{in } \Omega \tag{1.1}$$

and

$$u = 0, \quad \partial_\nu u = \text{const} \qquad \text{on } \partial\Omega. \tag{1.2}$$

Here ν is the unit outer normal on $\partial\Omega$. More precisely, in [23] Serrin proved that, if Ω is a bounded domain of class C^2 such that (1.1), (1.2) admits a solution, then Ω is a ball. The problem (1.1), (1.2) arises in e.g. in fluid dynamics and the linear theory of torsion, and we refer the reader to [23, 25] for a detailed account on its relevance. As we shall discuss further below, it is also related to the notion of Cheeger sets, which in turn has applications in the denoising problem in image processing. The proof of Serrin's classification result for (1.1), (1.2) relies on the moving plane method, and it extends to the much more general problem where (1.1) is replaced by

$$- \Delta u = f(u), \quad u > 0 \qquad \text{in} \quad \Omega \tag{1.3}$$

with a locally Lipschitz continuous nonlinearity f. We note that the positivity assumption in (1.3) is essential, and by the strong maximum principle it is automatically satisfied for solutions of (1.1), (1.2). The moving plane method, which Serrin established in a PDE context, is inspired by Alexandrov's reflection principle [1] for constant mean curvature hypersurfaces. On the other hand, Weinberger [28] found a simpler

Key words and phrases. Overdetermined problems, Cheeger sets, periodic domains, Serrin problem.

argument to prove Serrin's result for problem (1.1), (1.2) without the moving plane method, but his argument does not cover the more general problem (1.2), (1.3).

The result of Serrin parallels an earlier important result by Alexandrov [1] which states that closed embedded hypersurfaces with constant mean curvature (CMC hypersurfaces in short) are round spheres. This rigidity result for bounded embedded CMC hypersurfaces stands in striking contrast to the rich structure of unbounded CMC hypersurfaces which has been explored in classical and more recent papers. For a survey, we refer the reader to [16]. We recall in particular that already in 1841, Delaunay [4] constructed and classified unbounded surfaces of revolution in $\mathbb{R}^3$ with constant mean curvature. As Delaunay's construction shows, these surfaces bifurcate from a straight cylinder (see also [20, Section 2] for a different proof of the latter statement).

For some time, it was unknown whether the problem (1.2), (1.3) admits solutions in nontrivial unbounded domains. In fact, in [2], Berestycki, Caffarelli and Nirenberg conjectured that if $\Omega \subset \mathbb{R}^N$ is an *unbounded* sufficiently regular domain such that $\mathbb{R}^N \setminus \overline{\Omega}$ is connected and $f : [0, \infty) \to \mathbb{R}$ is a local Lipschitz continuous function such that the overdetermined problem (1.2), (1.3) admits a solution, then Ω is an affine half space or the complement B^c of a ball $B \subset \mathbb{R}^N$ or a product of the form $\mathbb{R}^j \times B$ bzw. $\mathbb{R}^j \times B^c$ with a ball $B \subset \mathbb{R}^{N-j}$ (after a suitable rotation).

This conjecture has been disproved by Sicbaldi [24] in dimensions $N \geq 3$. More precisely, in the case where $f(u) = \lambda_1 u$ with $\lambda_1 > 0$ suitably chosen, it was shown in [24] that there exist periodic domains of revolution such that the problem (1.3), (1.2) admits a positive solution. Moreover, these domains bifurcate from the straight cylinder $\mathbb{R} \times B$, where $B \subset \mathbb{R}^{N-1}$ is a ball. The construction in [24] relies on topological degree theory and therefore does not give rise to a smooth branch of domains; moreover, the case $N = 2$ was not included. Later in [20], Sicbaldi and Schlenk extended the result to dimensions $N \geq 2$, and they obtained a smooth branch of domains via the use of the Crandall-Rabinowitz bifurcation theorem (see [3]).

In further recent papers, different types of nonlinearities f and unbounded domains have been considered in the context of the general Serrin problem (1.2), (1.3). In [13,27], the authors study examples of unbounded domains where (1.2), (1.3) is solvable with $f = 0$, i.e. with harmonic functions. Moreover, in [5], the authors consider a monostable nonlinearity f, and they construct domains whose boundary is close to dilations of a given CMC-hypersurface and such that (1.2), (1.3) is solvable. It is clear from these works that the existence and shape of such domains depend in a crucial way on the function f. For negative results, excluding the solvability of (1.2), (1.3) in nontrivial unbounded domains belonging to certain domain classes (e.g. epigraphs), we refer the reader to [8–11, 21, 22] and the references therein.

In the present paper, we wish to analyze the original form of Serrin's problem (1.1), (1.2), i.e. the case $f \equiv 1$, in unbounded domains. More precisely, we study domains of the form

$$\Omega_\phi := \{(z, t) \in \mathbb{R}^n \times \mathbb{R}^m \: : \: |z| < \phi(t)\} \subset \mathbb{R}^N,$$

where $N = n + m$ and $\phi : \mathbb{R}^m \to (0, \infty)$ is an even and $2\pi\mathbb{Z}^m$-periodic function. The following is our main result.

Theorem 1.1. *For each $n, m \geq 1$ and $\alpha \in (0, 1)$, there exists $\lambda_* = \lambda_*(n) > 0$ and a smooth map*

$$(-\varepsilon_0, \varepsilon_0) \longrightarrow (0, \infty) \times C^{2,\alpha}(\mathbb{R}^m)$$
$$s \longmapsto (\lambda_s, \varphi_s)$$

with $\varphi_0 \equiv 0$, $\lambda_0 = \lambda_*$ and such that for all $s \in (-\varepsilon_0, \varepsilon_0)$, letting $\phi_s = \lambda_s + \varphi_s$, there exists a solution $u \in C^{2,\alpha}(\overline{\Omega_{\phi_s}})$ of the overdetermined problem

$$
\begin{cases}
-\Delta u = 1 & in \quad \Omega_{\phi_s} \\
u = 0, \quad \partial_\nu u = -\dfrac{\lambda_s}{n} & on \quad \partial\Omega_{\phi_s}
\end{cases}
\tag{1.4}
$$

in the domain

$$\Omega_{\phi_s} = \left\{ (z, t) \in \mathbb{R}^n \times \mathbb{R}^m : \quad |z| < \phi_s(t) \right\}. \tag{1.5}$$

Moreover, for every $s \in (-\varepsilon_0, \varepsilon_0)$, the function φ_s is even in $t_1, \ldots, t_m$, 2π-periodic in $t_1, \ldots, t_m$ and invariant with respect to permutations of the variables $t_1, \ldots, t_m$. Furthermore, we have

$$\varphi_s(t) = s\left(\sum_{j=1}^m \cos(t_j) + \mu_s(t) \right) \qquad for \ s \in (-\varepsilon_0, \varepsilon_0)$$

with a smooth map $(-\varepsilon_0, \varepsilon_0) \to C^{2,\alpha}(\mathbb{R}^m)$, $s \mapsto \mu_s$ satisfying

$$\int_{[0,2\pi]^m} \mu_s(t) \cos(t_j) \, dt = 0 \qquad for \ s \in (-\varepsilon_0, \varepsilon_0), \ j = 1, \ldots, m,$$

and $\mu_0 \equiv 0$.

Note here that, since the domain Ω_{ϕ_s} is radially symmetric in z for every fixed $s \in (-\varepsilon_0, \varepsilon_0)$, the corresponding solution u is also radially symmetric in the z-variable. We also remark that the bifurcation value λ_* in Theorem 1.1 is the unique zero of the function

$$\sigma : (0, \infty) \to \mathbb{R}, \quad \sigma(\rho) = \begin{cases} \rho \tanh(\rho) - 1, & \text{in case } n = 1, \\ \dfrac{1}{n}\left(\rho\dfrac{I_{\nu+1}(\rho)}{I_\nu(\rho)} - 1 \right), & \text{in case } n \geq 2, \end{cases}$$

see Propositions 3.1 and 4.1 below. Here I_ν is the modified Bessel function of the first kind of order $\nu = \frac{n-2}{2}$. Numerically, λ_* is given by

$$\lambda_* \approx \begin{cases} 1.199 & \text{in case } n = 1; \\ 1.608 & \text{in case } n = 2; \\ 1.915 & \text{in case } n = 3. \end{cases}$$

As remarked above, Sicbaldi and Schlenk [20] have derived – in the special case $m = 1$ – a result analogous to Theorem 1.1 where (1.1) is replaced by (1.3) with $f(u) = \lambda_1 u$. Our proof of Theorem 1.1 is partly inspired by [20] and also relies on the Crandall-Rabinowitz Theorem, but there are key differences due to the special form of (1.1). We believe that our approach can also be generalized to study Serrin's overdetermined problem on Riemannian manifolds. Related to this, we mention the recent works [7,18]. In [7, Theorem 5], necessary conditions for the solvability of some overdetermined problems on manifolds are given, and in [18] the case $f(u) = \lambda_1 u$ is studied in the product manifolds $S^N \times \mathbb{R}$ and $H^N \times \mathbb{R}$.

