# PERIODIC PATTERNS FOR A MODEL INVOLVING SHORT-RANGE AND LONG-RANGE INTERACTIONS 

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#### Abstract

We consider a physical model where the total energy is governed by surface tension and attractive screened Coulomb potential on the 3-dimensional space. We obtain different periodic equilibrium patterns i.e. stationary sets for this energy, under some volume constraints. The patterns bifurcate smoothly from straight lamellae, lattices of round solid cylinders and lattices of round balls.


## 1. Introduction and main results

In this paper, we are interested with a class of periodic stationary sets in $\mathbb{R}^{3}$ for the following energy functional

$$
\begin{equation*}
\mathcal{P}_{\gamma}(\Omega):=|\partial \Omega|+\gamma \int_{\Omega} \int_{\Omega} G_{\kappa}(|x-y|) d x d y \tag{1.1}
\end{equation*}
$$

where $\kappa, \gamma>0$ and $G_{\kappa}(r)=\frac{1}{r} e^{-\kappa r}$ is the repulsive Yukawa potential (or the screened repulsive Coulomb potential). The Yukawa potential $\bar{G}_{\kappa}(x)=G_{\kappa}(|x|)$ satisfies

$$
\begin{equation*}
-\Delta \bar{G}_{\kappa}+\kappa^{2} \bar{G}_{\kappa}=4 \pi \delta_{0} \quad \text { in } \mathbb{R}^{3} \tag{1.2}
\end{equation*}
$$

It plays an important role in the theory of elementary particles, see 54]. For this energy $\mathcal{P}_{\gamma}$, comprising short-range interaction (surface tension) and attractive Yukawa potential, a stationary set $\Omega$ may separate into well ordered disjoint components. This gives rise to formation of patterns in the limit of several nonlocal reaction-diffusion models from physics, biology and chemistry. Indeed, systems with competing short range and Coulomb-type attractive interaction provides sharp interfaces to the Ohta-Kawasaki model of block polymers and the activator-inhibitor reaction-diffusion system, see for for instance [22,35,52] and the references therein. The paper of Muratov [29] contains a rigorous derivation of (1.1) from the general mean-field free energy functional, with $\gamma \approx \kappa^{2}$ and $\gamma, \kappa \rightarrow 0$. Equilibrium patterns such as spots, stripes and the annuli, together with morphlogical instabilities have been also studied in [29]. At certain energy levels, (1.1) provides the sharp interface for the Ohta-Kawasaki model of diblock copolymer melt, see Muratov [28] and the work of Goldmann, Muratov and Serfaty in [20, 21] for related variant of $\Gamma$-convergence on the 2 -dimensional torus.

In [34], Oshita showed that (1.1) is the $\Gamma$-limit for a class of activator-inhibitor reactiondiffusion system, see also Petrich and Goldstein in [39], for the two dimensional case.

The question of minimizing $\mathcal{P}_{\gamma}$ has been also raised by H. Knüpfer, C. Muratov and M. Novaga in [36], for physical and mathematical perspectives. They speculated that bifurcations from trivial to nontirival solutions can be expected. Moreover they predict the

[^0]"pearl-necklace morphology" (or string-of-pearl) exhibited by long polyelectrolyte molecules in poor solvents, [15, 19]. In a geometric point of view, this is alike the cylinder-to-sphere transition, and known as Delauney surfaces (or unduloids). The unduloids are constant mean curvature surfaces of revolution interpolating between a straight cylinder to a string of tangent round spheres. Here, we will consider the limit configuration, and we prove existence of stationary sets for $\mathcal{P}_{\gamma}$ formed by a periodic string of nearly round balls.

It is known, see for instance [17, 29], that a stationary set (or critical point) $\Omega$ for the functional $\mathcal{P}_{\gamma}$ satisfies the following Euler-Lagrange equation

$$
\begin{equation*}
H_{\partial \Omega}(x)+\gamma \int_{\Omega} G_{\kappa}(|x-y|) d y=\text { Const. } \quad \text { for all } x \in \partial \Omega \tag{1.3}
\end{equation*}
$$

where $H_{\partial \Omega}$ is the mean curvature (positive for a ball) of $\partial \Omega$. Here and in the following, for every set $\Omega$ with $C^{2}$ boundary (not necessarily bounded), we define the function

$$
\mathcal{H}_{\Omega}: \partial \Omega \rightarrow \mathbb{R}
$$

given by

$$
\begin{equation*}
\mathcal{H}_{\Omega}(x)=H_{\partial \Omega}(x)+\gamma \int_{\Omega} G_{\kappa}(|x-y|) d y, \quad \text { for every } x \in \partial \Omega . \tag{1.4}
\end{equation*}
$$

In this paper, equilibrium patterns are sets $\Omega \subset \mathbb{R}^{3}$ for which $\mathcal{H}_{\Omega} \equiv$ Const on $\partial \Omega$.
Our aim is to construct nontrivial unbounded equilibrium patterns $\Omega$. These sets are multiply periodic and bifurcate from lattices of round spheres, straight cylinders and slabs. In particular, we obtain stationary sets for $\mathcal{P}_{\gamma}$ made by lattices of near-spheres centered at any $M$-dimensional Bravais lattice. When $M=1$, these are adjacent nearly-spherical sets centered at a straight line with equal gaps between them. The cylinders, result from the 2-dimensional existence of spherical patterns with the corresponding Yukawa potential given by a modified Bessel function. These configurations appear for all $\gamma \in\left(0, \gamma_{N}\right)$, for some positive constant $\gamma_{N}$. Finally, we obtain lamellar structures (modulated slabs) bifurcating from parallel planes. The bifurcations of these modulated slabs occur as long as $\frac{\kappa \sqrt{\kappa^{2}+1}}{\sqrt{\kappa^{2}+1}-\kappa}<$ $2 \pi \gamma<\left(\kappa^{2}+1\right)^{3 / 2}$.

We first describe our doubly periodic slabs. We consider domains of the form,

$$
\begin{equation*}
\Omega_{\varphi}=\left\{(t, z) \in \mathbb{R}^{2} \times \mathbb{R}:-\varphi(t)<z<\varphi(t)\right\} \subset \mathbb{R}^{3}, \tag{1.5}
\end{equation*}
$$

where $\varphi: \mathbb{R}^{2} \rightarrow(0, \infty)$ is $2 \pi \mathbb{Z}^{2}$-periodic and satisfies some symmetry properties. Indeed, we let $C^{k, \gamma}\left(\mathbb{R}^{2}\right)$ denotes the space of $C^{k}\left(\mathbb{R}^{2}\right)$ bounded functions $u$, with bounded derivatives up to order $k$ and with $D^{k} u$ having finite Hölder seminorm of order $\alpha \in(0,1)$. The space $C_{p}^{k, \alpha}\left(\mathbb{R}^{2}\right)$ denotes the space of functions in $C^{k, \alpha}\left(\mathbb{R}^{2}\right)$ which are $2 \pi \mathbb{Z}^{2}$-periodic. Finally, we define

$$
C_{p, \mathcal{R}}^{k, \alpha}=\left\{\varphi \in C_{p}^{k, \alpha}\left(\mathbb{R}^{2}\right): \varphi\left(t_{1}, t_{2}\right)=\varphi\left(t_{2}, t_{1}\right)=\varphi\left(-t_{1}, t_{2}\right), \text { for all }\left(t_{1}, t_{2}\right) \in \mathbb{R}^{2}\right\} .
$$

The spaces $C_{p, \mathcal{R}}^{k, \alpha}$ and $C_{p}^{k, \alpha}\left(\mathbb{R}^{2}\right)$ are equipped with the standard Hölder norm of $C^{k, \alpha}\left(\mathbb{R}^{2}\right)$.
Our first main result is the following.

Theorem 1.1. Let $\kappa>0$. Then for every $\gamma$ satisfying

$$
\begin{equation*}
\frac{1}{2 \pi} \frac{1}{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}}<\gamma<\frac{1}{2 \pi} \frac{1}{\frac{1}{\left(\kappa^{2}+1\right)^{3 / 2}}} \tag{1.6}
\end{equation*}
$$

there exists $\lambda_{*}=\lambda_{*}(\kappa, \gamma)>0$ and a smooth curve

$$
\left(-s_{0}, s_{0}\right) \rightarrow \mathbb{R}_{+} \times C_{p, \mathcal{R}}^{2, \alpha}, \quad s \mapsto\left(\lambda_{s}, \varphi_{s}\right)
$$

with the properties that $\lambda_{0}=\lambda_{*}$ and $\Omega_{\varphi_{s}}$ is an equilibrium pattern satisfying, for all $s \in$ $\left(-s_{0}, s_{0}\right)$,

$$
\mathcal{H}_{\Omega_{\varphi_{s}}}(x)=2 \pi \gamma \operatorname{arcsinh}\left(2 \lambda_{s} / \kappa\right) \quad \text { for every } x \in \partial \Omega_{\varphi_{s}}
$$

Moreover, for every $s \in\left(-s_{0}, s_{0}\right)$ and $\left(t_{1}, t_{2}\right) \in \mathbb{R}^{2}$,

$$
\varphi_{s}\left(t_{1}, t_{2}\right)=\lambda_{s}+s\left(\cos \left(t_{1}\right)+\cos \left(t_{2}\right)+v_{s}\left(t_{1}, t_{2}\right)\right)
$$

with a smooth curve $\left(-s_{0}, s_{0}\right) \rightarrow C_{p, \mathcal{R}}^{2, \alpha}\left(\mathbb{R}^{2}\right)$, $s \mapsto v_{s}$ satisfying

$$
\int_{[-\pi, \pi]^{2}} v_{s}(t) \cos \left(t_{j}\right) d t=0 \quad \text { for } j=1,2
$$

and $v_{0} \equiv 0$.
The domains $\Omega_{\varphi_{s}}$ in Theorem 1.1 bifurcates from the same slab $\left\{-\lambda_{*}<z<\lambda_{*}\right.$. For $\gamma$ satisfying $\frac{1}{2 \pi} \frac{1}{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}}<\gamma($ see the lower bound in (1.6) $)$, the constant $\lambda_{*}$ is the unique zero of the function

$$
\begin{equation*}
\lambda \mapsto 1-2 \pi \gamma\left\{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}-\frac{\exp \left(-2 \lambda \sqrt{\kappa^{2}+1}\right)}{\sqrt{\kappa^{2}+1}}-\frac{\exp (-2 \lambda \kappa)}{\kappa}\right\} \tag{1.7}
\end{equation*}
$$

The Crandall-Rabonowitz bifurcation theorem is the main tool we have used here to prove the existence of the branches $s \mapsto\left(\lambda_{s}, \varphi_{s}\right)$. It provides a bifurcation result from a unique simple zero eigenvalue.
We now discuss the condition (1.6). If $\gamma \leq \frac{1}{2 \pi} \frac{1}{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}}$ then the only solutions bifurcating from $\Omega_{\lambda}$, for $\lambda>0$, are the trivial ones. This follows from the fact that all eigenvalues of the lienearization of $\varphi \mapsto \mathcal{H}_{\Omega_{\varphi}}$ at $\lambda$ are positive. However, equilibrium patterns with different morphology (undulated cylinders), boundary of Delauney-type surfaces, exist for $\gamma<\frac{1}{2 \pi} \frac{1}{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}}$, see [23]. Now if $\gamma \geq \frac{1}{2 \pi} \frac{1}{\left(\kappa^{2}+1\right)^{3 / 2}}$, we do not know the number of zero eigenvalues of the linearized operator of $\varphi \mapsto \mathcal{H}_{\Omega_{\varphi}}$ about the straight slab $\Omega_{\lambda_{*}}$, while the uniqueness of the trivial eigenvalue is crucial in our analysis.

We describe next our (modulated) lamellae. For $\varepsilon \in \mathbb{R}_{*}$ close to zero, we consider the lamellae made by the slices of the slabs $\Omega_{\psi}$ above:

$$
\begin{equation*}
\mathcal{L}_{\psi}^{\varepsilon}=\Omega_{\psi}+\frac{1}{|\varepsilon|} \mathbb{Z} e_{3}=\bigcup_{p \in \mathbb{Z}}\left(\Omega_{\psi}+\left(0,0, \frac{p}{|\varepsilon|}\right)\right) \tag{1.8}
\end{equation*}
$$

where $\Omega_{\psi}$ is as defined in (1.5). It is easy to see that for small $|\varepsilon|$, the set $\mathcal{L}_{\psi}^{\varepsilon}$ is as smooth as $\psi$. Moreover the straight lamellae $\mathcal{L}_{1}^{\varepsilon}$ is an equilibrium pattern. More precisely, $\mathcal{H}_{\mathcal{L}_{\lambda}^{\varepsilon}} \equiv$ Const for every $\lambda>0$ and $|\varepsilon|<\frac{1}{2 \lambda}$. We establish the existence of nontrivial periodic patterns bifurcating from $\mathcal{L}_{\lambda_{*}}^{\varepsilon}$, with $\lambda_{*}$ given by Theorem 1.1 .