The overdetermined problem (1.1), (1.2) is related to a generalized notion of Cheeger sets. To define this generalized notion, let S, Ω be open subsets of $\mathbb{R}^N$. For a subset $A \subset S$ with Lipschitz boundary, we let $P(A, S) := H^{N-1}(\partial A \cap S)$ denote the relative perimeter of A in S. Here and in the following, H^{N-1} denotes the $N - 1$-dimensional Hausdorff measure. For an equivalent definition which extends to Borel subsets A of S, see e.g. [15, Def. 13.6]. We then define the Cheeger constant of Ω relative to S as

$$h(\Omega, S) := \inf_{A \subset \Omega \cap S} \frac{P(A, S)}{|A|}, \tag{1.6}$$

where the infimum is taken over subsets $A \subset \Omega \cap S$ with Lipschitz boundary. If this constant is attained by some subset $A \subset \Omega \cap S$ with Lipschitz boundary, then A will be called a Cheeger set of Ω relative to S. If Ω has a Lipschitz boundary and $A = \Omega \cap S$ attains the constant $h(\Omega, S)$ in (1.6), we say that Ω is self-Cheeger relative to S. Moreover, if $A = \Omega \cap S$ is the only set which attains $h(\Omega, S)$, we say that Ω is uniquely self-Cheeger relative to S. These notions generalize the classical notions of the Cheeger constant and Cheeger sets which correspond to the case $S = \mathbb{R}^N$, see e.g. [14].

We have the following corollary of Theorem 1.1.

Corollary 1.2. *For every $s \in (-\varepsilon_0, \varepsilon_0)$ and $a, b \in \pi \mathbb{Z}^m$ with $a_i < b_i$ for $i = 1, \ldots, m$, the set Ω_{ϕ_s} given in Theorem 1.1 is uniquely self-Cheeger relative to the set*

$$S_a^b := \mathbb{R}^n \times (a_1, b_1) \times \cdots \times (a_m, b_m) \quad \subset \quad \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m \tag{1.7}$$

with corresponding relative Cheeger constant $h(\Omega_{\phi_s}, S_a^b) = \dfrac{n}{\lambda_s}$.

As discussed in detail in the illuminating surveys [14, 19], self-Cheeger sets arise in various problems as e.g. the construction of prescribed mean curvature graphs or the regularization of noisy images within the ROF model.

The link between Serrin's over-determined problem (1.1), (1.2) and Cheeger sets on N-dimensional Riemannian manifolds was also studied by the second author in [17], where he proved the existence of a family of uniquely self-Cheeger sets $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ with classical Cheeger constant $h(\Omega_\varepsilon) = \frac{N}{\varepsilon}$.

The paper is organized as follows. In Section 2 we transform the overdetermined problem to an equivalent boundary value problem on a fixed underlying domain with a

ϕ-dependent metric. In Section 3.1, we then study the eigenvalues and eigenfunctions of the linearization of the problem at constant functions $\phi \equiv \lambda$. In particular, we study the dependence of the eigenvalues on $\lambda > 0$. In Section 4, we then complete the proof of Theorem 1.1 via the Crandall-Rabinowitz Theorem. Finally, in Section 5, we give the proof of Corollary 1.2.

2. The transformed problem and its linearization

We fix $\alpha \in (0, 1)$ in the following. For $j \in \mathbb{N} \cup \{0\}$, we consider the Banach space

$$C_{p,e}^{j,\alpha}(\mathbb{R}^m) := \left\{ \phi \in C^{j,\alpha}(\mathbb{R}^m) \; : \; \phi \text{ is even and } 2\pi\text{-periodic in } t_1, \ldots, t_m \right\}$$

Let

$$\mathcal{U} := \{\phi \in C_{p,e}^{2,\alpha}(\mathbb{R}^m) \; : \; \phi > 0\}.$$

For a function $\phi \in \mathcal{U}$, we define

$$\Omega_\phi := \{(z, t) \in \mathbb{R}^n \times \mathbb{R}^m \; : \; |z| < \phi(t)\}$$

as well as the spaces

$$C_{p,e}^{j,\alpha}(\Omega_\phi, \mathbb{R}^k) := \left\{ u \in C^{j,\alpha}(\Omega_\phi, \mathbb{R}^k) : u \text{ is even and } 2\pi\text{-periodic in } t_1, \ldots, t_m \right\},$$

$$C_{p,e}^{j,\alpha}(\partial\Omega_\phi, \mathbb{R}^k) := \left\{ u \in C^{j,\alpha}(\partial\Omega_\phi, \mathbb{R}^k) : u \text{ is even and } 2\pi\text{-periodic in } t_1, \ldots, t_m \right\}$$

for $j = 0, 1, 2$, $k \in \mathbb{N}$. If $k = 1$, we simply write $C_{p,e}^{j,\alpha}(\overline{\Omega}_\phi)$ and $C_{p,e}^{j,\alpha}(\partial\Omega_\phi)$. Moreover, in the special case $\phi \equiv 1$ we write

$$\Omega := \Omega_1 = \{(z, t) \in \mathbb{R}^n \times \mathbb{R}^m \; : \; |z| < 1\}.$$

Every $\phi \in \mathcal{U}$ gives rise to a locally $C^{2,\alpha}$-regular map

$$\Psi_\phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}, \qquad \Psi_\phi(z, t) = (\phi(t)z, t). \tag{2.1}$$

such that Ψ_ϕ maps Ω diffeomorphically onto Ω_ϕ. Let the metric g_ϕ be defined as the pull back of the euclidean metric g_{eucl} under the map Ψ_ϕ, so that $\Psi_\phi : (\overline{\Omega}, g_\phi) \to (\overline{\Omega}_\phi, g_{eucl})$ is an isometry. Hence our original problem is equivalent to the overdetermined problem consisting of the Dirichlet problem

$$\begin{cases} -\Delta_{g_\phi} u = 1 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.2}$$

and the additional Neumann condition

$$\partial_{\nu_\phi} u \equiv -c \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

Here

$$\nu_\phi : \partial\Omega \to \mathbb{R}^{n+m}$$

is the unit outer normal vector field on $\partial\Omega$ with respect to g_ϕ. Since $\Psi_\phi : (\overline{\Omega}, g_\phi) \to (\overline{\Omega}_\phi, g_{eucl})$ is an isometry, we have

$$\nu_\phi = [d\Psi_\phi]^{-1} \mu_\phi \circ \Psi_\phi \qquad \text{on } \partial\Omega, \tag{2.4}$$

where $\mu_\phi : \partial\Omega_\phi \to \mathbb{R}^{n+m}$ denotes the outer normal on $\partial\Omega_\phi$ with respect to the euclidean metric g_{eucl} given by

$$\mu_\phi(z,t) = \frac{(\frac{z}{|z|}, -\nabla\phi(t))}{\sqrt{1 + |\nabla\phi(t)|^2}} \in \mathbb{R}^{n+m} \qquad \text{for } (z,t) \in \partial\Omega_\phi. \tag{2.5}$$

Here and in the following, we distinguish different types of derivatives in our notation. If $f : O \to \mathbb{R}^\ell$ is a C^1-map defined on an open set $O \subset \mathbb{R}^k$, we write $df(x) \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^\ell)$ for its derivative at a point $x \in O$. In contrast, we shall use the symbols D or D_ϕ to denote functional derivatives. More precisely, if X, Y are infinite dimensional normed (function) spaces and $F \in C^1(O, Y)$, where $O \subset X$ is open, we let $DF(\phi)$ or $D_\phi F(\phi) \in \mathcal{L}(X, Y)$ denote the Fréchet derivative of F at a function $\phi \in O$.

The following lemma is concerned with the well-posedness of problem (2.2).

Lemma 2.1. *For any $\phi \in \mathcal{U}$, there is a unique solution $u_\phi \in C^{2;\alpha}_{p,e}(\overline{\Omega})$ of (2.2), and the map*

$$C^{2,\alpha}_{p,e}(\mathbb{R}^m) \to C^{2,\alpha}_{p,e}(\overline{\Omega}), \qquad \phi \mapsto u_\phi \tag{2.6}$$

is smooth. Moreover we have the following properties.

(i) *For any $\phi \in \mathcal{U}$, the functions $u_\phi : \overline{\Omega} \to \mathbb{R}$ and $\partial_{\nu_\phi} u_\phi : \partial\Omega \to \mathbb{R}$ are radially symmetric in the z-variable.*

(ii) *For a constant function $\phi \equiv \lambda > 0$, we have $u_\lambda(z,t) = \frac{\lambda^2 - |\lambda z|^2}{2n}$.*

(iii) *Let $\mathcal{P} \subset \mathcal{L}(\mathbb{R}^m)$ denote the subset of all coordinate permutations in $t_1, \ldots, t_m$. If $\phi \in \mathcal{U}$ satisfies*

$$\phi(\boldsymbol{p}(t)) = \phi(t) \qquad \text{for all } t \in \mathbb{R}^m, \, \boldsymbol{p} \in \mathcal{P}, \tag{2.7}$$

then

$$u_\phi(z, \boldsymbol{p}(t)) = u_\phi(z, t) \qquad \text{for all } (z,t) \in \Omega, \, \boldsymbol{p} \in \mathcal{P} \tag{2.8}$$

and

$$\partial_{\nu_\phi} u_\phi(z, \boldsymbol{p}(t)) = \partial_{\nu_\phi} u_\phi(z, t) \qquad \text{for all } (z,t) \in \partial\Omega, \, \boldsymbol{p} \in \mathcal{P}. \tag{2.9}$$

Proof. Let

$$X := \{u \in C^{2,\alpha}_{p,e}(\overline{\Omega}) \, : \, u = 0 \text{ on } \partial\Omega\} \qquad \text{and} \qquad Y = C^{0,\alpha}_{p,e}(\overline{\Omega}).$$

Moreover, let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators $X \to Y$, and let $\mathcal{I}(X, Y) \subset \mathcal{L}(X, Y)$ denote the subset of topological isomorphisms $X \to Y$. Since the metric coefficients of g_ϕ are smooth functions of ϕ and $\nabla\phi$, it is easy to see that the map

$$\Upsilon : \mathcal{U} \to \mathcal{L}(X, Y), \qquad \phi \mapsto \Upsilon(\phi) := -\Delta_{g_\phi}$$

is smooth. Moreover, for $\phi \in \mathcal{U}$, the definition of g_ϕ implies that Δ_{g_ϕ} is an elliptic, coercive second order differential operator in divergence form with $C^{1,\alpha}(\overline{\Omega})$-coefficients. This immediately implies that, by the maximum principle and elliptic regularity, $\Upsilon(\phi) \in \mathcal{I}(X, Y)$ for every $\phi \in \mathcal{U}$, and consequently the problem (2.2) has a

unique solution $u_\phi \in X$ for every $\phi \in \mathcal{U}$. We now recall that $\mathcal{I}(X,Y) \subset \mathcal{L}(X,Y)$ is an open set and that the inversion

$$\mathrm{inv} : \mathcal{I}(X,Y) \to \mathcal{I}(Y,X), \qquad \mathrm{inv}(A) = A^{-1}$$

is smooth. Since $u_\phi = \mathrm{inv}(\Upsilon(\phi))1$, the smoothness of the map in (2.6) follows.

Next, to show (i), we fix $\phi \in \mathcal{U}$ and note that $u_\phi = \tilde{u}_\phi \circ \Psi_\phi$, where $\tilde{u}_\phi$ is the unique solution of the problem

$$-\Delta \tilde{u}_\phi = 1 \quad \text{in } \Omega_\phi, \qquad \tilde{u}_\phi = 0 \quad \text{on } \partial\Omega_\phi.$$

Since Ω_ϕ is invariant under rotations in the z-variable, the uniqueness implies that $\tilde{u}_\phi$ is radially symmetric in z and hence u_ϕ is also radially symmetric by the definition of Ψ_ϕ. Moreover, the outer unit normal $\mu_\phi : \partial\Omega_\phi \to \mathbb{R}^{n+m}$ with respect to g_{eucl} is equivariant with respect to rotations in z by (2.5), i.e.,

$$\mu_\phi(Az, t) = \tilde{A}\mu(z, t) \qquad \text{for all } (z, t) \in \partial\Omega_\phi, \ A \in O(n),$$

where $\tilde{A} \in O(n+m)$ is defined by $\tilde{A}(z, t) = (Az, t)$. It then follows that the function $\partial_{\mu_\phi} \tilde{u}_\phi : \partial\Omega_\phi \to \mathbb{R}$ is also radially symmetric in z, whereas by (2.4) we have

$$\partial_{\nu_\phi} u_\phi = du_\phi \nu_\phi = [d\tilde{u}_\phi \circ \Psi_\phi][\mu_\phi \circ \Psi_\phi] = [\partial_{\mu_\phi} \tilde{u}_\phi] \circ \Psi_\phi \quad \text{on } \partial\Omega.$$

As a consequence, the function $\partial_{\nu_\phi} u_\phi$ is also radially symmetric in the z-variable, as claimed.

Next we note that (ii) follows from the fact that in case $\phi \equiv \lambda > 0$ we have

$$g_\lambda^{ij} = \lambda^{-2}\delta_{ij} \text{ if } 1 \leq i, j \leq n \text{ and } g_\lambda^{ij} = g_\lambda^{ji} = \delta_{ij} \text{ for } 1 \leq i \leq n+m, \ n+1 \leq j \leq n+m \tag{2.10}$$

and therefore

$$\Delta_{g_\lambda} = \lambda^{-2}\Delta_z + \Delta_t. \tag{2.11}$$

Consequently, the function $(z, t) \mapsto \frac{\lambda^2 - |\lambda z|^2}{2n}$ solves (2.2) for $\phi \equiv \lambda$, and thus it coincides with u_λ, as claimed.

Finally, to show (iii), we fix $\phi \in \mathcal{U}$ such that (2.7) holds, and we let $\tilde{u}_\phi$ be defined as in the proof of (i). Then Ω_ϕ is invariant under coordinate permutations, and by uniqueness this implies that

$$\tilde{u}_\phi(z, \mathbf{p}(t)) = \tilde{u}_\phi(z, t) \qquad \text{for all } (z, t) \in \Omega_\phi, \ \mathbf{p} \in \mathcal{P}.$$

By definition of Ψ_ϕ, we then conclude that the function u_ϕ satisfies (2.8), as claimed. Moreover, by a similar argument as in the proof of (i), we find that (2.9) holds as well.

$\square$

By Lemma 2.1, condition (2.3) is equivalent to

$$[\partial_{\nu_\phi} u_\phi](e_1, t) = -c \qquad \text{for } t \in \mathbb{R}^m, \tag{2.12}$$

where $e_1 \in \mathbb{R}^n$ is the first coordinate vector. It follows from (2.4) and (2.5) that the map

$$\mathcal{U} \to C^{1,\alpha}_{p,e}(\partial\Omega, \mathbb{R}^{n+m}), \qquad \phi \mapsto \nu_\phi$$

is smooth, and thus we have a smooth map

$$H : \mathcal{U} \to C^{1,\alpha}_{p,e}(\mathbb{R}^m), \qquad H(\phi)(t) = \partial_{\nu_\phi} u_\phi(e_1, t), \tag{2.13}$$

whereas (2.12) writes as

$$H(\phi) \equiv -c \qquad \text{on } \mathbb{R}^m. \tag{2.14}$$

In order to find solutions of the latter equation bifurcating from the trivial branch of solutions $\phi \equiv \lambda$, $\lambda > 0$, we need to study the linearization of H at constant functions. The following is the main result of this section.

Proposition 2.2. *At a constant function $\lambda > 0$, the operator $\mathcal{H}_\lambda := DH(\lambda) \in \mathcal{L}(C^{2,\alpha}_{p,e}(\mathbb{R}^m), C^{1,\alpha}_{p,e}(\mathbb{R}^m))$ is given by*

$$[\mathcal{H}_\lambda \omega](t) = \frac{1}{n}\Big(\partial_\nu \psi_{\omega,\lambda}(e_1, t) - \omega(t)\Big) \qquad \text{for } \omega \in C^{2,\alpha}_{p,e}(\mathbb{R}^m), \ t \in \mathbb{R}^m, \tag{2.15}$$

where $\psi_{\omega,\lambda} \in C^{2,\alpha}_{p,e}(\overline{\Omega})$ is the unique solution of the problem

$$\begin{cases} \Delta_z \psi_{\omega,\lambda}(z, t) + \lambda^2 \Delta_t \psi_{\omega,\lambda}(z, t) = 0 & (z, t) \in \Omega, \\ \psi_{\omega,\lambda}(z, t) = \omega(t) & (z, t) \in \partial\Omega \end{cases} \tag{2.16}$$

and ν is the outer unit normal on $\partial\Omega$ with respect to g_{eucl} given by $\nu(z, t) = (z, 0)$.

The remainder of this section is devoted to the proof of Proposition 2.2. In the following, we put

$$\tilde{\nu}_\phi(\omega) := [D_\phi \nu_\phi]\omega \in C^{1,\alpha}_{p,e}(\partial\Omega, \mathbb{R}^{n+m}) \qquad \text{for } \phi \in \mathcal{U}, \ \omega \in C^{2,\alpha}_{p,e}(\mathbb{R}^m).$$

We start with the following simple observations.

Lemma 2.3.

(i) *The map $\mathcal{U} \to C^{2,\alpha}_{p,e}(\overline{\Omega}, \mathbb{R}^{n+m})$, $\phi \mapsto \Psi_\phi$ is smooth (recall (2.1)). Moreover, the derivative $D_\phi \Psi_\phi \in \mathcal{L}\big(C^{2,\alpha}_{p,e}(\mathbb{R}^m), C^{2,\alpha}_{p,e}(\overline{\Omega}, \mathbb{R}^{n+m})\big)$ is given by*

$$[D_\phi \Psi_\phi]\omega(z, t) = (\omega(t)z, 0) \qquad \text{for } (z, t) \in \overline{\Omega}, \ \omega \in C^{2,\alpha}_{p,e}(\mathbb{R}^m). \tag{2.17}$$

(ii) *Let*

$$g : \mathcal{U} \to C^{2,\alpha}(\overline{\Omega}), \qquad \phi \mapsto g_\phi$$

be a smooth map. Then the map

$$\mathcal{G} : \mathcal{U} \to C^{1,\alpha}(\partial\Omega), \qquad \mathcal{G}(\phi) = \partial_{\nu_\phi} g_\phi$$

is smooth as well and satisfies

$$D_\phi \mathcal{G}(\phi)\omega = \partial_{\tilde{\nu}_\phi(\omega)} g_\phi + \partial_{\nu_\phi}\Big([D_\phi g_\phi]\omega\Big) \qquad for \ \omega \in C^{2,\alpha}_{p,e}(\mathbb{R}^m).$$

Proof. (i) follows immediately from the definition of Ψ_ϕ.
(ii) For $\phi \in \mathcal{U}$ and $(z,t) \in \partial\Omega$ we have

$$\mathcal{G}(\phi)(z,t) = \partial_{\nu_\phi} g_\phi(z,t) = dg_\phi(z,t)\nu_\phi(z,t).$$

Hence $\mathcal{G}$ is smooth as a bilinear form composed with two smooth functions, and its derivative is given by

$$D_\phi \mathcal{G}(\phi)\omega = d[(D_\phi g_\phi)\omega]\nu_\phi + dg_\phi\big(\tilde{\nu}_\phi(\omega)\big) = \partial_{\nu_\phi}\Big([D_\phi g_\phi]\omega\Big) + \partial_{\tilde{\nu}_\phi(\omega)} g_\phi,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next we consider the function

$$\bar{u} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad \bar{u}(z,t) = -\frac{|z|^2}{2n},$$

and the smooth map

$$\mathcal{U} \to C^{2,\alpha}(\overline{\Omega}), \qquad \phi \mapsto u^\phi := \bar{u} \circ \Psi_\phi.$$