Theorem 1.2. Let $\kappa, \gamma, \lambda_{s}>0, \varphi_{s}$ and $s_{0}>0$ be as in Theorem 1.1. Then there exists $\varepsilon_{0}>0$ and a smooth branch

$$
\left(-\varepsilon_{0}, \varepsilon_{0}\right) \times\left(-s_{0}, s_{0}\right) \rightarrow \mathbb{R}_{+} \times C_{p, \mathcal{R}}^{2, \alpha}\left(\mathbb{R}^{2}\right), \quad \varepsilon \mapsto\left(\lambda_{\varepsilon, s}, \psi_{\varepsilon, s}\right)
$$

such that the set $\mathcal{L}_{\psi_{\varepsilon, s}}^{\varepsilon}:=\Omega_{\psi_{\varepsilon, s}}+\frac{1}{|\varepsilon|} \mathbb{Z} e_{3}$ defined in (1.8) is an equilibrium pattern, with

$$
\mathcal{H}_{\mathcal{L}_{\psi_{\varepsilon, s}}^{\varepsilon}}(x)=\mathcal{H}_{\mathcal{L}_{\varepsilon, s}}^{\varepsilon} \quad \text { for every } x \in \partial \mathcal{L}_{\psi_{\varepsilon, s}}^{\varepsilon},
$$

where, $\lambda_{0, s}=\lambda_{s}, \psi_{0, s}=\varphi_{s}$. Moreover, for every $s \in\left(s_{0}, s_{0}\right), \varepsilon \in\left(-\varepsilon_{0}, \varepsilon_{0}\right)$ and $\left(t_{1}, t_{2}\right) \in \mathbb{R}^{2}$,

$$
\psi_{\varepsilon, s}\left(t_{1}, t_{2}\right)=\lambda_{\varepsilon, s}+s\left(\cos \left(t_{1}\right)+\cos \left(t_{2}\right)+v_{\varepsilon, s}\left(t_{1}, t_{2}\right)\right)
$$

with

$$
\int_{[-\pi, \pi]^{2}} v_{\varepsilon, s}(t) \cos \left(t_{j}\right) d t=0 \quad \text { for } j=1,2
$$

As mentioned earlier, we are also interested on the cylindrical and spherical stationary set for the energy $\mathcal{P}_{\gamma}$. These surfaces are obtained by perturbing balls in $\mathbb{R}^{N}$, with $N=2,3$, centered at any $M$-dimensional Bravais lattice for $1 \leq M \leq N$. The perturbation parameter is a scaling parameter which makes the distance between the lattice points sufficiently large. Let $M \in \mathbb{N}$ with $1 \leq M \leq 3$. Let $\left\{\mathbf{a}_{1} ; \ldots ; \mathbf{a}_{M}\right\}$ be a basis of $\mathbb{R}^{M}$. We consider the $M$ dimensional Bravais lattice in $\mathbb{R}^{3}$ :

$$
\mathscr{L}^{M}=\left\{\sum_{i=1}^{M} k_{i} \mathbf{a}_{i}: k=\left(k_{1}, \ldots, k_{M}\right) \in \mathbb{Z}^{M}\right\} \subset \mathbb{R}^{3} .
$$

Here and in the following, we identify $\mathbb{R}^{M}$ with $\mathbb{R}^{M} \times\left\{0_{\mathbb{R}^{3-M}}\right\} \subset \mathbb{R}^{3}$. Obviously $\mathscr{L}^{1}=c \mathbb{Z}$, for some nonzero real number $c$.

We define the perturbed cylinder

$$
C_{\varphi}:=\left\{(r, \theta, t) \in \mathbb{R}_{+} \times S^{1} \times \mathbb{R}: r<\varphi(\theta)\right\}
$$

where $\varphi: S^{1} \rightarrow(0, \infty)$ is an even function. For $\varepsilon>0$ small, we consider the lattice of perturbed cylinders centered at the lattice points $\mathscr{L}^{M}$ given by

$$
\begin{equation*}
\mathcal{C}_{\varphi}^{\varepsilon}:=C_{\varphi}+\frac{1}{\varepsilon} \mathscr{L}^{M} \tag{1.9}
\end{equation*}
$$

We obtain the following result.
Theorem 1.3. Let $\mathscr{L}^{M}$ be an $M$-dimensional Bravais lattices, with $M=1,2$. Then there exists $\gamma_{2}>0$ such that for every $\gamma \in\left(0, \gamma_{2}\right)$, there exists a smooth curve $\left(0, \varepsilon_{0}\right) \rightarrow C^{2, \alpha}\left(S^{1}\right)$, $\varepsilon \mapsto \varphi_{\varepsilon}$ such that for every $\varepsilon \in\left(0, \varepsilon_{0}\right)$, the lattice of perturbed cylinders $\mathcal{C}_{\varphi_{\varepsilon}}^{\varepsilon}$ given by (1.9) is an equilibrium pattern, with

$$
\mathcal{H}_{\mathcal{C}_{\varphi_{\varepsilon}}}(x)=\mathcal{H}_{C_{1}}(\cdot) \quad \text { for every } x \in \partial \mathcal{C}_{\varphi_{\varepsilon}}^{\varepsilon} .
$$

Moreover $\varphi_{\varepsilon}=1+\omega_{\varepsilon}$ with $\omega_{\varepsilon}$ even and $\left\|\omega_{\varepsilon}\right\|_{C^{2, \alpha}\left(S^{1}\right)} \leq e^{-\frac{c}{\varepsilon}}$ as $\varepsilon \rightarrow 0$, for some positive constant $c$ depending only on $\kappa, \alpha, \gamma$ and $\mathscr{L}^{M}$. When $M=1$, the function $\omega_{\varepsilon}$ is not constant.

We emphasize that Theorem 1.3 results from a two-dimensional study. Indeed spherical equilibrium patterns in 2-dimension with the corresponding 2-dimensional screened Coulomb potential (given by the second of modified Bessel function $2 K_{0}(\kappa r)$ ) yields cylinderical patterns in 3-dimension with the Yukawa potential $G_{\kappa}(r)$. In [10], Chen and Oshita showed
that among all 2-dimensional Bravais lattices of near-balls, the hexagonal one provides least possible energy.
We expect that the perturbation $\omega_{\varepsilon}$ is not constant on $S^{1}$. We are able to prove this fact only in the case of lower dimensional lattice, $M=1$, see Section 5.3 below. The fact that the perfect 2-dimensional lattice of cylinders $\mathcal{C}_{\varphi}^{\varepsilon}:=C_{1}+\frac{1}{\varepsilon} \mathscr{L}^{2}$ is not an equilibrium pattern remains an open question. However, numerical simulations give evidence that they are not constant, at least in the hexagonal configuration, see Muratov [29, for $0<\gamma=O\left(\kappa^{2}\right) \ll 1$.

To introduce the sphere lattices, we consider the $M$-dimensional Bravais $\mathscr{L}^{M}$ as above, with $M \leq 3$. Here, $S^{2}$ is the 2 -dimensional unit sphere. We consider the perturbed ball

$$
B_{\phi}^{3}:=\left\{(r, \theta) \in \mathbb{R}_{+} \times S^{2}: 0<r<\phi(\theta)\right\},
$$

where $\phi: S^{2} \rightarrow(0, \infty)$ is an even function. For $\varepsilon$ small enough, we consider the following disjoint union of perturbed balls

$$
\begin{equation*}
\mathcal{B}_{\phi}^{\varepsilon}:=B_{\phi}^{3}+\frac{1}{\varepsilon} \mathscr{L}^{M} . \tag{1.10}
\end{equation*}
$$

We have the following result.
Theorem 1.4. There exists $\gamma_{3}>0$ such that for every $\gamma \in\left(0, \gamma_{3}\right)$, there exists a smooth curve $\left(0, \varepsilon_{0}\right) \rightarrow C^{2, \alpha}\left(S^{2}\right), \varepsilon \mapsto \phi_{\varepsilon}$ such that for every $\varepsilon \in\left(0, \varepsilon_{0}\right)$, the set $\mathcal{B}_{\phi_{\varepsilon}}^{\varepsilon}$ defined by (1.10) is an equilibrium pattern, with

$$
\mathcal{H}_{\mathcal{B}_{\phi_{\varepsilon}}^{\varepsilon}}(x)=\mathcal{H}_{B_{1}^{3}}(\cdot) \quad \text { for every } x \in \partial \mathcal{B}_{\phi_{\varepsilon}}^{\varepsilon}
$$

Moreover $\phi_{\varepsilon}=1+w_{\varepsilon}$ with $\omega_{\varepsilon}$ even and $\left\|w_{\varepsilon}\right\|_{C^{2}, \alpha}\left(S^{2}\right) \leq e^{-\frac{c}{\varepsilon}}$ as $\varepsilon \rightarrow 0$, for some positive constant $c$ depending only on $\kappa, \alpha, \gamma$ and $\mathscr{L}^{M}$. When $M=2$, the function $w_{\varepsilon}$ is not constant.

The constants $\gamma_{N}$, for $N=2,3$, in Theorem 1.4 and Theorem 1.3 depend only on the eigenvalues of the linearized operator of the map $w \mapsto \mathcal{H}_{B_{w}^{N}}$, see Lemma 5.4 below.

In the case of near-sphere lattices also, we are able to prove that $w_{\varepsilon}$ is not constant on $S^{N-1}$ only in the lower dimensional lattices, $M \leq 2$, see Section 5.3 below. It is an open question to know whether for $M=3$, the function $w_{\varepsilon}$ is not constant i.e. $\mathcal{H}_{\mathcal{B}_{1}^{\varepsilon}}$ is not constant on $\partial B_{1}$.

Related to results in the present paper are the works, for $\kappa=0$, of Ren and Wei in their series of papers 4050 . Among many others, they build equilibrium patterns of type lamellar, ring/cylindrical, and spot/spherical solutions. See also [37, 38] and the references therein.
Next, we emphasize some similarities between equilibrium patterns derived here and the structures of Constant Nonlocal Mean Curvatures (CNMC) sets recently discovered. The notion of nonlocal mean curvature was recently introduced by Caffarelli and Souganidis in 9 . See Caffarelli, Roquejoffre, and Savin [8], it arises from the Euler-Lagrange equation for the fractional perimeter functional which in turn is a $\Gamma$-limit of some nonlocal phase transition problems, see 51.
The long range interaction in the expression of $\mathcal{H}_{\Omega}$ is, up to an additional constant, a weighted average of all " +1 " in $\Omega$ and all "-1" from outside $\Omega$, since,

$$
\begin{equation*}
\mathcal{H}_{\Omega}(x)=H_{\partial \Omega}(x)+\frac{\gamma}{2} \int_{\mathbb{R}^{3}}\left(1_{\Omega}(y)-1_{\mathbb{R}^{3} \backslash \Omega}(y)\right) G_{\kappa}(|x-y|) d y+\bar{c}, \tag{1.11}
\end{equation*}
$$

for $x \in \partial \Omega$, where $\bar{c}=\frac{1}{2} \int_{\mathbb{R}^{3}} G_{\kappa}(|x|) d x=\frac{2 \pi}{\kappa^{2}}$. Similarly, for $\alpha \in(0,1)$, the fractional or nonlocal mean curvature function $\mathcal{H}_{\Omega}^{\alpha}: \partial \Omega \rightarrow \mathbb{R}$ is given by

$$
\mathcal{H}_{\Omega}^{\alpha}(x):=-\frac{1-\alpha}{\left|B^{2}\right|} \int_{\mathbb{R}^{3}}\left(1_{\Omega}(y)-1_{\mathbb{R}^{3} \backslash \Omega}(y)\right)|x-y|^{-3-\alpha} d y
$$

(positive if $\Omega$ is a ball). The factor $\frac{1-\alpha}{\left|B^{2}\right|}$ implies that $H_{\partial \Omega}^{\alpha}(x)$ converges to the mean curvature $H_{\partial \Omega}(x)$ as $\alpha \rightarrow 1$, see [1, 14].

We can parallel the recently constructed CNMC sets and the equilibrium patterns obtained here. Indeed, by [6, 7], CNMC bands exist, and from which 1-periodic CNMC slabs can be derived. Modulated CNMC slabs are found in [27]. Finally from [5], there exist CNMC near-sphere lattices from which CNMC lattices of near-cylinders can be derived. Finally, we point out that resemblance between phase structures in diblock copolymer melts, and triply periodic constant mean curvature surfaces was also observed several years ago, see e.g. 4, 53].