Since $\bar{u}$ satisfies $-\Delta\bar{u} = 1$ in $\mathbb{R}^n \times \mathbb{R}^m$, for every $\phi \in \mathcal{U}$ we have

$$- \Delta_{g_\phi} u^\phi = 1 \qquad \text{in } \Omega. \tag{2.18}$$

Lemma 2.4. *The map*

$$\mathcal{U} \to C^{1,\alpha}(\mathbb{R}^m), \quad \phi \mapsto \partial_{\nu_\phi} u^\phi(e_1, \cdot)$$

is smooth, and its derivative at a constant function $\phi \equiv \lambda > 0$ *satisfies*

$$\big[D_\phi|_{\phi=\lambda} \, \partial_{\nu_\phi} u^\phi(e_1, \cdot)\big]\omega(t) = -\frac{1}{n}\omega(t) \qquad for \ \omega \in C^{2,\alpha}_{p,e}(\mathbb{R}^m) \ and \ t \in \mathbb{R}^m. \tag{2.19}$$

Proof. The smoothness follows directly from Lemma 2.3(ii). To see (2.19), we consider the smooth function

$$\mathcal{U} \to C^{1,\alpha}_{p,e}(\mathbb{R}^m), \qquad \phi \mapsto \tilde{\mu}_\phi(\cdot) := \mu_\phi(\Psi_\phi(e_1, \cdot)),$$

where $\mu_\phi : \partial\Omega_\phi \to \mathbb{R}^{n+m}$ is the unit outer normal with respect to g_{eucl} given in (2.5). Let $\omega \in C^{2,\alpha}_{p,e}(\mathbb{R}^m)$. Since the map $\mathcal{U} \to C^{1,\alpha}_{p,e}(\mathbb{R}^m)$, $\phi \mapsto |\tilde{\mu}_\phi(\cdot)|^2_{g_{eucl}}$ is constant, we have

$$0 = [D_\phi \, |\tilde{\mu}_\phi(\cdot)|^2_{g_{eucl}}]\omega = 2\langle[D_\phi \, \tilde{\mu}_\phi(\cdot)]\omega, \tilde{\mu}_\phi(\cdot)\rangle_{g_{eucl}} \qquad \text{on} \quad \mathbb{R}^m. \tag{2.20}$$

Moreover, by (2.4) we have $\tilde{\mu}_\phi(t) = d\Psi_\phi(e_1, t)\nu_\phi(e_1, t)$ for $t \in \mathbb{R}^m$ and therefore

$$\partial_{\nu_\phi} u^\phi(e_1, \cdot) = du^\phi(e_1, \cdot)\nu_\phi(e_1, \cdot) = d\bar{u}(\Psi_\phi(e_1, \cdot))d\Psi_\phi(e_1, \cdot)\nu_\phi(e_1, \cdot) = d\bar{u}(\Psi_\phi(e_1, \cdot))\tilde{\mu}_\phi(\cdot)$$

$$= \langle\tilde{\mu}_\phi(\cdot), \nabla_{g_{eucl}}\bar{u}(\Psi_\phi(e_1, \cdot))\rangle_{g_{eucl}}$$

on $\mathbb{R}^m$. Consequently,

$$[D_\phi|_{\phi=\lambda} \, \partial_{\nu_\phi} u^\phi(e_1, \cdot)]\omega = \langle [D_\phi|_{\phi=\lambda} \, \tilde{\mu}_\phi(\cdot)]\omega, \nabla_{g_{eucl}} \bar{u}(\Psi_\lambda(e_1, \cdot)) \rangle_{g_{eucl}} \qquad (2.21)$$
$$+ \langle \tilde{\mu}_\lambda(\cdot), [D_\phi|_{\phi=\lambda} [\nabla_{g_{eucl}} \bar{u}(\Psi_\phi(e_1, \cdot))]\omega \rangle_{g_{eucl}}$$
$$=: I_1(\omega) + I_2(\omega). \qquad (2.22)$$

Since, by definition, $\nabla_{g_{eucl}} \bar{u}(z, t) = -\frac{1}{n}(z, 0)$ for $(z, t) \in \mathbb{R}^n \times \mathbb{R}^m$, we get

$$\nabla_{g_{eucl}} \bar{u}(\Psi_\phi(e_1, t)) = -\frac{1}{n}(\phi(t)e_1, 0) \qquad \text{for } t \in \mathbb{R}^m \qquad (2.23)$$

and thus

$$[D_\phi|_{\phi=\lambda}[(\nabla_{g_{eucl}} \bar{u}(\Psi_\phi(e_1, \cdot))]\omega(t) = -\frac{1}{n}(\omega(t)e_1, 0) \qquad t \in \mathbb{R}^m.$$

Noting also that $\tilde{\mu}_\lambda \equiv (e_1, 0) \in \mathbb{R}^{n+m}$ on $\mathbb{R}^m$, we infer that

$$I_2(\omega)(t) = \langle \tilde{\mu}_\lambda(t), -\frac{1}{n}(\omega(t)e_1, 0) \rangle_{g_{eucl}} = -\frac{1}{n}\omega(t) \qquad \text{for } t \in \mathbb{R}^m. \qquad (2.24)$$

From (2.23) we also deduce that

$$(\nabla_{g_{eucl}} \bar{u}(\Psi_\lambda(e_1, \cdot)) \equiv -\frac{1}{n}(\lambda e_1, 0) \equiv -\frac{\lambda}{n}\tilde{\mu}_\lambda(\cdot),$$

so the identity (2.20) with $\phi = \lambda$ implies that

$$I_1(\omega) = 0 \qquad \text{on } \mathbb{R}^m. \qquad (2.25)$$

Now (2.19) follows by combining (2.22), (2.24) and (2.25). □

We are now in a position to complete the

Proof of Proposition 2.2. For $\phi \in \mathcal{U}$, we note that the function $a_\phi := u^\phi - u_\phi \in C_{p,e}^{2,\alpha}(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta_{g_\phi} a_\phi = 0 & \text{in } \Omega \\ a_\phi = u^\phi & \text{on } \partial\Omega. \end{cases} \qquad (2.26)$$

Moreover, in the case where $\phi \equiv \lambda > 0$, we have $u^\lambda(z, t) = -\frac{|\lambda z|^2}{2n}$ and therefore

$$a_\lambda(z, t) = -\frac{\lambda^2}{2n} \qquad \text{for } (z, t) \in \overline{\Omega} \qquad (2.27)$$

by Lemma 2.1(ii). Now, consider the smooth map $T : \mathcal{U} \to C^{0,\alpha}(\overline{\Omega})$ given by

$$T(\phi) = \Delta_{g_\phi} a_\phi = g_\phi^{ij} \partial_{ij} a_\phi + \frac{1}{\sqrt{|g_\phi|}} \partial_i \left(\sqrt{|g_\phi|} g_\phi^{ij} \right) \partial_j a_\phi.$$

By (2.26) we have $T \equiv 0$ on $\mathcal{U}$. Thus for every $\omega \in C_{p,e}^{2,\alpha}(\mathbb{R}^m)$ we have

$$0 = DT(\phi)\omega = \Delta_{g_\phi}[D_\phi a_\phi]\omega + h_\phi^{ij} \partial_{ij} a_\phi + \ell_\phi^j \partial_j a_\phi \qquad (2.28)$$

with

$$h_\phi^{ij} := [D_\phi g_\phi^{ij}]\omega \qquad \text{and} \qquad \ell_\phi^j := [D_\phi \frac{1}{\sqrt{|g_\phi|}} \partial_i (\sqrt{|g_\phi|} g_\phi^{ij})]\omega.$$

Evaluating (2.28) at $\phi = \lambda$ and using that the function a_λ is constant in Ω by (2.27), we find that the function

$$\tau_{\omega,\lambda} := [D_\phi|_{\phi=\lambda} a_\phi]\omega \in C_{p,e}^{2,\alpha}(\Omega)$$

satisfies

$$\Delta_z \tau_{\omega,\lambda} + \lambda^2 \Delta_t \tau_{\omega,\lambda} = \lambda^2 \Delta_{g_\lambda} \tau_{\omega,\lambda} = 0 \qquad \text{in } C^{0,\alpha}(\overline{\Omega}).$$

(here the first equality follows from (2.11)). Moreover, differentiating the boundary condition in (2.26) and using Lemma 2.3(i) gives

$$\tau_{\omega,\lambda}(z,t) = [D_\phi|_{\phi=\lambda} \bar{u} \circ \Psi_\phi]\omega(z,t) = d\bar{u}(\Psi_\lambda(z,t))[D_\phi|_{\phi=\lambda} \Psi_\phi]\omega(z,t)$$

$$= d\bar{u}(\lambda z, t)(\omega(t)z, 0) = -\frac{\lambda}{n}\omega(t) \qquad \text{for } (z,t) \in \partial\Omega.$$

By Lemma 2.3(ii) and since a_λ is constant in $\overline{\Omega}$, we also have that

$$[D_\phi|_{\phi=\lambda} \partial_{\nu_\phi} a_\phi]\omega = \partial_{\tilde{\nu}_\phi(\omega)} a_\lambda + \partial_{\nu_\lambda} \tau_{\omega,\lambda} = \frac{1}{\lambda}\partial_\nu \tau_{\omega,\lambda} \qquad \text{on } \partial\Omega, \qquad (2.29)$$

where ν is the outer unit normal on $\partial\Omega$ with respect to g_{eucl} given by $\nu(z,t) = (z,0)$. Combining (2.29) with Lemma 2.4, we thus find that

$$[D_\phi|_{\phi=\lambda} H(\phi)]\omega(t) = [D_\phi|_{\phi=\lambda} \partial_{\nu_\phi}(u^\phi - a_\phi)(e_1,\cdot)]\omega(t) = -\frac{1}{n}\omega(t) - \frac{1}{\lambda}\partial_\nu \tau_{\omega,\lambda}(e_1,t)$$

for $t \in \mathbb{R}^m$. Putting $\psi_{\omega,\lambda} := -\frac{n}{\lambda}\tau_{\omega,\lambda}$, we then see that (2.15) and (2.16) hold, as claimed.
□

3. SPECTRAL PROPERTIES OF THE LINEARIZATION

In this section we study the spectral properties of the linearized operators $\mathcal{H}_\lambda = DH(\lambda) \in \mathcal{L}(C_{p,e}^{2,\alpha}(\mathbb{R}^m), C_{p,e}^{1,\alpha}(\mathbb{R}^m))$, $\lambda > 0$ considered in Proposition 2.2. We start with the following observation.

Proposition 3.1. *Let* $\lambda > 0$. *The functions*

$$\omega_k \in C_{p,e}^{2,\alpha}(\mathbb{R}^m), \quad \omega_k(s) = \prod_{j=1}^m \cos(k_j s_j), \quad k \in (\mathbb{N} \cup \{0\})^m \qquad (3.1)$$

are eigenfunctions of $\mathcal{H}_\lambda$ *in the sense that*

$$[\mathcal{H}_\lambda \omega_k](t) = \sigma(|k|\lambda)\,\omega_k(t) \qquad \text{for } t \in \mathbb{R}^m \text{ with } |k| = \sqrt{k_1^2 + \cdots + k_m^2}. \qquad (3.2)$$

Here the function σ *is defined by*

$$\sigma : [0,\infty) \to \mathbb{R}, \qquad \sigma(\rho) = \frac{1}{n}\Big(\rho \frac{h'(\rho)}{h(\rho)} - 1\Big), \qquad (3.3)$$

where $h : [0, \infty) \to \mathbb{R}$ is the unique solution of the initial value problem

$$\begin{cases} h''(\rho) + \dfrac{n-1}{\rho} h'(\rho) - h(\rho) = 0 \\ h(0) = 1, \quad h'(0) = 0. \end{cases} \tag{3.4}$$

Furthermore, in case $n = 1$ we have

$$\sigma(\rho) = \rho \tanh(\rho) - 1, \tag{3.5}$$

and in case $n \geq 2$ we have

$$\sigma(\rho) = \frac{1}{n}\left(\rho \frac{I_{\nu+1}(\rho)}{I_\nu(\rho)} - 1\right), \tag{3.6}$$

where I_ν is the modified Bessel function of the first kind of order $\nu = \frac{n-2}{2}$.

Proof. Fix $\lambda > 0$ and $k \in (\mathbb{N} \cup \{0\})^m$. For $\omega = \omega_k \in C^{2,\alpha}_{p,e}(\mathbb{R}^m)$ as defined in (3.1), the unique solution $\psi_{\omega_k,\lambda}$ of (2.16) is given by $\psi_{\omega_k,\lambda}(z,t) = b(|z|)\omega_k(t)$, where the function $b : [0, 1] \to \mathbb{R}$ is the unique solution to

$$\begin{cases} b'' + \dfrac{n-1}{r} b' - \lambda^2 |k|^2 b = 0, \qquad r \in (0, 1) \\ b'(0) = 0, \quad b(1) = 1. \end{cases} \tag{3.7}$$

Hence we have that

$$[\mathcal{H}_\lambda \omega_k](e_1, t) = \frac{1}{n}\left(\partial_\nu \psi_{\omega_k,\lambda}(e_1, t) - \omega_k(t)\right) = \frac{1}{n}(b'(1) - 1)\omega_k(t) \qquad \text{for } t \in \mathbb{R}^m.$$

Now putting $\rho_0 = \lambda |k|$ and considering $\tilde{h} : [0, \rho_0] \to \mathbb{R}$ defined by $\tilde{h}(\rho) := b(\frac{\rho}{\rho_0})$, we see that $\tilde{h}$ satisfies

$$\begin{cases} \tilde{h}''(\rho) + \dfrac{n-1}{\rho} \tilde{h}'(\rho) - \tilde{h}(\rho) = 0, \qquad \rho \in (0, \rho_0), \\ \tilde{h}'(0) = 0, \qquad \tilde{h}(\rho_0) = 1. \end{cases} \tag{3.8}$$

Consequently, $\tilde{h} = \frac{h}{h(\rho_0)}$ in $[0, \rho_0]$, where $h : [0, \infty) \to \mathbb{R}$ is the unique solution of the initial value problem (3.4). Moreover,

$$\frac{1}{n}(b'(1) - 1) = \frac{1}{n}\left(\rho_0 \tilde{h}'(\rho_0) - 1\right) = \frac{1}{n}\left(\rho_0 \frac{h'(\rho_0)}{h(\rho_0)} - 1\right) = \sigma(\rho_0),$$

as claimed in (3.3). Now in case $n = 1$ we have $h(\rho) = \cosh(\rho)$ for $\rho > 0$ and thus (3.3) follows.

In case $n \geq 2$, we consider $g(\rho) := \rho^\nu h(\rho)$ with $\nu := \frac{n-2}{2}$, so that (3.4) transforms into the following (modified) Bessel equation:

$$g''(\rho) + \frac{1}{\rho} g'(\rho) - \left(1 + \frac{\nu^2}{\rho^2}\right) g(\rho) = 0.$$

Up to a constant, the unique locally bounded solution to this equation is the modified Bessel function of the first kind I_ν. Since $I_\nu > 0$ on $(0, \infty)$, we thus have

$$h(\rho) = c\,\rho^{-\nu} I_\nu(\rho) \qquad \text{for } \rho > 0 \text{ with a constant } c > 0. \tag{3.9}$$

In fact, it will follow from (3.4) and (3.11) below that $c = 2^\nu \Gamma(\nu + 1)$, but we do not need this. Using (3.9) together with the recurrence formula $\rho I_\nu'(\rho) - \nu I_\nu(\rho) = \rho I_{\nu+1}(\rho)$ (see e.g. [6, Section 7.11]), we find that

$$\frac{h'(\rho)}{h(\rho)} = \frac{I_\nu'(\rho) - \frac{\nu}{\rho} I_\nu(\rho)}{I_\nu(\rho)} = \frac{I_{\nu+1}(\rho)}{I_\nu(\rho)} \qquad \text{for } \rho > 0.$$

Therefore (3.3) yields

$$\sigma(\rho) = \frac{1}{n}\left(\rho \frac{I_{\nu+1}(\rho)}{I_\nu(\rho)} - 1\right),$$

as claimed. $\square$

The following lemma gives the asymptotic behavior of the function σ.

Lemma 3.2. *The function* $\sigma : (0, \infty) \longrightarrow \mathbb{R}$ *has the following asymptotic properties:*

(i) $\displaystyle\lim_{\rho \to \infty} \frac{\sigma(\rho)}{\rho} = \frac{1}{n}$,

(ii) $\displaystyle\lim_{\rho \to 0} \sigma(\rho) = -\frac{1}{n}$.

Proof. In case $n = 1$, both (i) and (ii) follow immediately from (3.5). In case $n \geq 2$, (i) follows from (3.6) and the asymptotic formula

$$\lim_{\rho \to +\infty} \frac{I_\tau(\rho)}{\frac{1}{\sqrt{2\pi\rho}} e^\rho} = 1 \qquad \text{for every } \tau \geq 0,$$

see [6, Section 7.13.1]. Moreover (see e.g. [6, Section 7.2.2]), for $\tau \geq 0$, we have the power series expression of I_τ given by

$$I_\tau(\rho) = \sum_{i=0}^{\infty} \frac{(\frac{1}{2}\rho)^{\tau+2i}}{i!\,\Gamma(\tau + i + 1)}, \qquad \text{for } \tau \geq 0 \text{ and } \rho > 0. \tag{3.10}$$

We can write

$$I_\tau(\rho) = (\frac{1}{2}\rho)^\tau \left[\frac{1}{\Gamma(\tau + 1)} + \sum_{i=1}^{\infty} \frac{(\frac{1}{2}\rho)^{2i}}{i!\,\Gamma(\tau + i + 1)}\right], \tag{3.11}$$

which shows that $\displaystyle\lim_{\rho \to 0} \rho \frac{I_{\tau+1}(\rho)}{I_\tau(\rho)} = 0$ for every $\tau \geq 0$. Together with (3.6) this gives (ii).$\square$

Next, we show that the functions $\lambda \mapsto \sigma(\lambda)$ are strictly increasing on $(0, \infty)$.