## 2. Preliminary and notations

We let $K_{\nu}$ be the second kind of modified Bessel function of order $\nu \in \mathbb{R}$. See e.g. [16], for $\nu \in \mathbb{R}$, there holds

$$
K_{\nu}^{\prime}(r)=-\frac{\nu}{r} K_{\nu}(r)-K_{\nu-1}(r)
$$

and $K_{-\nu}=K_{\nu}$ for $\nu<0$. Therefore

$$
\begin{equation*}
K_{0}^{\prime}(r)=-K_{1}(r), \quad\left(r K_{1}\right)^{\prime}(r)=-r K_{0}(r) . \tag{2.1}
\end{equation*}
$$

See also [16], for $\nu \geq 0$,

$$
\begin{equation*}
K_{\nu}(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-1 / 2} e^{-r} \quad \text { as } r \rightarrow+\infty \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
K_{0}(r) \sim-\log (r), \quad K_{\nu}(r) \sim \frac{1}{2} \Gamma(\nu)(r / 2)^{-\nu} \quad \text { as } r \rightarrow 0 . \tag{2.3}
\end{equation*}
$$

We now collect some useful formula, where most of them can be found in [24]. In the following, we let $\alpha, \beta, \delta \geq 0$ and $\kappa>0$. By [24, page 491, ET I 16(27)], we have

$$
\begin{equation*}
\int_{\mathbb{R}} \cos (\beta r) G_{\kappa}\left(\sqrt{r^{2}+\alpha^{2}}\right) d r=2 K_{0}\left(\alpha \sqrt{\kappa^{2}+\beta^{2}}\right) \tag{2.4}
\end{equation*}
$$

See also [24, page 719, WA 425(10)a, MO 48], we have

$$
\begin{equation*}
\int_{\mathbb{R}} \cos (\beta r) K_{0}\left(r \sqrt{\kappa^{2}+\alpha^{2}}\right) d r=\frac{\pi}{\sqrt{\kappa^{2}+\beta^{2}+\alpha^{2}}} \tag{2.5}
\end{equation*}
$$

By computing, we have

$$
\begin{equation*}
\int_{\mathbb{R}^{2}} G_{\kappa}(|x|) d x=\frac{2 \pi}{\kappa} . \tag{2.6}
\end{equation*}
$$

From [24, page 722, ET I 56(43)], we get

$$
\begin{equation*}
\int_{\mathbb{R}} \cos (\beta r) K_{0}\left(\sqrt{r^{2}+\delta^{2}} \sqrt{\alpha^{2}+\kappa^{2}}\right) d r=\pi \frac{\exp \left(-\delta \sqrt{\beta^{2}+\alpha^{2}+\kappa^{2}}\right)}{\sqrt{\beta^{2}+\alpha^{2}+\kappa^{2}}} . \tag{2.7}
\end{equation*}
$$

Combining (2.4) and (2.7), we obtain

$$
\begin{equation*}
\int_{\mathbb{R}^{2}} G_{\kappa}\left(\sqrt{|x|^{2}+\delta^{2}}\right) d x=2 \pi \frac{\exp (-\delta \kappa)}{\kappa} \tag{2.8}
\end{equation*}
$$

For a finite set $\mathcal{N}$, we let $|\mathcal{N}|$ denote the length (cardinal) of $\mathcal{N}$. It will be understood that $|\emptyset|=0$. Let $Z$ be a Banach space and $U$ a nonempty open subset of $Z$. If $T \in C^{k}(U, \mathbb{R})$ and $u \in U$, then $D^{k} T(u)$ is a continuous symmetric $k$-linear form on $Z$ whose norm is given by

$$
\left\|D^{k} T(u)\right\|=\sup _{u_{1}, \ldots, u_{k} \in Z} \frac{\left|D^{k} T(u)\left[u_{1}, \ldots, u_{k}\right]\right|}{\prod_{j=1}^{k}\left\|u_{j}\right\|_{Z}}
$$

If, $L: Z \rightarrow \mathbb{R}$ is a linear map, we have

$$
\begin{equation*}
D^{|\mathcal{N}|}\left(L T_{2}\right)(u)\left[u_{i}\right]_{i \in \mathcal{N}}=L(u) D^{|\mathcal{N}|} T_{2}(u)\left[u_{i}\right]_{i \in \mathcal{N}}+\sum_{j \in \mathcal{N}} L\left(u_{j}\right) D^{|\mathcal{N}|-1} T_{2}(u)\left[u_{i}\right]_{\substack{i \in \mathcal{N} \\ i \neq j}} \tag{2.9}
\end{equation*}
$$

We let $T$ be as above, $V \subset \mathbb{R}$ open with $T(U) \subset V$ and $g: V \rightarrow \mathbb{R}$ be a $k$-times differentiable map. The Faá de Bruno formula states that

$$
\begin{equation*}
D^{k}(g \circ T)(u)\left[u_{1}, \ldots, u_{k}\right]=\sum_{\Pi \in \mathscr{P}_{k}} g^{(|\Pi|)}(T(u)) \prod_{P \in \Pi} D^{|P|} T(u)\left[u_{j}\right]_{j \in P} \tag{2.10}
\end{equation*}
$$

for $u, u_{1}, \ldots, u_{k} \in U$, where $\mathscr{P}_{k}$ denotes the set of all partitions of $\{1, \ldots, k\}$, see e.g. [26].

## 3. Perodic SLabs

3.1. Doubly periodic slabs. In this section we construct equilibrium patterns with period cell given by a doubly periodic slab. That is domains of the form

$$
\Omega_{\varphi}:=\left\{(t, z) \in \mathbb{R}^{2} \times \mathbb{R}:-\varphi(t)<z<\varphi(t)\right\} \subset \mathbb{R}^{3}
$$

where $\varphi: \mathbb{R}^{2} \rightarrow(0, \infty)$, with $\varphi \in C^{2, \alpha}\left(\mathbb{R}^{2}\right)$. Recall that for $x \in\{-1,1\}$, the mean curvature is given by

$$
H_{\partial \Omega_{\varphi}}((t, \varphi(t) x))=-\operatorname{div} \frac{\nabla \varphi(t)}{\sqrt{1+|\nabla \varphi(t)|^{2}}}
$$

Moreover, by a change of variable, the Yukawa interaction takes the form

$$
\begin{aligned}
& \int_{\Omega_{\varphi}} G_{\kappa}(|(t, \varphi(t) x)-X|) d X=\int_{\mathbb{R}^{2}} \int_{-\varphi(s)}^{\varphi(s)} G_{\kappa}(|(t, \varphi(t) x)-(s, z)|) d z d s \\
& =\int_{\mathbb{R}^{2}} \int_{-1}^{1} \varphi(s) G_{\kappa}(|(t, \varphi(t) x)-(s, \varphi(s) y)|) d y d s \\
& =\int_{\mathbb{R}^{2}} \int_{-1}^{1} \varphi(s) G_{\kappa}(|(t, \varphi(t))-(s, \varphi(s) y)|) d y d s
\end{aligned}
$$

Next, we define the open set

$$
\mathcal{O}:=\left\{\varphi \in C^{2, \alpha}\left(\mathbb{R}^{2}\right): \varphi>0\right\}
$$

and the $\operatorname{map} \mathcal{F}: \mathcal{O} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$ by

$$
\begin{equation*}
\mathcal{F}(\varphi)(t):=-\operatorname{div} \frac{\nabla \varphi(t)}{\sqrt{1+|\nabla \varphi(t)|^{2}}}+\gamma \int_{\mathbb{R}^{2}} \int_{-1}^{1} \varphi(s) G_{\kappa}(|(t, \varphi(t))-(s, \varphi(s) y)|) d y d s \tag{3.1}
\end{equation*}
$$

so that, for every $x \in\{-1,1\}$ and $t \in \mathbb{R}^{2}$,

$$
\mathcal{H}_{\Omega_{\varphi}}((t, \varphi(t) x))=\mathcal{F}(\varphi)(t)
$$

Our aim is to apply the Crandall-Rabinowitz bifurcation theorem to solve the equation

$$
\mathcal{F}(\lambda+u)=\text { const }
$$

for $\lambda>0$ and $u \in C^{2, \alpha}\left(\mathbb{R}^{2}\right)$ is small. We will achieve this by solving the equation

$$
\begin{equation*}
\mathcal{F}(\lambda+u)=\mathcal{F}(\lambda) \tag{3.2}
\end{equation*}
$$

We note that by some computation and (2.8), we have

$$
\begin{equation*}
\mathcal{F}(\lambda)=\gamma \lambda \int_{-1}^{1} \int_{\mathbb{R}^{2}} G_{\kappa}\left(\sqrt{|r|^{2}+\lambda^{2}(1-y)^{2}}\right) d r d y=2 \pi \gamma \operatorname{arcsinh}(2 \lambda / \kappa) \tag{3.3}
\end{equation*}
$$

Let us study the linearized operator $D \mathcal{F}(\lambda)$, for $\lambda>0$. We consider the map

$$
\mathcal{O} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right), \quad \mathcal{F}_{1}(\varphi)(t):=\int_{\mathbb{R}^{2}} \varphi(s) \int_{-1}^{1} G_{\kappa}\left(\left(|t-s|^{2}+(\varphi(t)-\varphi(s) y)^{2}\right)^{1 / 2}\right) d y d s
$$

We have the following result.
Lemma 3.1. The map $\mathcal{F}: \mathcal{O} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$ is smooth, with

$$
\left\|D^{k} \mathcal{F}(\varphi)\right\| \leq c\left(1+\|\varphi\|_{C^{2, \alpha}\left(\mathbb{R}^{2}\right)}\right)^{c}
$$

for some positive constant $c=c(\kappa, \alpha, \delta, k)$. Moreover for every $\varphi \in \mathcal{O}$ and $w \in C^{2, \alpha}\left(\mathbb{R}^{2}\right)$,

$$
\begin{align*}
D \mathcal{F}_{1}(\varphi)[w](t)= & \int_{\mathbb{R}^{2}}(w(t-r)-w(t)) G_{\kappa}\left(\left(|r|^{2}+(\varphi(t)-\varphi(t-r))^{2}\right)^{1 / 2}\right) d r \\
& +\int_{\mathbb{R}^{2}}(w(t-r)+w(t)) G_{\kappa}\left(\left(|r|^{2}+(\varphi(t)+\varphi(t-r))^{2}\right)^{1 / 2}\right) d r \tag{3.4}
\end{align*}
$$

In particular, for every $\lambda>0$,

$$
\begin{align*}
& D \mathcal{F}(\lambda)[w](t)=-\Delta w(t)-\gamma \int_{\mathbb{R}^{2}}(w(t)-w(t-r)) G_{\kappa}(|r|) d r \\
& \quad+\gamma(w(t)+w(t-r)) \int_{\mathbb{R}^{2}} G_{\kappa}\left(\left(|r|^{2}+4 \lambda^{2}\right)^{1 / 2}\right) d r \tag{3.5}
\end{align*}
$$

Proof. We notice that

$$
\begin{align*}
& \mathcal{F}(\varphi)(t)=-\operatorname{div} \frac{\nabla \varphi(t)}{\sqrt{1+|\nabla \varphi(t)|^{2}}} \\
& \quad+\gamma \int_{\mathbb{R}^{2}} \varphi(s) \int_{-1}^{1} G_{\kappa}\left(\left(|t-s|^{2}+(\varphi(t)-\varphi(s) y)^{2}\right)^{1 / 2}\right) d y d s \tag{3.6}
\end{align*}
$$

The regularity of the map

$$
C^{2, \alpha}\left(\mathbb{R}^{2}\right) \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right), \quad \varphi \mapsto \operatorname{div} \frac{\nabla \varphi(t)}{\sqrt{1+|\nabla \varphi(t)|^{2}}}
$$

is easy, we therefore consider the second term. Fro every $\delta>0$, we define

$$
\mathcal{O}_{\delta}=\left\{\varphi \in C^{2, \alpha}\left(\mathbb{R}^{2}\right) \varphi>\delta\right\}
$$

Let us first compute the formal derivative of this map which we will show that it is smooth. Once this is done we can easily deduce the regularity of $\mathcal{F}_{1}$ and hence of $\mathcal{F}$. We have

$$
\begin{equation*}
D \mathcal{F}_{1}(\varphi)[w](t)=\int_{\mathbb{R}^{2}} w(s) \int_{-1}^{1} f(y) d y d s+\int_{\mathbb{R}^{2}} \int_{-1}^{1}(w(s) y-w(t)) f^{\prime}(y) d y d s \tag{3.7}
\end{equation*}
$$

where

$$
f(y)=G_{\kappa}(|(t, \varphi(t))-(s, \varphi(s) y)|)
$$

which satisfy

$$
f^{\prime}(y)=\varphi(s)(\varphi(s) y-\varphi(t)) G_{\kappa}^{\prime}(|(t, \varphi(t))-(s, \varphi(s) y)|) .
$$

Integration by parts give

$$
\int_{-1}^{1}(w(s) y-w(t)) f^{\prime}(y) d y=-\int_{-1}^{1} w(s) f(y) d y+\left.(w(s) \theta-w(t)) f(\theta)\right|_{\theta=-1} ^{\theta=1}
$$

Inserting this in (3.7), we deduce that

$$
\begin{aligned}
D \mathcal{F}_{1}(\varphi)[w](t)= & \left.\int_{\mathbb{R}^{2}}(w(s) \theta-w(t)) f(\theta) d s\right|_{\theta=-1} ^{\theta=1} \\
= & \int_{\mathbb{R}^{2}}(w(s)-w(t)) f(1) d s-\int_{\mathbb{R}^{2}}(-w(s)-w(t)) f(-1) d s \\
= & \int_{\mathbb{R}^{2}}(w(s)-w(t)) G_{\kappa}(|(t, \varphi(t))-(s, \varphi(s))|) d s \\
& +\int_{\mathbb{R}^{2}}(w(s)+w(t)) G_{\kappa}(|(t, \varphi(t))-(s,-\varphi(s))|) d s
\end{aligned}
$$

This gives (3.4) after the change of variable $s=t-r$. We now have

$$
\begin{align*}
D \mathcal{F}_{1}(\varphi)[w](t)= & \int_{\mathbb{R}^{2}}(w(t-r)-w(t)) G_{\kappa}\left(\left|\left(|r|^{2}+(\varphi(t)-\varphi(t-r))^{2}\right)^{1 / 2}\right|\right) d s \\
& +\int_{\mathbb{R}^{2}}(w(t-r)+w(t)) G_{\kappa}\left(\left|\left(|r|^{2}+(\varphi(t)+\varphi(t-r))^{2}\right)^{1 / 2}\right|\right) d s \\
= & \int_{\mathbb{R}^{2}} \frac{w(t-r)-w(t)}{|r|} G_{\kappa|r|}\left(\left(1+\left(\Lambda_{0}(\varphi, t, r)\right)^{2}\right)^{1 / 2}\right) d r \\
& +\int_{\mathbb{R}^{2}}(w(t-r)+w(t)) G_{\kappa}\left(\left(|r|^{2}+(\varphi(t)+\varphi(t-r))^{2}\right)^{1 / 2}\right) d r \tag{3.8}
\end{align*}
$$

where $\Lambda_{0}: C^{2, \alpha}\left(\mathbb{R}^{2}\right) \times \mathbb{R}^{4} \rightarrow \mathbb{R}$ is given by

$$
\Lambda_{0}(\varphi, t, r)=\int_{0}^{1} \nabla \varphi(t-\varrho r) \cdot \frac{r}{|r|} d \varrho .
$$

For every fixed $w \in C^{2, \alpha}\left(\mathbb{R}^{2}\right)$, we can define $\mathcal{J}_{i}: \mathcal{O}_{\delta} \rightarrow L^{\infty}\left(\mathbb{R}^{2}\right)$ by

$$
\mathcal{J}_{1}(\varphi)(t):=\int_{\mathbb{R}^{2}} \frac{w(t-r)-w(t)}{|r|} G_{\kappa|r|}\left(\left|\left(1+\left(\Lambda_{0}(\varphi, t, r)\right)^{2}\right)^{1 / 2}\right|\right) d r
$$

and

$$
\mathcal{J}_{2}(\varphi)(t):=\int_{\mathbb{R}^{2}}(w(t-r)+w(t)) G_{\kappa}\left(\left|\left(|r|^{2}+(\varphi(t)+\varphi(t-r))^{2}\right)^{1 / 2}\right|\right) d r
$$

Therefore by (3.8)

$$
\begin{equation*}
D \mathcal{F}_{1}(\varphi)[w]=\mathcal{J}_{1}(\varphi)+\mathcal{J}_{2}(\varphi) . \tag{3.9}
\end{equation*}
$$

Thanks to (2.9) and the fact that the integrand in $\mathcal{J}_{2}$ is bounded away from zero, it becomes straightforward to see that the map $\mathcal{J}_{2}: \mathcal{O}_{\delta} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$, is smooth and for every $k \in \mathbb{N}$,

$$
\begin{equation*}
\left\|D^{k} \mathcal{J}_{2}(\varphi)\right\| \leq c\left(1+\|\varphi\|_{C^{2, \alpha}\left(\mathbb{R}^{2}\right)}\right)^{c}\|w\|_{C^{2, \alpha}\left(\mathbb{R}^{2}\right)} \tag{3.10}
\end{equation*}
$$

where $c=c(\kappa, \alpha, \delta, k)>0$. Next, we prove the regularity of $\mathcal{J}_{1}: \mathcal{O}_{\delta} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$. For $|r|>0$, we consider the function

$$
h_{r}(x)=G_{\kappa|r|}\left(\left(1+x^{2}\right)^{1 / 2}\right)
$$

We observe for every $k \in \mathbb{N}$, there exists a constant $c=c(\kappa, k)$ such that

$$
\left|h_{r}^{(k)}(x)\right| \leq c G_{\kappa|r|}(1 / 2) \quad \text { for every } x \in \mathbb{R}, r \in \mathbb{R}^{2} \backslash\{0\}
$$

Letting $\mathcal{K}_{1}=C^{2, \alpha}\left(\mathbb{R}^{2}\right) \times \mathbb{R}^{4} \rightarrow \mathbb{R}$, given by

$$
\mathcal{K}_{1}(\varphi, t, r):=G_{\kappa|r|}\left(\left|\left(1+\left(\Lambda_{0}(\varphi, t, r)\right)^{2}\right)^{1 / 2}\right|\right)
$$

We can use (2.10), $g(x)=h_{r}(x)$ and $T(\varphi)=\mathcal{K}(\varphi, t, r)$ to get the following estimates. For every $(t, r) \in \mathbb{R}^{2} \times \mathbb{R}^{2}$,

$$
\left\|D_{\varphi}^{k} \mathcal{K}_{1}(\varphi, t, r)\right\| \leq c\left(1+\|\varphi\|_{2, \alpha}\right)^{c} G_{\kappa|r|}(1 / 2)
$$

with $c=c(\alpha, \kappa, k)>0$. Since also $\int_{\mathbb{R}^{2}}|r|^{-1} G_{\kappa|r|}(1 / 2)<\infty$, by using (2.9), we get, for every $k \in \mathbb{N}$,

$$
\left\|D^{k} \mathcal{J}_{1}(\varphi)\right\| \leq c\left(1+\|\varphi\|_{C^{2, \alpha}\left(\mathbb{R}^{2}\right)}\right)^{c}\|w\|_{C^{2, \alpha}\left(\mathbb{R}^{2}\right)}
$$

with $c=c(\kappa, \alpha, \delta, k)>0$. This then shows that $\mathcal{J}_{1}: \mathcal{O}_{\delta} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$.
From this together with (3.10) and (3.9), we deduce that $D \mathcal{F}_{1}: \mathcal{O}_{\delta} \rightarrow$ $\mathcal{B}\left(C^{2, \alpha}\left(\mathbb{R}^{2}\right), C^{0, \alpha}\left(\mathbb{R}^{2}\right)\right)$ is smooth. Hence $\mathcal{F}$ is smooth in $\mathcal{O}_{\delta}$ and

$$
\left\|D^{k} \mathcal{F}_{1}(\varphi)\right\| \leq c\left(1+\|\varphi\|_{C^{2, \alpha}\left(\mathbb{R}^{2}\right)}\right)^{c}, \quad \text { for some positive constant } c=c(\kappa, \alpha, \delta, k)
$$

In addition, we have that

$$
\begin{aligned}
D^{k} \mathcal{F}_{1}(\varphi)[w](t)= & \int_{\mathbb{R}^{2}} \frac{w(t-r)-w(t)}{|r|} D_{\varphi}^{k} \mathcal{K}_{1}(\varphi, t, r) d r \\
& +\int_{\mathbb{R}^{2}}(w(t-r)+w(t)) D_{\varphi}^{k} \mathcal{K}_{2}(\varphi, t, r) d r
\end{aligned}
$$

where $\mathcal{K}_{2}(\varphi, t, r)=G_{\kappa}\left(\left|\left(|r|^{2}+(\varphi(t)+\varphi(t-r))^{2}\right)^{1 / 2}\right|\right)$. Since $\delta$ was arbitrarily chosen, this proves the smoothness of $\mathcal{F}_{1}$ on $\mathcal{O}$.

Finally, letting $\varphi \equiv \lambda$ in (3.8), we obtain (3.5).
We now study the spectral properties of $D \mathcal{F}(\lambda)$.
Lemma 3.2. For every $\lambda>0$ we define the linear operator

$$
\begin{equation*}
L_{\lambda}:=D \mathcal{F}(\lambda): C_{p, \mathcal{R}}^{2, \alpha} \rightarrow C_{p, \mathcal{R}}^{0, \alpha} \tag{3.11}
\end{equation*}
$$

Then the functions $e_{k} \in C_{p, \mathcal{R}}^{2, \alpha}$, given by

$$
\begin{equation*}
e_{k}(t):=\frac{1}{\pi} \prod_{i=1}^{2} \cos \left(k_{i} t_{i}\right) \tag{3.12}
\end{equation*}
$$

for $k \in \mathbb{N}^{2}$, are the eigenfunctions of $L_{\lambda}$ and moreover

$$
\begin{equation*}
L_{\lambda}\left(e_{k}\right)=\sigma_{\lambda, \gamma}(|k|) e_{k}, \tag{3.13}
\end{equation*}
$$

where $|k|=\sqrt{k_{1}^{2}+k_{2}^{2}}$ and for every $\ell \in \mathbb{N}$,

$$
\begin{equation*}
\sigma_{\lambda, \gamma}(\ell)=\ell^{2}-2 \pi \gamma\left\{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+\ell^{2}}}-\frac{\exp \left(-2 \lambda \sqrt{\kappa^{2}+\ell^{2}}\right)}{\sqrt{\kappa^{2}+\ell^{2}}}-\frac{\exp (-2 \lambda \kappa)}{\kappa}\right\} . \tag{3.14}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\partial_{\lambda} \sigma_{\lambda, \gamma}(1)<0 \quad \text { for every } \gamma, \lambda>0 \tag{3.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} \frac{\sigma_{\lambda, \gamma}(\ell)}{\ell^{2}}=1 \tag{3.16}
\end{equation*}
$$

Proof. By direct computations using Fourier decomposition, we have

$$
\begin{align*}
\sigma_{\lambda, \gamma}(|k|)= & |k|^{2}-\gamma \int_{\mathbb{R}^{2}}\left(1-\prod_{i=1}^{2} \cos \left(k_{i} r_{i}\right)\right) G_{\kappa}(|r|) d r \\
& +\gamma \int_{\mathbb{R}^{2}}\left(1+\prod_{i=1}^{2} \cos \left(k_{i} r_{i}\right)\right) G_{\kappa}\left(\left(|r|^{2}+4 \lambda^{2}\right)^{1 / 2}\right) d r \tag{3.17}
\end{align*}
$$

By (2.4), we have

$$
\int_{\mathbb{R}} \cos \left(k_{2} r_{2}\right) \int_{\mathbb{R}} \cos \left(k_{1} r_{1}\right) G_{\kappa}\left(\left(r_{1}^{2}+r_{2}^{2}\right)^{1 / 2}\right) d r_{1} d r_{2}=2 \int_{\mathbb{R}} \cos \left(k_{2} r_{2}\right) K_{0}\left(r_{2} \sqrt{\kappa^{2}+k_{1}^{2}}\right) d r_{2}
$$

and by (2.5)

$$
\int_{\mathbb{R}} \cos \left(k_{2} r_{2}\right) K_{0}\left(r_{2} \sqrt{\kappa^{2}+k_{1}^{2}}\right) d r_{2}=\frac{2 \pi}{2 \sqrt{\kappa^{2}+|k|^{2}}} .
$$

We then deduce that

$$
\int_{\mathbb{R}} \cos \left(k_{2} r_{2}\right) \int_{\mathbb{R}} \cos \left(k r_{1}\right) G_{\kappa}\left(\left(r_{1}^{2}+r_{2}^{2}\right)^{1 / 2}\right) d r_{1} d r_{2}=\frac{2 \pi}{\sqrt{\kappa^{2}+|k|^{2}}} .
$$

Moreover, taking $k=0$ in the above expression, we have

$$
\int_{\mathbb{R}^{2}} G_{\kappa}(|r|) d r=\frac{2 \pi}{\kappa} .
$$

From the two qualities above, we deduce that

$$
\begin{equation*}
\int_{\mathbb{R}^{2}}\left(1-\prod_{i=1}^{2} \cos \left(k_{i} r_{i}\right)\right) G_{\kappa}(|r|) d r=\frac{2 \pi}{\kappa}-\frac{2 \pi}{\sqrt{\kappa^{2}+|k|^{2}}} . \tag{3.18}
\end{equation*}
$$

By (2.4) and (2.7), we get

$$
\begin{equation*}
\int_{\mathbb{R}^{2}} \prod_{i=1}^{2} \cos \left(k_{i} r_{i}\right) G_{\kappa}\left(\left(|r|^{2}+4 \lambda^{2}\right)^{1 / 2}\right) d r=2 \pi \frac{\exp \left(-2 \lambda \sqrt{|k|^{2}+\kappa^{2}}\right)}{\sqrt{|k|^{2}+\kappa^{2}}} \tag{3.19}
\end{equation*}
$$

and letting $k=0$, we have

$$
\int_{\mathbb{R}^{2}} G_{\kappa}\left(\left(|r|^{2}+4 \lambda^{2}\right)^{1 / 2}\right) d r=2 \pi \frac{\exp (-2 \lambda \kappa)}{\kappa} .
$$

Using this with (3.18) and (3.19) in (3.17), we get the desired expression of $\sigma_{\lambda, \gamma}(|k|)$ in (3.14).