Lemma 3.3. *We have $\sigma'(\rho) > 0$ for $\rho > 0$. Moreover, σ has exactly one zero in $(0, \infty)$.*

Proof. By Lemma 3.2, we only need to show that $\sigma'(\rho) > 0$ for $\rho > 0$. In case $n = 1$, (3.5) gives that $\sigma'(\rho) = \frac{\rho}{\cosh^2(\rho)} + \tanh(\rho) > 0$ for $\rho > 0$. In case $n \geq 2$, we use (3.3) and calculate that

$$nh^2(\rho)\sigma'(\rho) = h'(\rho)h(\rho) + \rho\left(h''(\rho)h(\rho) - h'(\rho)^2\right) = (2-n)h'(\rho)h(\rho) + \rho(h^2(\rho) - h'(\rho)^2)$$

for $\rho > 0$. For the latter equality, we used the fact that

$$h''(\rho) = h(\rho) + \frac{1-n}{\rho}h'(\rho) \qquad \text{for } \rho > 0 \tag{3.12}$$

as a consequence of (3.4). It then suffices to show that the function

$$\rho \mapsto j(\rho) := \rho^{n-1}nh^2(\rho)\sigma'(\rho) = (2-n)\rho^{n-1}h'(\rho)h(\rho) + \rho^n(h^2(\rho) - h'(\rho)^2)$$

is positive on $(0, \infty)$. Since $j(0) = 0$, it suffices to show that $j'(\rho) > 0$ for $\rho > 0$. Using (3.12) again, we find that

$$\begin{aligned}
j'(\rho) &= (n-1)(2-n)\rho^{n-2}h'(\rho)h(\rho) + (2-n)\rho^{n-1}\left(h''(\rho)h(\rho) + h'(\rho)^2\right) \\
&\quad + n\rho^{n-1}\left(h^2(\rho) - h'(\rho)^2\right) + 2\rho^n\left(h(\rho)h'(\rho) - h'(\rho)h''(\rho)\right) \\
&= (n-1)(2-n)\rho^{n-2}h'(\rho)h(\rho) + (2-n)\rho^{n-1}\left(h^2(\rho) + \frac{1-n}{\rho}h'(\rho)h(\rho) + h'(\rho)^2\right) \\
&\quad + n\rho^{n-1}\left(h^2(\rho) - h'(\rho)^2\right) + 2(n-1)\rho^{n-1}h'(\rho)^2 \\
&= \rho^{n-1}\left(2h^2(\rho) + (2-n)h'(\rho)^2 - nh'(\rho)^2 + 2(n-1)h'(\rho)^2\right) \\
&= 2\rho^{n-1}h^2(\rho) = 2c^2\rho I_\nu^2(\rho) \qquad \text{for } \rho > 0
\end{aligned}$$

with $\nu = \frac{n-2}{2}$ and $c > 0$ as in (3.9). Since $I_\nu(\rho) > 0$ for $\rho > 0$, we thus conclude that $j'(\rho) > 0$ for $\rho > 0$, as required. $\qquad\square$

In the following, we consider the Sobolev spaces

$$H_{p,e}^j := \left\{v \in H_{loc}^j(\mathbb{R}^m) \ : \ v \text{ even, } 2\pi\text{-periodic in } t_1, \dots, t_m\right\}, \qquad j \in \mathbb{N} \cup \{0\}, \tag{3.13}$$

and we put $L_{p,e}^2 := H_{p,e}^0$. Note that $L_{p,e}^2$ is a Hilbert space with scalar product

$$(u, v) \mapsto \langle u, v\rangle_{L^2} := \int_{[0,2\pi]^m} u(t)v(t) \, dt \qquad \text{for } u, v \in L_{p,e}^2.$$

We denote the induced norm by $\|\cdot\|_{L^2}$. For the functions ω_k in (3.1) we then have

$$\|\omega_0\|_{L^2} = (2\pi)^{m/2}, \qquad \|\omega_k\|_{L^2} = \pi^{m/2} \quad \text{for } k \in (\mathbb{N} \cup \{0\})^m, \ k \neq 0,$$

and the functions $\frac{\omega_k}{\|\omega_k\|_{L^2}}$ form an orthonormal basis for $L^2_{p,e}$. Moreover, $H^j_{p,e} \subset L^2_{p,e}$ is characterized as the subspace of all functions $v \in L^2_{p,e}$ such that

$$\sum_{k \in (\mathbb{N} \cup \{0\})^m} (1 + |k|^2)^j \langle v, \omega_k \rangle^2_{L^2} < \infty.$$

Thus, $H^j_{p,e}$ is also a Hilbert space with scalar product

$$(u, v) \mapsto \langle u, v \rangle_{H^j} := \sum_{k \in (\mathbb{N} \cup \{0\})^m} (1 + |k|^2)^j \langle u, \omega_k \rangle_{L^2} \langle v, \omega_k \rangle_{L^2} \qquad \text{for } u, v \in H^j_{p,e}. \quad (3.14)$$

In the following, we also consider the subspaces

$$V_\ell := \langle \omega_k \; : \; |k| = \ell \rangle \subset \bigcap_{j \in \mathbb{N}} H^j_{p,e}, \quad (3.15)$$

the corresponding $\langle \cdot, \cdot \rangle_{L^2}$-orthogonal projections $P_\ell : L^2_{p,e} \to L^2_{p,e}$ on V_ℓ, and the complements

$$Z^j_\ell := \{ v \in H^j_{p,e} \; : \; P_\ell v = 0 \} \subset H^j_{p,e}, \qquad \ell \in \mathbb{N} \cup \{0\}. \quad (3.16)$$

Since the latter spaces are closed subspaces of $H^j_{p,e}$, they are also Hilbert spaces with respect to the scalar product in (3.14).

Proposition 3.4. *For fixed $\lambda > 0$, the linear map $\mathcal{H}_\lambda$ defined in (2.15) extends to a continuous linear map*

$$\mathcal{H}_\lambda : H^2_{p,e} \to H^1_{p,e}, \qquad \mathcal{H}_\lambda v = \sum_{\ell \in \mathbb{N} \cup \{0\}} \sigma(\lambda \ell) P_\ell v.$$

Moreover, for any $\ell \in \mathbb{N} \cup \{0\}$, the operator

$$\mathcal{H}_\lambda - \sigma(\lambda \ell) \, \mathrm{id} \; : \; Z^2_\ell \to Z^1_\ell \qquad \text{is an isomorphism.}$$

Proof. This follows from Proposition 3.1, Lemma 3.2, Lemma 3.3 and the remarks above. $\qquad \square$

Remark 3.5. *The extension $\mathcal{H}_\lambda : H^2_{p,e} \to H^1_{p,e}$ given in Proposition 3.4 can be characterized as follows. For $k \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$, we consider the space*

$$W^{k,p}_b(\Omega) := \{ \psi \in W^{k,p}_{loc}(\Omega) \; : \; \psi \in W^{k,p}(\Omega') \text{ for every bounded subset } \Omega' \subset \Omega \} \quad (3.17)$$

Given $\omega \in H^2_{p,e}$, standard elliptic theory shows that there is a unique solution $\psi \in W^{2,2}_b(\Omega)$ of the problem

$$\begin{cases} \Delta_z \psi(z, t) + \lambda^2 \Delta_t \psi(z, t) = 0 & (z, t) \in \Omega, \\ \psi(z, t) = \omega(t) & (z, t) \in \partial \Omega \end{cases} \quad (3.18)$$

which is even and 2π-periodic in $t_1, \ldots, t_m$. Then $\mathcal{H}_\lambda \omega$ is given by

$$[\mathcal{H}_\lambda \omega](t) = \frac{1}{n} \Big(\partial_\nu \psi(e_1, t) - \omega(t) \Big) \qquad \text{for a.e. } t \in \mathbb{R}^m,$$

where ν is the outer unit normal on $\partial\Omega$ with respect to g_{eucl} and $\partial_\nu\psi$ is considered in the sense of traces. This can be easily seen by approximating ω in $H_{p,e}^2$ with functions in $C_{p,e}^{2,\alpha}(\mathbb{R}^m)$ and using standard elliptic estimates.

4. Proof of Theorem 1.1

In the following, we let $\mathcal{P} \subset \mathcal{L}(\mathbb{R}^m)$ denote the subset of all coordinate permutations, and we consider the spaces

$$X := \{\varphi \in C_{p,e}^{2,\alpha}(\mathbb{R}^m) \ : \ \varphi(t) = \varphi(\mathbf{p}(t)) \text{ for all } t \in \mathbb{R}^m, \mathbf{p} \in \mathcal{P}\},$$
$$Y := \{\varphi \in C_{p,e}^{1,\alpha}(\mathbb{R}^m) \ : \ \varphi(t) = \varphi(\mathbf{p}(t)) \text{ for all } t \in \mathbb{R}^m, \mathbf{p} \in \mathcal{P}\}.$$

We also consider the nonlinear operator H defined in (2.13), and we note that H maps $\mathcal{U} \cap X$ into Y by Lemma 2.1(iii). Consider the open set

$$\mathcal{O} := \{(\lambda, \varphi) \in \mathbb{R} \times X \ : \ \lambda > 0, \ \varphi > -\lambda\} \subset \mathbb{R} \times X. \tag{4.1}$$

The proof of Theorem 1.1 will be completed by applying the Crandall-Rabinowitz Bifurcation theorem to the smooth nonlinear operator

$$G : \mathcal{O} \subset \mathbb{R} \times X \to Y, \qquad G(\lambda, \varphi) = H(\lambda + \varphi) + \frac{\lambda}{n}. \tag{4.2}$$

Recalling the formula of u_λ in Lemma 2.1, we have

$$G(\lambda, 0)(t) = H(\lambda)(t) + \frac{\lambda}{n} = \partial_{\nu_\lambda} u_\lambda(e_1, t) + \frac{\lambda}{n} = 0 \qquad \text{for } t \in \mathbb{R}^m, \lambda > 0.$$

Moreover,

$$D_\varphi G(\lambda, 0) = DH(\lambda)\big|_X = \mathcal{H}_\lambda|_X \in \mathcal{L}(X, Y). \tag{4.3}$$

We have the following.

Proposition 4.1. *There exists a unique $\lambda_* = \lambda_*(n) > 0$ such that $\sigma(\lambda_*) = 0$, where the function σ is defined in Proposition 3.1. Moreover, the linear operator*

$$\mathcal{H}_* := \mathcal{H}_{\lambda_*}\big|_X \in \mathcal{L}(X, Y)$$

has the following properties.