Next, we give some qualitative properties of $\sigma_{\lambda, \gamma}$.
Lemma 3.3. Let $\kappa>0$ and let $\gamma$ be such that

$$
\begin{equation*}
\frac{1}{2 \pi} \frac{1}{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}}<\gamma<\frac{1}{2 \pi} \frac{1}{\frac{1}{\left(\kappa^{2}+1\right)^{3 / 2}}} . \tag{3.20}
\end{equation*}
$$

Then there exists a unique $\lambda_{*}=\lambda_{*}(\gamma, \kappa)>0$ such that

$$
\begin{gather*}
\sigma_{\lambda_{*}, \gamma}(1)=0  \tag{3.21}\\
\sigma_{\lambda_{*}, \gamma}(0)=\frac{4 \pi \gamma \exp \left(-2 \lambda_{*} \kappa\right)}{\kappa}>0 \tag{3.22}
\end{gather*}
$$

and

$$
\begin{equation*}
\sigma_{\lambda_{*}, \gamma}(\ell+1)>\sigma_{\lambda_{*}, \gamma}(\ell) \quad \text { for every } \ell \geq 1 . \tag{3.23}
\end{equation*}
$$

Proof. We first note that, by elementary calculations, the strict inequality

$$
\frac{1}{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}}<\frac{1}{\frac{1}{\left(\kappa^{2}+1\right)^{3 / 2}}}
$$

is equivalent to

$$
\left(\kappa^{2}+1\right)^{2}-\left(\kappa^{4}+2 \kappa^{2}\right)>0,
$$

which holds true for every $\kappa>0$. Therefore we can always pick $\gamma$ as in (3.20).
In view of (3.14), we have

$$
\sigma_{\lambda, \gamma}(1)=1-2 \pi \gamma\left\{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}-\frac{\exp \left(-2 \lambda \sqrt{\kappa^{2}+1}\right)}{\sqrt{\kappa^{2}+1}}-\frac{\exp (-2 \lambda \kappa)}{\kappa}\right\} .
$$

It is easy to see that

$$
\lim _{\lambda \rightarrow 0} \sigma_{\lambda, \gamma}(1)=1+\frac{4 \pi \gamma}{\sqrt{\kappa^{2}+1}}>0
$$

Now if $\gamma$ satisfies the lower bound in (3.20), we have that

$$
\lim _{\lambda \rightarrow+\infty} \sigma_{\lambda, \gamma}(1)=1-2 \pi \gamma\left\{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+1}}\right\}<0
$$

Therefore (3.21) follows, for a unique $\lambda_{*}=\lambda_{*}(\gamma, \kappa)$ thanks to (3.15). We now prove (3.23), where the assumption on the upper bound in (3.21) will be used.

Indeed, we consider the smooth function $g_{\lambda, \gamma}:[1, \infty) \rightarrow \mathbb{R}$, given by

$$
g_{\lambda, \gamma}(x)=x^{2}-2 \pi \gamma\left\{\frac{1}{\kappa}-\frac{1}{\sqrt{\kappa^{2}+x^{2}}}-\frac{\exp \left(-2 \lambda \sqrt{\kappa^{2}+x^{2}}\right)}{\sqrt{\kappa^{2}+x^{2}}}-\frac{\exp (-2 \lambda \kappa)}{\kappa}\right\} .
$$

Then

$$
g_{\lambda, \gamma}^{\prime}(x)=2 x\left(1-\frac{\pi \gamma}{\left(\kappa^{2}+x^{2}\right)^{3 / 2}}-\frac{\pi \gamma \exp \left(-2 \lambda \sqrt{\kappa^{2}+x^{2}}\right)}{\left(\kappa^{2}+x^{2}\right)^{3 / 2}}-\frac{2 \pi \gamma \lambda \exp \left(-2 \lambda \sqrt{\kappa^{2}+x^{2}}\right)}{\kappa^{2}+x^{2}}\right) .
$$

We want $g_{\lambda, \gamma}^{\prime}(x)>0$ for every $\lambda>0$ and $x \geq 1$, which holds true provided

$$
\pi \gamma<\frac{1}{\sup _{\substack{\lambda>0 \\ x \geq 1}}\left(\frac{1}{\left(\kappa^{2}+x^{2}\right)^{3 / 2}}+\frac{\exp \left(-2 \lambda \sqrt{\kappa^{2}+x^{2}}\right)}{\left(\kappa^{2}+x^{2}\right)^{3 / 2}}+\frac{2 \lambda \exp \left(-2 \lambda \sqrt{\kappa^{2}+x^{2}}\right)}{\kappa^{2}+x^{2}}\right)}
$$

But since the (maximized) function in the denominator is decreasing in $x$ and $\lambda$, this is equivalent to

$$
\pi \gamma<\frac{1}{\frac{2}{\left(\kappa^{2}+1\right)^{3 / 2}}}
$$

which is our assumed upper bound in (3.20). It is now clear hat as long as $\gamma$ satisfies (3.20), we have that $g_{\lambda, \gamma}^{\prime}(x)>0$ for every $\lambda>0$ and $x \geq 1$. This completes the proof of (3.23) since $\sigma_{\lambda, \gamma}(\ell)=g_{\lambda, \gamma}(\ell)$ for every $\ell \geq 1$.

We will consider the subspaces of Sobolev spaces which are invariant under the rotations

$$
\mathcal{R}(2):=\left\{\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right),\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)\right\} .
$$

In the following, for $j \in \mathbb{N}$, we define

$$
H_{p, \mathcal{R}}^{j}:=\left\{v \in H_{l o c}^{j}\left(\mathbb{R}^{2}\right): v \text { is } 2 \pi \mathbb{Z}^{2} \text {-periodic and } v(M t)=v(t), M \in \mathcal{R}(2), t \in \mathbb{R}^{2}\right\}
$$

and we put $L_{p, \mathcal{R}}^{2}:=H_{p, \mathcal{R}}^{0}$. Note that $L_{p, \mathcal{R}}^{2}$ is a Hilbert space with scalar product

$$
(u, v) \mapsto\langle u, v\rangle_{L^{2}\left([-\pi, \pi]^{2}\right)}:=\int_{[-\pi, \pi]^{2}} u(t) v(t) d t \quad \text { for } u, v \in L_{p, \mathcal{R}}^{2}
$$

Then the functions $e_{k}$ in (3.12) form an orthonormal basis for $L_{p, \mathcal{R}}^{2}$. Moreover, $H_{p, \mathcal{R}}^{j} \subset L_{p, \mathcal{R}}^{2}$ is characterized as the subspace of all functions $v \in L_{p, \mathcal{R}}^{2}$ such that

$$
\sum_{k \in \mathbb{N}^{2}}\left(1+|k|^{2}\right)^{j}\left\langle v, e_{k}\right\rangle_{L^{2}\left([-\pi, \pi]^{2}\right)}^{2}<\infty
$$

Therefore, $H_{p, \mathcal{R}}^{j}$ is also a Hilbert space with scalar product

$$
(u, v) \mapsto\langle u, v\rangle_{H_{p, \mathcal{R}}^{j}}:=\sum_{k \in \mathbb{N}^{2}}\left(1+|k|^{2}\right)^{j}\left\langle u, e_{k}\right\rangle_{L^{2}\left([-\pi, \pi]^{2}\right)}\left\langle v, e_{k}\right\rangle_{L^{2}\left([-\pi, \pi]^{2}\right)} \quad \text { for } u, v \in H_{p, \mathcal{R}}^{j} .
$$

We consider the eigenscapces corresponding to $\sigma_{\lambda, \gamma}(\ell)$, given by

$$
\begin{equation*}
V_{\ell}:=\operatorname{span}\left\{e_{k}:|k|=\ell\right\} \tag{3.24}
\end{equation*}
$$

and $P_{\ell}: L_{p, \mathcal{R}}^{2} \rightarrow V_{\ell}$ be the $L^{2}$-orthogonal projection on $V_{\ell}$. We will denote by $V_{\ell}^{\perp}$ to $L^{2}$ complement of $V_{\ell}$, which is given by

$$
V_{\ell}^{\perp}:=\left\{v \in V_{\ell}: P_{\ell} v=0\right\} .
$$

Obviously, the direct-sum $\bigoplus_{\ell \in \mathbb{N}} V_{\ell}$ provides a complete decomposition of $L_{p, \mathcal{R}}^{2}$. More precisely for any $v \in L_{p, \mathcal{R}}^{2}$, we can write it as

$$
v=\sum_{\ell \in \mathbb{N}} P_{\ell} v .
$$

Moreover, by (3.13), we have

$$
\begin{equation*}
L_{\lambda}(v)=\sigma_{\lambda, \gamma}(\ell) v \quad \text { for all } v \in V_{\ell} \tag{3.25}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
V_{1} \cap C_{p, \mathcal{R}}^{0, \alpha}=\operatorname{span}\{\bar{v}\}=\mathbb{R} \bar{v} \quad \text { with } \quad \bar{v}\left(t_{1}, t_{2}\right)=\cos \left(t_{1}\right)+\cos \left(t_{2}\right) . \tag{3.26}
\end{equation*}
$$

We also consider the nonlinear operator $\mathcal{F}$ defined in (3.1), and we note that $\mathcal{F}$ maps $\mathcal{O} \cap C_{p, \mathcal{R}}^{2, \alpha}$ into $C_{p, \mathcal{R}}^{0, \alpha}$. Consider the open set

$$
\mathcal{U}:=\left\{(\lambda, u) \in \mathbb{R} \times C_{p, \mathcal{R}}^{2, \alpha}: \lambda>0, u>-\lambda\right\} \subset \mathbb{R} \times C_{p, \mathcal{R}}^{2, \alpha}
$$

The proof of our main theorem will be completed by applying the Crandall-Rabinowitz bifurcation theorem to the smooth nonlinear operator

$$
\begin{equation*}
\mathcal{G}: \mathcal{U} \subset \mathbb{R} \times C_{p, \mathcal{R}}^{2, \alpha} \rightarrow C_{p, \mathcal{R}}^{0, \alpha}, \quad \mathcal{G}(\lambda, u)=\mathcal{F}(\lambda+u)-\mathcal{F}(\lambda) . \tag{3.27}
\end{equation*}
$$

We have the following
Proposition 3.4. Let $\gamma$ satisfy (3.20). Then there exists a unique $\lambda_{*}=\lambda_{*}(\gamma, \kappa)>0$ such that the linear operator $L_{\lambda_{*}}=D_{u} \mathcal{G}\left(\lambda_{*}, 0\right)$ has the following properties.
(i) The kernel $N\left(L_{\lambda_{*}}\right)$ of $L_{\lambda_{*}}$ is given by

$$
\begin{equation*}
N\left(L_{\lambda_{*}}\right)=V_{1}=\mathbb{R} \bar{v}, \quad \bar{v}\left(t_{1}, t_{2}\right)=\cos \left(t_{1}\right)+\cos \left(t_{2}\right) . \tag{3.28}
\end{equation*}
$$

(ii) The range $R\left(L_{\lambda_{*}}\right)$ of $L_{\lambda_{*}}$ is given by

$$
R\left(L_{\lambda_{*}}\right)=C_{p, \mathcal{R}}^{0, \alpha} \cap V_{1}^{\perp} .
$$

(iii) Finally,

$$
\begin{equation*}
\left.\partial_{\lambda}\right|_{\lambda=\lambda_{*}} L_{\lambda} \bar{v}=\left.\partial_{\lambda}\right|_{\lambda=\lambda_{*}} \sigma_{\lambda, \gamma}(1) \bar{v} \notin R\left(L_{\lambda_{*}}\right) . \tag{3.29}
\end{equation*}
$$

Proof. For $\gamma$ satisfying (3.20), by Lemma [3.3, there exists a unique $\lambda_{*}=\lambda_{*}(\gamma, \kappa)>0$ such that $\sigma_{\lambda_{*}, \gamma}(1)=0$ and $\sigma_{\lambda_{*}, \gamma}(\ell)>0$ for all $\ell \in \mathbb{N} \backslash\{1\}$. Hence by (3.25), we have that $L_{\lambda_{*}} v=0$ if and only if $v \in V_{1}$. This with (3.26) prove (i).