(i) *The kernel $N(\mathcal{H}_*)$ of $\mathcal{H}_*$ is spanned by the function*

$$v_0 \in X, \qquad v_0(t) = \cos(t_1) + \cdots + \cos(t_m). \tag{4.4}$$

(ii) *The range of $\mathcal{H}_*$ is given by*

$$R(\mathcal{H}_*) = \Big\{v \in Y \ : \ \int_{[0,2\pi]^m} v(t)v_0(t)\, dt = 0\Big\}.$$

Moreover,

$$\partial_\lambda\big|_{\lambda=\lambda_*} \mathcal{H}_\lambda v_0 \notin R(\mathcal{H}_*). \tag{4.5}$$

Proof. By Lemma 3.3, there exists a unique $\lambda_* = \lambda_*(n) > 0$ such that $\sigma(\lambda_*) = 0$, which by Proposition 3.1 is equivalent to $\mathcal{H}_{\lambda_*} v_0 = 0$. We put $\mathcal{H}_* := \mathcal{H}_{\lambda_*}$ in the following. Consider the subspaces

$$X^* := \left\{ v \in X \ : \ \int_{[0,2\pi]^m} v(t)v_0(t) \, dt = 0 \right\} \subset X, \qquad (4.6)$$

$$Y^* := \left\{ v \in Y \ : \ \int_{[0,2\pi]^m} v(t)v_0(t) \, dt = 0 \right\} \subset Y.$$

To show properties (i) and (ii), it clearly suffices to prove that

$$\mathcal{H}_* \text{ defines an isomorphism between } X^* \text{ and } Y^*. \qquad (4.7)$$

To prove (4.7), we need to introduce further spaces. We recall the definition of $H_{p,e}^j$ in (3.13) and put

$$H_{\mathcal{P}}^j := \left\{ v \in H_{p,e}^j \ : \ v(\mathbf{p}(t)) = v(t) \text{ for } t \in \mathbb{R}^m \text{ and } \mathbf{p} \in \mathcal{P} \right\}, \qquad j \in \mathbb{N} \cup \{0\}, \quad (4.8)$$

noting that $X = H_{\mathcal{P}}^2 \cap C^{2,\alpha}(\mathbb{R}^m)$ and $Y = H_{\mathcal{P}}^1 \cap C^{1,\alpha}(\mathbb{R}^m)$. Proposition 3.4 implies that $\mathcal{H}_*$ defines a continuous linear operator

$$\mathcal{H}_* : H_{\mathcal{P}}^2 \to H_{\mathcal{P}}^1, \qquad \mathcal{H}_* v = \sum_{\ell \in \mathbb{N} \cup \{0\}} \sigma(\lambda_* \ell) P_\ell v, \qquad (4.9)$$

Next we put

$$\tilde{V}^j := H_{\mathcal{P}}^j \cap V_1, \quad \tilde{Z}^j := H_{\mathcal{P}}^j \cap Z_1^j \quad \subset H_{\mathcal{P}}^j \qquad \text{for } j = 1, 2,$$

where the spaces V_1 resp. Z_1^j are defined in (3.15) and (3.16), respectively. We note that $\tilde{V}^1$ is one-dimensional and spanned by the function v_0 defined in (4.4). Since the spaces $\tilde{V}^j$ and $\tilde{Z}^j$ are invariant with respect to coordinate permutations $\mathbf{p} \in \mathcal{P}$, we deduce from Proposition 3.4 and our choice of λ_* that

$$\mathcal{H}_* \text{ defines an isomorphism } \tilde{Z}^2 \to \tilde{Z}^1. \qquad (4.10)$$

Moreover, since $X^* = \tilde{Z}^2 \cap X$ and $Y^* = \tilde{Z}^1 \cap Y$, we see that $\mathcal{H}_* : X^* \to Y^*$ is well defined and injective. To establish surjectivity, let $f \in Y^*$. By (4.10), there exists $\omega \in \tilde{Z}^2 \subset H_{p,e}^2$ such that $\mathcal{H}_* \omega = f$. As noted in Remark 3.5, we then have

$$\partial_\nu \psi(\pm e_1, t) - \omega(t) = nf(t) \quad \text{for a.e. } t \in \mathbb{R}^m, \qquad (4.11)$$

where $\psi \in W_b^{2,2}(\Omega)$ is the unique solution of (3.18) which is even and 2π-periodic in $t_1, \dots, t_m$. We claim that

$$\psi \in C_{p,e}^{2,\alpha}(\overline{\Omega}). \qquad (4.12)$$

Indeed, this follows from [12, Theorem 6.2.3.1] once we have shown that

$$\psi \in W_b^{2,p}(\Omega) \qquad \text{for some } p > N, \qquad (4.13)$$

where the space $W_b^{2,p}(\Omega)$ is defined in (3.17). To see this, we show by induction that

$$\psi \in W_b^{2,p_k}(\Omega) \qquad (4.14)$$

for a sequence of numbers $p_k \in [2, \infty)$ satisfying $p_0 = 2$ and $p_{k+1} \geq \frac{N-1}{N-2} p_k$ for $k \geq 0$. We already know that (4.14) holds for $p_0 = 2$. So let us assume that (4.14) holds for some $p_k \geq 2$. We distinguish two cases.

If $p_k < N$, then the trace theorem implies that

$$\psi|_{\partial\Omega} \in W_{loc}^{1,p_{k+1}}(\partial\Omega) \qquad \text{with } p_{k+1} := (\frac{N-1}{N-p_k}) p_k \geq \frac{N-1}{N-2} p_k,$$

so that $\omega \in W_{loc}^{1,p_{k+1}}(\mathbb{R}^m)$. Since also $f \in C^{1,\alpha}(\mathbb{R}^m) \subset W_{loc}^{1,p_{k+1}}(\mathbb{R}^m)$ and, by (4.11),

$$\partial_\nu \psi(z,t) + \psi(z,t) = \partial_\nu \psi(z,t) + \omega(t) = g(t) \quad \text{for } (z,t) \in \partial\Omega$$

with

$$g = nf + 2\omega \in W_{loc}^{1,p_{k+1}}(\mathbb{R}^m),$$

we may deduce from [12, Theorem 2.4.2.6] that $\psi \in W_b^{2,p_{k+1}}(\Omega)$.

If $p_k \geq N$, the trace theorem implies that $W_{loc}^{1,p}(\partial\Omega)$ for any $p > 2$, and then we may repeat the above argument with arbitrarily chosen $p_{k+1} \geq \frac{N-1}{N-2} p_k$ to infer again that $\psi \in W_b^{2,p_{k+1}}(\Omega)$.

We thus conclude that (4.13) holds, and hence (4.12) follows. By passing to the trace again, we then conclude that $\omega \in C_{p,e}^{2,\alpha}(\mathbb{R}^m)$. Consequently, $\omega \in C_{p,e}^{2,\alpha}(\mathbb{R}^m) \cap \tilde{Z}^2 = X^*$, and thus $\mathcal{H}_* : X^* \to Y^*$ is also surjective. Hence (4.7) is true.

It remains to prove (4.5), which follows from Lemma 3.3 and the identity

$$\partial_\lambda\Big|_{\lambda=\lambda_*} \mathcal{H}_\lambda v_0 = \partial_\lambda\Big|_{\lambda=\lambda_*} \sigma(\lambda) v_0 = \sigma'(\lambda_*) v_0. \tag{4.15}$$

$\square$

Proof of Theorem 1.1 (completed). Recalling (4.2) and (2.13), we shall apply the Crandall-Rabinowitz Bifurcation Theorem to solve the equation

$$G(\lambda, \varphi) = H(\lambda + \varphi) + \frac{\lambda}{n} = \partial_{\nu_\phi} u_\phi(e_1, \cdot) + \frac{\lambda}{n} = 0, \tag{4.16}$$

where $\phi = \lambda + \varphi \in \mathcal{U}$ and the function $u_\phi \in C^{2,\alpha}(\overline{\Omega_\phi})$ is the unique solution to the Dirichlet boundary value problem

$$\begin{cases} -\Delta_{g_\phi} u_\phi = 1 & \text{in } \Omega \\ u_\phi = 0 & \text{on } \partial\Omega, \end{cases}$$

see Lemma 2.1. Once this is done, (2.14) follows and thus we get (2.12), which is equivalent to (2.3) with $c = \frac{\lambda}{n}$.

To solve equation (4.16), we let $\lambda_* = \lambda_*(n)$ be defined as in Proposition 4.1, and let X^* be defined as in (4.6). By Proposition 4.1 and the Crandall-Rabinowitz Theorem (see [3, Theorem 1.7]), we then find $\varepsilon_0 > 0$ and a smooth curve

$$(-\varepsilon_0, \varepsilon_0) \to \mathcal{O}, \qquad s \mapsto (\lambda_s, \varphi_s)$$

such that

(i) $G(\lambda_s, \varphi_s) = 0$ for $s \in (-\varepsilon_0, \varepsilon_0)$,

(ii) $\lambda(0) = \lambda_*$, and

(iii) $\varphi_s = s(v_0 + \mu_s)$ for $s \in (-\varepsilon_0, \varepsilon_0)$ with a smooth curve

$$(-\varepsilon_0, \varepsilon_0) \to X^*, \qquad s \mapsto \mu_s.$$

satisfying $\mu_0 = 0$.

Here $\mathcal{O}$ is defined as in (4.1). Since (λ_s, φ_s) is a solution to (4.16) for every $s \in (-\varepsilon_0, \varepsilon_0)$, the function $u_{\phi_s} \in C^{2,\alpha}(\overline{\Omega})$ solves the overdetermined boundary value problem

$$\begin{cases} -\Delta_{g_{\phi_s}} u_{\phi_s} = 1 & \text{in} \quad \Omega, \\[2mm] u_{\phi_s} = 0, \quad \partial_\nu u_{\phi_s} = -\dfrac{\lambda_s}{n} & \text{on} \quad \partial\Omega, \end{cases}$$

where $\phi_s = \lambda_s + \varphi_s$. Recalling (2.1), we thus find that the map $s \mapsto (\lambda_s, \varphi_s)$ and the function $u := u_{\phi_s} \circ \Psi_{\phi_s}^{-1} : \Omega_{\phi_s} \to \mathbb{R}$ have the properties asserted in Theorem 1.1. $\quad\square$

5. PERIODIC CHEEGER SETS

In this section, we prove Corollary 1.2. Considering the notation of Theorem 1.1, we therefore fix $s \in (-\varepsilon_0, \varepsilon_0)$. Moreover, we recall that Theorem 1.1 yields a solution u of the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega_{\phi_s}, \\[2mm] u = 0, \quad \partial_\nu u = -\dfrac{\lambda_s}{n} & \text{on} \quad \partial\Omega_{\phi_s}, \end{cases} \tag{5.1}$$

where $\phi_s = \lambda_s + \varphi_s$. In the following, we put $E_s := \Omega_{\phi_s}$. We need the following property which follows by a very simple application of the P-function method, see e.g. [26]. We include a proof for the convenience of the reader.

Lemma 5.1. *We have $|\nabla u| < \frac{\lambda_s}{n}$ in E_s.*

Proof. Consider the function

$$P : \overline{E}_s \to \mathbb{R}, \qquad P(x) := |\nabla u(x)|^2.$$

It is clear, by standard elliptic regularity, that P is of class C^2. Moreover, in E_s we have, since $-\Delta u = 1$,

$$\Delta P = 2 \sum_{i,j=1}^{N} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \geq 2 \sum_{i=1}^{N} \left(\frac{\partial^2 u}{\partial x_i^2} \right)^2 \geq \frac{2}{N} (\Delta u)^2 = \frac{2}{N}.$$

Hence $\Delta P > 0$ in E_s, and thus P attains its maximum only on ∂E_s by the strong maximum principle. Since $P \equiv \frac{\lambda_s^2}{n^2}$ on ∂E_s by (5.1), the claim follows. $\quad\square$

Proof of Corollary 1.2 (completed). Since the domain E_s is 2π-periodic and symmetric in $t_1, \ldots, t_m$, the solution u of (5.1) is 2π-periodic and even in $t_1, \ldots, t_m$. Next, let $a, b \in \pi\mathbb{Z}^m$ with $a_i < b_i$ for $i = 1, \ldots, m$, and let S_a^b be defined as in (1.7). Then ∂S_a^b can be decomposed into a disjoint union $\partial S_\tau = K \cup S^1 \cup \cdots \cup S^m$, where

$$S^i := \mathbb{R}^n \times \{t \in \mathbb{R}^m \ : \ t_i \in \{a_i, b_i\}, \ t_j \in (a_j, b_j) \text{ for } j \neq i\} \qquad \text{for } i = 1, \ldots, m,$$

and K has zero $(N-1)$-dimensional Hausdorff measure. By the properties of u listed above, we then have

$$\frac{\partial u}{\partial t_i} \equiv 0 \qquad \text{on } S^i \text{ for } i = 1, \ldots, m. \tag{5.2}$$

Next, let $A \subset E_s \cap S_a^b$ be a Lipschitz open set. Then H^{N-1}-almost everywhere on ∂A the outer unit normal ν_A of A is well-defined, and H^{N-1}-almost everywhere on $\partial A \cap S^i$ it coincides with $(0, e_i)$ or $(0, -e_i)$, where $0 \in \mathbb{R}^n$ and e_i denotes the i-th coordinate vector in $\mathbb{R}^m$. Consequently, (5.2) implies that

$$\partial_{\nu_A} u \equiv 0 \qquad H^{N-1}\text{-almost everywhere on } \partial A \cap \partial S_a^b. \tag{5.3}$$

Since u satisfies (5.1), the divergence theorem and (5.3) yield the inequality

$$|A| = -\int_A \Delta u \, dx = -\int_{\partial A} \partial_{\nu_A} u \, d\sigma = -\int_{\partial A \cap S_a^b} \partial_{\nu_A} u \, d\sigma \leq \int_{\partial A \cap S_a^b} |\nabla u| d\sigma.$$

Hence Lemma 5.1 implies that

$$|A| \leq \frac{\lambda_s}{n} H^{N-1}(\partial A \cap S_a^b) = \frac{\lambda_s}{n} P(A, S_a^b),$$

whereas equality holds if and only if $H^{N-1}(\partial A \cap S_a^b \cap E_s) = 0$, i.e. if $A = E_s \cap S_a^b$. This implies that E_s is uniquely self-Cheeger relative to S_a^b with corresponding relative Cheeger constant $h(E_s, S_a^b) = \frac{n}{\lambda_s}$, as claimed. The proof of Corollary 1.2 is finished. $\square$

Conflict of Interest: The authors declare that they have no conflict of interest.

REFERENCES

[1] A. D. Alexandrov, Uniqueness Theorem for surfaces in large I, Vestnik Leningrad Univ. Math. 11 (1956), 5-17.
[2] H. Berestycki, L. A. Caffarelli and L. Nirenberg, Monotonicity for elliptic equations in unbounded Lipschitz domains. Comm. Pure Appl. Math. 50 (1997) 1089-1111.
[3] M. Crandall and P. Rabinowitz, Bifurcation from simple eigenvalues. J. Functional Analysis 8, 321-340 (1971).

[4] Ch. Delaunay, Sur la surface de révolution dont la courbure moyenne est constante, J. Math. Pures Appl. 1re. serie 6 (1841), 309-315.

[5] M. Del Pino, F. Pacard, J.C. Wei, Serrin's overdetermined problem and constant mean curvature surfaces. Duke Math. J. 164 (2015), no. 14, 2643-2722.

[6] Erdclyi et al. (Eds.), *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York 1953-1955.

[7] A. Farina, L. Mari, E. Valdinoci, Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds. Comm. Partial Differential Equations 38 (2013), no. 10, 1818-62.

[8] A. Farina, E. Valdinoci, Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems. Arch. Ration. Mech. Anal. 195 (2010), no. 3, 1025-58.

[9] A. Farina, E. Valdinoci, Overdetermined problems in unbounded domains with Lipschitz singularities. Rev. Mat. Iberoam. 26 (2010), no. 3, 965-4.

[10] A. Farina, E. Valdinoci, Partially and globally overdetermined problems of elliptic type. Adv. Nonlinear Anal. 1 (2012), no. 1, 27-45.

[11] A. Farina, E. Valdinoci, On partially and globally overdetermined problems of elliptic type. Amer. J. Math. 135 (2013), 1699-1726.

[12] P. Grisvard, Elliptic Problems in Nonsmooth Domains, SIAM Classics in Applied Mathematics 69, Philadelphia, PA 2011.

[13] L. Hauswirth, Laurent, F. Hélein, F. Pacard, On an overdetermined elliptic problem. Pacific J. Math. 250 (2011), no. 2, 319-334.

[14] G. P. Leonardi, An overview on the Cheeger problem, to appear on the Proceedings of the conference, New trends in shape optimization, held in Erlangen, September 2013.

[15] G. Leoni, A first course in Sobolev spaces. Graduate Studies in Mathematics, 105. American Mathematical Society, Providence, RI, 2009.

[16] R. Mazzeo, Recent advances in the global theory of constant mean curvature surfaces. In: Noncompact problems at the intersection of geometry, analysis, and topology, 179-199 , Contemp. Math., 350, Amer. Math. Soc., Providence, RI, 2004.

[17] I. A. Minlend, Existence of self-Cheeger sets on Riemannian manifolds. Preprint (2016) http://arxiv.org/abs/1603.00204.

[18] F. Morabito and P. Sicbaldi, Delaunay type domains for an overdetermined elliptic problem in $S^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$. ESAIM: COCV, 22 1 (2016) 1-28.

[19] E. Parini, An introduction to the Cheeger problem, Surveys Math. Appl. 6 (2011), 9-22.

[20] F. Schlenk and P. Sicbaldi, Bifurcating extremal domains for the first eigenvalue of the Laplacian. Adv. Math. 229 (2012) 602-632.

[21] A. Ros, P. Sicbaldi, Geometry and topology of some overdetermined elliptic problems. J. Differential Equations 255 (2013), no. 5, 951-7.

[22] A. Ros, D. Ruiz, P. Sicbaldi, A rigidity result for overdetermined elliptic problems in the plane. Preprint (2015), http://arxiv.org/abs/ arXiv:1505.05707.

[23] J. Serrin, A Symmetry Theorem in Potential Theory. Arch. Rational Mech. Anal. 43 (1971), 304-318.

[24] P. Sicbaldi, New extremal domains for the first eigenvalue of the Laplacian in flat tori. Calc. Var. (2010) 37:329-344.

[25] B. Sirakov, Overdetermined Elliptic Problems in Physics. In: Nonlinear PDE's in Condensed Matter and Reactive Flows, ed. by Henri Berestycki and Yves Pomeau, Springer 2002.

[26] R. P. Sperb, Maximum principles and their applications. Mathematics in Science and Engineering, 157. Academic Press, New York-London, 1981.

[27] M. Traizet, Classification of the solutions to an overdetermined elliptic problem in the plane. Geom. Funct. Anal. 24 (2014), 690-720.

[28] H. F. Weinberger, Remark on the preceding paper of Serrin. Arch. Rational Mech. Anal. 43 (1971), 319-320.

M. M. F.: African Institute for Mathematical Sciences in Senegal, KM 2, Route de Joal, B.P. 14 18. Mbour, Senegal.

E-mail address: `mouhamed.m.fall@aims-senegal.org`

I. A. M.:African Institute for Mathematical Sciences in Senegal, KM 2, Route de Joal, B.P. 14 18. Mbour, Senegal.

E-mail address: `ignace.a.minlend@aims-senegal.org`

T.W.: Goethe-Universität Frankfurt, Institut für Mathematik. Robert-Mayer-Str. 10 D-60054 Frankfurt, Germany.

E-mail address: `weth@math.uni-frankfurt.de`