We now prove (ii). We pick $f \in C_{p, \mathcal{R}}^{0, \alpha} \cap V_{1}^{\perp} \subset L_{p, \mathcal{R}}^{2} \cap V_{1}^{\perp}$. Then using Fourier decomposition, (3.25), (3.16) and (3.23), we have a unique $w \in H_{p, \mathcal{R}}^{2} \cap V_{1}^{\perp}$, such that

$$
L_{\lambda_{*}} w=f \quad \text { in } \mathbb{R}^{2}
$$

The proof of (ii) finishes once we show that $w \in C_{l o c}^{2, \alpha}\left(\mathbb{R}^{2}\right)$.
By Morrey's embedding, $w \in H_{l o c}^{2}\left(\mathbb{R}^{2}\right) \subset C_{l o c}^{0, \alpha}\left(\mathbb{R}^{2}\right)$, for every $\alpha \in(0,1)$. We then have

$$
\begin{equation*}
-\Delta w=\gamma w(t) \int_{\mathbb{R}^{2}} G_{\kappa}(|r|) d r-\gamma w(t) \int_{\mathbb{R}^{2}} G_{\kappa}\left(\left(|r|^{2}+4 \lambda^{2}\right)^{1 / 2}\right) d r-F(t), \tag{3.30}
\end{equation*}
$$

where

$$
F(t):=\int_{\mathbb{R}^{2}} w(t-r) G_{\kappa}(|r|) d r+\int_{\mathbb{R}^{2}} w(t-r) G_{\kappa}\left(\left(|r|^{2}+4 \lambda_{*}^{2}\right)^{1 / 2}\right) d r .
$$

Then for every $t \in \mathbb{R}^{2}$

$$
\begin{aligned}
|F(t)| & \leq\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)}\left(\int_{\mathbb{R}^{2}} G_{\kappa}(|r|) d r+\int_{\mathbb{R}^{2}} G_{\kappa}\left(\left(|r|^{2}+4 \lambda_{*}^{2}\right)^{1 / 2}\right) d r\right) \\
& \leq C\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)}
\end{aligned}
$$

It is also easy to see that for $t, \bar{t} \in \mathbb{R}^{2}$

$$
|F(t)-F(\bar{t})| \leq C\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)}|t-\bar{t}|^{\alpha} .
$$

We then obtain that $F \in C^{0, \alpha}\left(\mathbb{R}^{2}\right)$, for every $\alpha \in(0,1)$. Now by (3.30) and standard elliptic regularity theory, $w \in C_{\text {loc }}^{2, \alpha}\left(\mathbb{R}^{2}\right)$ for every $\alpha \in(0,1)$. Hence (ii) follows. It remains to prove (3.29), which follows from (3.15) and the identity

$$
\begin{equation*}
\left.\partial_{\lambda}\right|_{\lambda=\lambda_{*}} L_{\lambda} \bar{v}=\left.\partial_{\lambda}\right|_{\lambda=\lambda_{*}} \sigma_{\lambda, \gamma}(1) \bar{v} . \tag{3.31}
\end{equation*}
$$

3.2. Proof of Theorem 1.1. Thanks to (3.4), we can apply the Crandall-Rabinowitz Theorem (see [13, Theorem 1.7]), to have

Proposition 3.5. There exists $s_{0}>0$, only depending on $\kappa$ and $\gamma$, and a smooth curve

$$
\begin{equation*}
\left(-s_{0}, s_{0}\right) \rightarrow(0, \infty) \times C_{p, \mathcal{R}}^{2, \alpha}, \quad s \mapsto\left(\lambda_{s}, u_{s}\right) \tag{3.32}
\end{equation*}
$$

such that
(i) For every $s \in\left(-s_{0}, s_{0}\right)$

$$
\begin{equation*}
\mathcal{G}\left(\lambda_{s}, u_{s}\right)=\mathcal{F}\left(\lambda_{s}+u_{s}\right)-\mathcal{F}\left(\lambda_{s}\right)=0 \tag{3.33}
\end{equation*}
$$

(ii) $\lambda_{0}=\lambda_{*}$.
(iii) $u_{s}=s\left(\bar{v}+v_{s}\right)$ for all $s \in\left(-s_{0}, s_{0}\right)$, with a smooth curve

$$
\left(-s_{0}, s_{0}\right) \rightarrow C_{p, \mathcal{R}}^{2, \alpha}, \quad s \mapsto v_{s}
$$

satisfying $v_{0}=0$ and $\int_{[-\pi, \pi]^{2}} v_{s}(t) \bar{v}(t) d t=0$.

## 4. Periodic lamellae and proof of Theorem 1.2

For $\varepsilon \in \mathbb{R} \backslash\{0\}$, with $|\varepsilon|$ small, we define

$$
\mathcal{L}_{\psi}^{\varepsilon}=\Omega_{\psi}+\frac{1}{|\varepsilon|} \mathbb{Z} e_{3}=\bigcup_{p \in \mathbb{Z}}\left(\Omega_{\psi}+\left(0,0, \frac{p}{|\varepsilon|}\right)\right),
$$

where as above $\psi \in C_{p, \mathcal{R}}^{2, \alpha} \cap \mathcal{O}$. For $x \in\{-1,1\}$, the mean curvature takes the form

$$
\begin{equation*}
H_{\partial \mathcal{L}_{\psi}^{\varepsilon}}((t, \psi(t) x))=-\operatorname{div} \frac{\nabla \psi(t)}{\sqrt{1+|\nabla \psi(t)|^{2}}} \tag{4.1}
\end{equation*}
$$

and by a change of variable, the nonlocal part takes the form

$$
\begin{aligned}
& \int_{\mathcal{L}_{\psi}^{\varepsilon}} G_{\kappa}(|(t, \psi(t) x)-\xi|) d \xi=\int_{\mathbb{R}^{2}} \int_{-1}^{1} \psi(s) G_{\kappa}(|(t, \psi(t) x)-(s, \psi(s) y)|) d y d s \\
& \quad+\sum_{p \in \mathbb{Z}_{*}} \int_{\mathbb{R}^{2}} \int_{-1}^{1} \psi(s) G_{\kappa}\left(\left|(t, \psi(t) x)-(s, \psi(s) y)-\left(0,0,|\varepsilon|^{-1} p\right)\right|\right) d y d s
\end{aligned}
$$

Now using the fact that $\mathbb{Z}_{*}=-\mathbb{Z}_{*}$ and the change of variable $y$ to $-y$, we see that

$$
\int_{\mathcal{L}_{\psi}^{\varepsilon}} G_{\kappa}(|(t,-\psi(t))-\xi|) d \xi=\int_{\mathcal{L}_{\psi}^{\varepsilon}} G_{\kappa}(|(t, \psi(t))-\xi|) d \xi
$$

This implies, in particular, that for every $x \in\{-1,1\}, t \in \mathbb{R}^{2}$ and $|\varepsilon|<\frac{1}{2 \lambda}$,

$$
H_{\partial \mathcal{L}_{\psi}^{\varepsilon}}((t, \lambda x))=\gamma \lambda \sum_{p \in \mathbb{Z}} \int_{\mathbb{R}^{2}} \int_{-1}^{1} G_{\kappa}\left(\left(|r|^{2}+\left(\lambda(1-y)-|\varepsilon|^{-1} p\right)^{2}\right)^{1 / 2}\right) d y d r
$$

Namely $\mathcal{L}_{\lambda}^{\varepsilon}$ is an equilibrium pattern for every $\lambda>0$, with $|\varepsilon|<\frac{1}{2 \lambda}$.
Recalling (3.1), we get

$$
\begin{align*}
& \mathcal{H}_{\mathcal{L}_{\psi}^{\varepsilon}}((t, \psi(t) x))=H_{\partial \mathcal{L}_{\psi}^{\varepsilon}}((t, \psi(t)))+\int_{\mathcal{L}_{\psi}^{\varepsilon}} G_{\kappa}(|(t, \psi(t))-\xi|) d \xi \\
& =\mathcal{F}(\psi)(t)+\sum_{p \in \mathbb{Z}_{*}} \int_{\mathbb{R}^{2}} \int_{-1}^{1} \psi(s) G_{\kappa}\left(\left|(t, \psi(t))-(s, \psi(s) y)-\left(0,0,|\varepsilon|^{-1} p\right)\right|\right) d y d s \tag{4.2}
\end{align*}
$$

where $\mathcal{F}: \mathcal{O} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$ is as in the previous section, is given by

$$
\begin{equation*}
\mathcal{F}(\psi)(t):=-\operatorname{div} \frac{\nabla \psi(t)}{\sqrt{1+|\nabla \psi(t)|^{2}}}+\gamma \int_{\mathbb{R}^{2}} \int_{-1}^{1} \psi(s) G_{\kappa}(|(t, \psi(t))-(s, \psi(s) y)|) d y d s \tag{4.3}
\end{equation*}
$$

We now define $\widehat{\mathcal{F}}: \mathbb{R} \backslash\{0\} \times \mathcal{O} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$ by

$$
\begin{equation*}
\widehat{\mathcal{F}}(\varepsilon, \psi)(t):=\sum_{p \in \mathbb{Z}_{*}} \int_{\mathbb{R}^{2}} \int_{-1}^{1} \psi(s) G_{\kappa}\left(\left|(t, \psi(t))-(s, \psi(s) y)-\left(0,0,|\varepsilon|^{-1} p\right)\right|\right) d y d s \tag{4.4}
\end{equation*}
$$

By similar arguments as in the proof of Lemma 3.1, we can deduce that

$$
\begin{aligned}
& D \widehat{\mathcal{F}}(\varepsilon, \psi)[w](t) \\
& =\sum_{p \in \mathbb{Z}_{*}} \int_{\mathbb{R}^{2}}(w(t-r)-w(t)) G_{\kappa}\left(\left(|r|^{2}+\left(\psi(t)-\psi(t-r)+|\varepsilon|^{-1} p\right)^{2}\right)^{1 / 2}\right) d r \\
& +\sum_{p \in \mathbb{Z}_{*}} \int_{\mathbb{R}^{2}}(w(t-r)+w(t)) G_{\kappa}\left(\left(|r|^{2}+\left(\psi(t)+\psi(t-r)+|\varepsilon|^{-1} p\right)^{2}\right)^{1 / 2}\right) d r .
\end{aligned}
$$

Let $\mathcal{U}_{\lambda}$ be a neighborhood of $\lambda>0$ in $\mathcal{O}$. Then there exists $\bar{\varepsilon}, c>0$, only depending on $\mathcal{U}_{\lambda}$ and $\kappa$, such that for $|\varepsilon| \in(0, \bar{\varepsilon})$ and $\psi \in \mathcal{U}_{\lambda}$, we have

$$
G_{\kappa}\left(\left(|r|^{2}+\left(\varphi(t)-\varphi(t-r)+|\varepsilon|^{-1} p\right)^{2}\right)^{1 / 2}\right) \leq G_{\kappa}\left(\left(|r|^{2}+c^{2}|\varepsilon|^{-2} p^{2}\right)^{1 / 2}\right)
$$

Therefore by a change of variable and (2.8), if $|\varepsilon| \in(0, \bar{\varepsilon})$ and $\psi \in \mathcal{U}_{\lambda}$, then

$$
\begin{aligned}
\|D \widehat{\mathcal{F}}(\varepsilon, \varphi)[w]\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)} & \leq 4\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)} \sum_{p \in \mathbb{Z}_{*}} \int_{\mathbb{R}^{2}} G_{\kappa}\left(\left(|r|^{2}+c^{2}|\varepsilon|^{-2} p^{2}\right)^{1 / 2}\right) d r \\
& \leq 4 c\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)} \sum_{p \in \mathbb{Z}_{*}}|\varepsilon|^{-1}|p| \int_{\mathbb{R}^{2}} G_{\kappa c|p|}^{|\varepsilon|}\left(\left(|t|^{2}+1\right)^{1 / 2}\right) d t \\
& \leq 4 c\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)}|\varepsilon| \sum_{p \in \mathbb{Z}_{*}}|p|^{-1} \exp \left(-\kappa c|p \| \varepsilon|^{-1}\right) \\
& \leq 4 c\|w\|_{C^{0, \alpha}\left(\mathbb{R}^{2}\right)|\varepsilon|} \exp \left(-\frac{\kappa}{2} c|\varepsilon|^{-1}\right) .
\end{aligned}
$$

Up to decreasing $\bar{\varepsilon}$ and choosing a smaller neighborhood of $\lambda$, we can easily see that $\widehat{\mathcal{F}}$ extends to a smooth function $(-\bar{\varepsilon}, \bar{\varepsilon}) \times \mathcal{U}_{\lambda} \rightarrow C^{0, \alpha}\left(\mathbb{R}^{2}\right)$ with $D^{k} \widehat{\mathcal{F}}(0, \psi)=0$ for all $\psi \in \mathcal{U}_{\lambda}$ and $k \in \mathbb{N}$. In view of (4.3), (4.4) and (4.2), we define

$$
\mathcal{N}_{\lambda}:(-\bar{\varepsilon}, \bar{\varepsilon}) \times \mathcal{U}_{\lambda} \cap C_{p, \mathcal{R}}^{2, \alpha} \rightarrow C_{p, \mathcal{R}}^{0, \alpha}
$$

by

$$
\begin{equation*}
(\varepsilon, \psi) \mapsto \mathcal{N}_{\lambda}(\varepsilon, \psi)(t):=\mathcal{F}(\psi)(t)+\gamma \widehat{\mathcal{F}}(\varepsilon, \psi)(t)=\mathcal{H}_{\mathcal{L}_{\lambda}^{\varepsilon}}((t, \psi(t))) \tag{4.5}
\end{equation*}
$$

In the following, we let $\gamma$ satisfy (3.20),
Instead of applying the Crandall-Rabinowitz Theorem [13, Theorem 1.7], we will use the same argument of its proof, mainly based on the Implicit Function Theorem. More precisely, we apply the latter theorem to solve the equation $\mathcal{M}=0$, with the function

$$
\mathcal{M}:(-\bar{\varepsilon}, \bar{\varepsilon}) \times\left(-\lambda_{*} / 8, \lambda_{*} / 8\right) \times(1 / 2,3 / 2) \times C_{p, \mathcal{R}}^{2, \alpha} \rightarrow C_{p, \mathcal{R}}^{0, \alpha}
$$

given by

$$
\begin{align*}
\mathcal{M}(\varepsilon, s, \delta, v)(t) & :=\frac{\mathcal{N}_{\lambda_{*}}\left(\varepsilon, \delta \lambda_{*}+s(\bar{v}+v)\right)(t)-\mathcal{N}_{\lambda_{*}}\left(\varepsilon, \delta \lambda_{*}\right)}{s}  \tag{4.6}\\
& \left.=\int_{0}^{1} D_{\psi} \mathcal{N}_{\lambda_{*}}\left(\varepsilon, \delta \lambda_{*}+\varrho s(\bar{v}+v)\right)\right)[\bar{v}+v](t) d \varrho .
\end{align*}
$$

It is clear that $\mathcal{M}$ is smooth in $(-\bar{\varepsilon}, \bar{\varepsilon}) \times\left(-\lambda_{*} / 8, \lambda_{*} / 8\right) \times(1 / 2,3 / 2) \times B_{C_{p, \mathcal{R}}^{2, \alpha}}(0,1 / 8)$. Now from Proposition $3.4($ i $)$, we have that $\mathcal{M}(0,0,1,0)=L_{\lambda_{*}}(\bar{v})=0$. We also have that

$$
D_{v} \mathcal{M}(0,0,1,0)=L_{\lambda_{*}}: C_{p, \mathcal{R}}^{2, \alpha} \cap V_{1}^{\perp} \rightarrow C_{p, \mathcal{R}}^{0, \alpha} \cap V_{1}^{\perp}
$$

is an isomorphism by Proposition 3.4(ii). Moreover, by Proposition 3.4(iii),

$$
D_{\delta} \mathcal{M}(0,0,1,0): \mathbb{R} \rightarrow V_{1}
$$

is an isomorphism. From these, we can now apply the implicit function theorem to get $s_{0}, \varepsilon_{0}>0$ and a smooth curve

$$
\left(-\varepsilon_{0}, \varepsilon_{0}\right) \times\left(-s_{0}, s_{0}\right) \rightarrow(1 / 2,3 / 2) \times C_{p, \mathcal{R}}^{2, \alpha} \cap V_{1}^{\perp}, \quad(\varepsilon, s) \mapsto\left(\delta_{\varepsilon, s}, v_{\varepsilon, s}\right)
$$

with $\delta_{0,0}=1$ and $v_{0,0}=0$ such that

$$
\mathcal{M}\left(\varepsilon, s, \delta_{\varepsilon, s}, v_{\varepsilon, s}\right)=0 \quad \text { on } C_{p, \mathcal{R}}^{0, \alpha} .
$$

From this together with (4.6) and (4.5), we conclude that, for every $t \in \mathbb{R}^{2}$,

$$
\begin{equation*}
\mathcal{F}\left(\lambda_{\varepsilon, s}+s\left(\bar{v}+v_{\varepsilon, s}\right)\right)(t)+\gamma \widehat{\mathcal{F}}\left(\varepsilon, \lambda_{\varepsilon, s}+s\left(\bar{v}+v_{\varepsilon, s}\right)\right)(t)=\mathcal{F}\left(\lambda_{\varepsilon, s}\right)+\gamma \widehat{\mathcal{F}}\left(\varepsilon, \lambda_{\varepsilon, s}\right) \tag{4.7}
\end{equation*}
$$

where we have set $\lambda_{\varepsilon, s}:=\delta_{\varepsilon, s} \lambda_{*}$. This ends the first part of the proof of Theorem [1.2, It follows from the uniqueness, inherent to the construction, that $\lambda_{0, s}=\lambda_{s}$ and $v_{0, s}=v_{s}$, for all $s \in\left(-s_{0}, s_{0}\right)$, where $\lambda_{s}$ and $v_{s}$ are given by Theorem 1.1.
4.0.1. Proof of Theorem 1.2 (completed). The existence of the smooth curve of equilibrium patterns $\mathcal{L}_{\psi_{\varepsilon, s}}^{\varepsilon}$ follows from the identity (4.7). Obviously $\psi_{0, s}=\varphi_{s}$ as a consequence of the uniqueness form the Crandall-Rabinowitz theorem.

## 5. Periodic lattice of near-cylinders and near-spheres

5.1. Some preliminary and notations. Let $M \in \mathbb{N}$ with $1 \leq M \leq N$. Let $\left\{\mathbf{a}_{1} ; \ldots ; \mathbf{a}_{M}\right\}$ be a basis of $\mathbb{R}^{M}$. We recall from the first section the $M$-dimensional Bravais lattice in $\mathbb{R}^{N}$ :

$$
\begin{equation*}
\mathscr{L}^{M}=\left\{\sum_{i=1}^{N} k_{i} \mathbf{a}_{i}: k=\left(k_{1}, \ldots, k_{M}\right) \in \mathbb{Z}^{N}\right\} \subset \mathbb{R}^{N} . \tag{5.1}
\end{equation*}
$$

Throughout this paper, we will put $\mathscr{L}_{*}^{M}:=\mathscr{L}^{M} \backslash\{0\}$ and $\mathbb{Z}_{*}^{M}:=\mathbb{Z}^{M} \backslash\{0\}$.
We let $S^{N-1}$ be the unit sphere on $\mathbb{R}^{N}$. We denote by $\mathcal{S}_{k}$ the finite dimensional subspace of spherical harmonics of degree $k$ on $S^{N-1}$, and by $\left(Y_{k}^{i}\right)_{k \in \mathbb{N}, i=1, \ldots, n_{k}}$ an orthonormal basis of $\mathcal{S}_{k}$. It is well known that $n_{0}=1$ with $Y_{0}^{1}(\theta)=\frac{1}{\sqrt{\left|S^{N-1}\right|}}$ and $n_{1}=N$ with $Y_{1}^{i}(\theta)=\frac{\theta_{i}}{\sqrt{\left|B^{N}\right|}}$, for each $i=1, \ldots, N$. We recall that $-\Delta_{S^{N-1}} Y=k(k+N-2) Y$, for every $Y \in \mathcal{S}_{k}$. Moreover for a function $f:(0,2) \rightarrow \mathbb{R}_{+}$, with $\int_{0}^{2} f(t)(t(2-t))^{\frac{N-3}{2}} d t<\infty$, by the Funck-Hecke formula, see [16, page 247], for $k \geq 1$ and $\theta \in S^{N-1}$, we have

$$
\int_{S^{N-1}}(Y(\theta)-Y(\sigma)) f(|\theta-\sigma|) d \sigma=\mu_{k} Y(\theta), \quad \text { for every } Y \in \mathcal{S}_{k}
$$

for some real number $\mu_{k} \geq 0$. From this, we deduce that

$$
\mu_{k}=\frac{1}{2} \int_{S^{N-1}} \int_{S^{N-1}}\left(Y_{k}^{i}(\theta)-Y_{k}^{i}(\sigma)\right)^{2} f(|\theta-\sigma|) d \sigma d \theta \quad \text { for every } i=1, \ldots, n_{k}
$$

For a function $w \in L^{2}\left(S^{N-1}\right)$, we can decompose it as

$$
w=\sum_{k \in \mathbb{N}} \sum_{i=1}^{n_{k}} w_{k}^{i} Y_{k}^{i}, \quad \text { with } \quad w_{k}^{i}=\int_{S^{N-1}} w(\sigma) Y_{k}^{i}(\sigma) d \sigma
$$

For $j \in \mathbb{N}$, we define the Sobolev spaces

$$
\begin{equation*}
H^{j}\left(S^{N-1}\right)=\left\{w \in L^{2}\left(S^{N-1}\right): \sum_{k \in \mathbb{N}} \sum_{i=1}^{n_{k}}\left(1+|k|^{2}\right)^{j}\left(w_{k}^{i}\right)^{2}<\infty\right\} \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\mathbf{e}}^{j}\left(S^{N-1}\right)=\left\{w \in H^{j}\left(S^{N-1}\right): w \text { is even }\right\} \tag{5.3}
\end{equation*}
$$

We define also Hölder spaces

$$
C_{\mathbf{e}}^{j, \alpha}\left(S^{N-1}\right):=\left\{\phi \in C^{j, \alpha}\left(S^{N-1}\right), \quad \phi(-\theta)=\phi(\theta)\right\}
$$

endowed with usual Hölder norm of $C^{j, \alpha}\left(S^{N-1}\right)$.
Following [5], for a fixed $e \in S^{N-1}$ and a Lipschitz continuous map of rotations $S^{N-1} \mapsto$ $S O(N), \theta \mapsto R_{\theta}$ with the property that

$$
\begin{equation*}
S_{e}:=\left\{\theta \in S^{N-1}: \theta \cdot e \geq 0\right\} \subset\left\{\theta \in S^{N-1}: R_{\theta} e=\theta\right\} . \tag{5.4}
\end{equation*}
$$

Example 5.1 ( 5 ). Consider the map $\theta \mapsto R_{\theta}$ defined as follows. For $\theta \in S^{N-1}$ with $\theta \cdot e \geq 0$, we let $R_{\theta}$ be the rotation of the angle $\arccos \theta \cdot e$ which maps e to $\theta$ and keeps all vectors perpendicular to $\theta$ and e fixed. We then extend the $\operatorname{map} \theta \mapsto R_{\theta}$ to all of $S^{N-1}$ as an even map with respect to reflection at the hyperplane $\left\{\theta \in \mathbb{R}^{N}: \theta \cdot e=0\right\}$.

It is easy to see that

$$
\begin{equation*}
\left|R_{\theta} \sigma-\theta\right|=|e-\sigma| \quad \text { for all } \theta \in S_{e} \text { and } \sigma \in S^{N-1} . \tag{5.5}
\end{equation*}
$$

Moreover, the Lipschitz property of the map $\theta \mapsto R_{\theta}$ implies that there is a constant $C>0$ with

$$
\begin{equation*}
\left\|R_{\theta_{1}}-R_{\theta_{2}}\right\| \leq C\left|\theta_{1}-\theta_{2}\right| \quad \text { for all } \theta_{1}, \theta_{2} \in S^{N-1} \tag{5.6}
\end{equation*}
$$

where, here and in the following, $\|\cdot\|$ denotes the usual operator norm. We put

$$
\mathcal{D}:=\left\{\phi \in C^{2, \alpha}\left(S^{N-1}\right): \phi>0\right\} .
$$

For every $\varphi \in \mathcal{D}$, we let

$$
\begin{equation*}
F_{\varphi}: \mathbb{R}_{+} \times S^{N-1} \rightarrow \mathbb{R}^{N}, \quad F_{\varphi}(r, \sigma):=r \sigma \varphi(\sigma) . \tag{5.7}
\end{equation*}
$$

The map $F_{\varphi}$ defines a parameterization of the set

$$
\begin{equation*}
B_{\varphi}^{N}:=F_{\varphi}\left([0,1) \times S^{N-1}\right) . \tag{5.8}
\end{equation*}
$$

We will denote $S_{\varphi}^{N-1}:=\partial B_{\varphi}^{N}=F\left(1, S^{N-1}\right)$. See e.g. [5], the unit outer normal of $B_{\varphi}^{N}$ at a point $F_{\varphi}(1, \sigma), \sigma \in S^{N-1}$, is given by

$$
\begin{equation*}
\nu_{\varphi}\left(F_{\varphi}(1, \sigma)\right)=\frac{\phi(\sigma) \sigma-\nabla \varphi(\sigma)}{\sqrt{\phi^{2}(\sigma)+|\nabla \varphi(\sigma)|^{2}}} \tag{5.9}
\end{equation*}
$$

and for every continuous function $f$ on $\mathbb{R}^{N}$, we have

$$
\begin{equation*}
\int_{S_{\varphi}^{N-1}} f(y) d V(y)=\int_{S^{N-1}}\left[f \circ F_{\varphi}(1, \cdot)\right](\sigma) \Gamma_{\varphi}(\sigma) d \sigma \quad \text { with } \quad \Gamma_{\varphi}=\varphi^{N-2} \sqrt{\varphi^{2}+|\nabla \varphi|^{2}} \tag{5.10}
\end{equation*}
$$

Moreover letting

$$
\mathcal{A}(\varphi)=\int_{S^{N-1}} \Gamma_{\varphi}(\sigma) d \sigma
$$

then differentiating $\mathcal{A}$, we get

$$
D \mathcal{A}(\varphi)[v]=\int_{S^{N-1}} \mathcal{H}(\varphi)(\sigma) v(\sigma) d \sigma
$$

where $\mathcal{H}(\varphi)(\sigma)=H_{S_{\varphi}^{N-1}}(\varphi(\sigma) \sigma)$ is the mean curvature operator of $\partial B_{\varphi}^{N}=S_{\phi}^{N-1}$ and is given by

$$
\begin{equation*}
\mathcal{H}(\varphi)=-\operatorname{div} \frac{\varphi^{N-2} \nabla \varphi}{\sqrt{\varphi^{2}+|\nabla \varphi|^{2}}}+(N-2) \frac{\varphi^{N-3}}{\sqrt{\varphi^{2}+|\nabla \varphi|^{2}}}+\frac{\varphi^{N-1}}{\left(\varphi^{2}+|\nabla \varphi|^{2}\right)^{-1 / 2}} \tag{5.11}
\end{equation*}
$$

5.1.1. From $2 D$ near-balls to $3 D$ near-cylinders. We define the perturbed cylinder

$$
\begin{equation*}
C_{\varphi}:=\left\{(r, \theta, t) \in \mathbb{R}_{+} \times S^{1} \times \mathbb{R}: r<\varphi(\theta)\right\}=B_{\varphi}^{2} \times \mathbb{R} \tag{5.12}
\end{equation*}
$$

We then consider the lattice of cylinders

$$
\mathcal{C}_{\varphi}^{\varepsilon}:=\bigcup_{p \in \mathscr{L}^{M}}\left(C_{\varphi}+\frac{p}{|\varepsilon|}\right)=\bigcup_{p \in \mathscr{L}^{M}}\left(C_{\varphi}+\left(\frac{p}{|\varepsilon|}, 0\right)\right)
$$

where $\varphi: S \rightarrow(0, \infty)$ is an even and $\mathscr{L}^{M}$ is the $M$-dimensional Bravais lattice, with $M \leq 2$. We note that the integral of the Yukawa interaction $G_{\kappa}$ in the $t$ direction gives $2 K_{0}$ which is the 2 -dimensional screened Coulomb potential. Our problem is therefore reduced to find spherical lattices of equilibrium pattern in $\mathbb{R}^{2}$. Indeed, for a point $Z_{0}=(\varphi(\theta) \theta, t) \in \partial C_{\varphi}$, the Yukawa interaction takes the form

$$
\begin{aligned}
\int_{\mathcal{C}_{\varphi}^{\varepsilon}} G_{\kappa}\left(\left|Z_{0}-Z\right|\right) d Z= & \sum_{p \in \mathscr{L}^{M}} \int_{C_{\varphi}} G_{\kappa}\left(\left|Z_{0}-Z-\frac{p}{|\varepsilon|}\right|\right) d Z \\
= & \int_{\mathcal{C}_{\varphi}^{\varepsilon}} G_{\kappa}\left(\left|Z_{0}-Z\right|\right) d Z+\sum_{p \in \mathscr{L}^{M}} \int_{C_{\varphi}} G_{\kappa}\left(\left|Z_{0}-Z-\frac{p}{|\varepsilon|}\right|\right) d Z \\
= & \int_{B_{\varphi}^{2}} \int_{\mathbb{R}} G_{\kappa}\left(\left|(\varphi(\theta) \theta, t)-(y, s)-\frac{p}{|\varepsilon|}\right|\right) d s d y \\
& +\sum_{p \in \mathscr{L}^{M}} \int_{B_{\varphi}^{2}} \int_{\mathbb{R}} G_{\kappa}(|(\varphi(\theta) \theta, t)-(y, s)|) d s d y
\end{aligned}
$$

By (2.4),

$$
\int_{\mathbb{R}} G_{\kappa}\left(\left(r^{2}+\delta^{2}\right)^{1 / 2}\right) d r=2 K_{0}(\kappa \delta)
$$

where as before $K_{0}$ is the modified Bessel function of the second kind of order 0 . We then deduce that

$$
\begin{aligned}
\int_{\mathcal{C}_{\varphi}} G_{\kappa}\left(\left|Z_{0}-Z\right|\right) d Z & =2 \int_{B_{\varphi}^{2}} K_{0}(\kappa|(\varphi(\theta) \theta)-y|) d y \\
& +2 \sum_{p \in \mathscr{L}_{*}^{M}} \int_{B_{\varphi}^{2}} K_{0}\left(\kappa\left|(\varphi(\theta) \theta)-y-\frac{p}{|\varepsilon|}\right|\right) d y .
\end{aligned}
$$

Since the mean curvature $H_{C_{\varphi}}$ does not depend on $t$, from the computations above, $\mathcal{H}_{\mathcal{C}}^{\varepsilon}$ does not depend on $t$. We are therefore reduce to find near-ball lattices $B_{\varphi}^{2}+\frac{1}{|\varepsilon|} \mathscr{L}^{M}$ in $\mathbb{R}^{2}$ with Yukawa potential (or screened Coulomb potential) $2 K_{0}$. We will therefore treat simultaneously the 2D and 3D cases. For that, we define

$$
G_{\kappa, N}(r):=\left\{\begin{array}{cc}
2 K_{0}(\kappa r) & \text { for } N=2  \tag{5.13}\\
G_{\kappa}(r) & \text { for } N=3 .
\end{array}\right.
$$

5.2. Lattices of perturbed balls in 2D and 3D. For $|\varepsilon|$ small and $\phi \in \mathcal{D}$, we then define

$$
\mathcal{B}_{\phi}^{\varepsilon}:=\bigcup_{p \in \mathscr{L}^{M}}\left(B_{\phi}^{N}+\frac{p}{|\varepsilon|}\right)=B_{\phi}^{N}+\frac{1}{|\varepsilon|} \mathscr{L}^{M} .
$$

We will consider in this section the Yukawa potential $G_{\kappa, N}$ defined in (5.13). For $Z_{0}=$ $\theta \phi(\theta) \in \partial B_{\phi}^{N}$, we have

$$
\begin{equation*}
\mathcal{H}_{\mathcal{B}_{\phi}^{\varepsilon}}\left(Z_{0}\right)=H_{\partial B_{\phi}^{N}}\left(Z_{0}\right)+\gamma \int_{\mathcal{B}_{\phi}^{\varepsilon}} G_{\kappa, N}\left(\left|Z_{0}-Z\right|\right) d Z \tag{5.14}
\end{equation*}
$$

The Yukawa interaction takes the form

$$
\begin{aligned}
\int_{\mathcal{B}_{\phi}^{\varepsilon}} G_{\kappa, N}\left(\left|Z_{0}-Z\right|\right) d Z & =\sum_{p \in \mathscr{L}^{M}} \int_{B_{\phi}^{N}} G_{\kappa, N}\left(\left|Z_{0}-Z-\frac{p}{|\varepsilon|}\right|\right) d Z \\
& =\int_{B_{\phi}^{N}} G_{\kappa, N}\left(\left|Z_{0}-Z\right|\right) d Z+\sum_{p \in \mathscr{L}_{*}^{M}} \int_{B_{\phi}^{N}} G_{\kappa, N}\left(\left|Z_{0}-Z-\frac{p}{|\varepsilon|}\right|\right) d Z .
\end{aligned}
$$

We define the following maps

$$
\begin{align*}
& \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right), \quad \mathcal{F}_{0}(\phi)(\theta):=\int_{B_{\phi}^{N}} G_{\kappa, N}\left(\left|Z_{0}-Z\right|\right) d Z,  \tag{5.15}\\
& \mathbb{R} \backslash\{0\} \times \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right), \quad \mathcal{F}_{p}(\varepsilon, \phi)(\theta):=\int_{B_{\phi}^{N}} G_{\kappa, N}\left(\left|Z_{0}-Z-\frac{p}{|\varepsilon|}\right|\right) d Z \tag{5.16}
\end{align*}
$$

and

$$
\begin{equation*}
\mathcal{F}: \mathbb{R} \backslash\{0\} \times \mathcal{D} \rightarrow C^{0, \alpha}(S), \quad \mathcal{F}(\varepsilon, \phi)(\theta):=\sum_{p \in \mathscr{L}_{*}^{M}} \int_{B_{\phi}^{N}} G_{\kappa, N}\left(\left|Z_{0}-Z-\frac{p}{|\varepsilon|}\right|\right) d Z . \tag{5.17}
\end{equation*}
$$

We observe that for any neighborhood $\mathcal{U}$ of 1 in $\mathcal{D}$, there exist constants $C, \varepsilon_{0}>0$ (depending only on $\kappa$ and $\left.\mathscr{L}^{M}\right)$ such that for every $|\varepsilon| \in\left(0, \varepsilon_{0}\right)$ and any $p \in \mathscr{L}_{*}^{M}$

$$
\left|\mathcal{F}_{p}(\varepsilon, \phi)(\theta)\right| \leq C|p|^{-1 / 2}|\varepsilon|^{1 / 2} \exp \left(\frac{-\kappa|p|}{2|\varepsilon|}\right), \quad \text { for every } \varphi \in \mathcal{U} \text { and } \theta \in S^{N-1}
$$

and

$$
\begin{equation*}
\|\mathcal{F}(\varepsilon, \phi)\|_{C^{0, \alpha}\left(S^{N-1}\right)} \leq \exp \left(-c|\varepsilon|^{-1}\right), \quad \text { for every } \varphi \in \mathcal{U} \text { and } \theta \in S^{N-1} \tag{5.18}
\end{equation*}
$$

for some $c>0$ depending only on $\alpha, N, \kappa$ and $\mathscr{L}^{M}$.
It follows that $\mathcal{F}$ extends smoothly in a neighborhood of $(0,1)$ in $\mathbb{R} \times \mathcal{D}$. Now we define the mean curvature operator $\mathcal{H}: \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right)$ (computed in (5.11)) by

$$
\begin{equation*}
\mathcal{H}(\phi)(\theta)=H_{\partial B_{\phi}^{N}}(\phi(\theta)) . \tag{5.19}
\end{equation*}
$$

We define $\mathcal{Q}: \mathbb{R} \times \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right)$ by

$$
\begin{equation*}
\mathcal{Q}(\varepsilon, \phi):=\mathcal{H}(\phi)+\gamma \mathcal{F}_{0}(\phi)+\gamma \mathcal{F}(\varepsilon, \phi) \tag{5.20}
\end{equation*}
$$

and we note that from (5.14),

$$
\mathcal{H}_{\mathcal{B}_{\phi}^{\varepsilon}}(\theta \phi(\theta))=\mathcal{Q}(\varepsilon, \phi)(\theta) .
$$

Our aim is to apply the implicit function theorem to solve the equation

$$
\begin{equation*}
\mathcal{Q}(\varepsilon, \phi)-\mathcal{Q}(0,1)=0 \tag{5.21}
\end{equation*}
$$

Let us then study the linearization of $D_{\phi} \mathcal{Q}(0,1)$.
Lemma 5.2. The map $\mathcal{Q}: \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right)$ is smooth. Moreover for every $\lambda>0$ and $w \in C^{2, \alpha}\left(S^{N-1}\right)$, we have

$$
\begin{align*}
D_{\phi} \mathcal{Q}(0,1)[w](\theta)= & -\Delta_{S^{N-1}} w(\theta)-(N-1) w(\theta)-\gamma \int_{S^{N-1}}(w(\theta)-w(\sigma)) G_{\kappa, N}(|\theta-\sigma|) d \sigma \\
& +\gamma w(\theta) \int_{S^{N-1}}(1-\theta \cdot \sigma) G_{\kappa, N}(|\theta-\sigma|) d \sigma \tag{5.22}
\end{align*}
$$

The proof of this result is not immediate (at least for $N=2$ ). We therefore postpone its proof to Section 6 below.

Lemma 5.3. We define the linear operator

$$
L:=D_{u} \mathcal{Q}(0,1): C^{2, \alpha}\left(S^{N-1}\right) \rightarrow C^{0, \alpha}\left(S^{N-1}\right) .
$$

Then the spherical harmonics $Y_{k}^{i}$ are the eigenfunctions of $L$ and moreover

$$
L\left(Y_{k}^{i}(\theta)\right)=\sigma_{\gamma}(k) Y_{k}^{i}(\theta)
$$

with eigenvalues

$$
\begin{equation*}
\sigma_{\gamma}(k)=\lambda_{k}-\lambda_{1}-\gamma\left(\mu_{k}-\mu_{1}\right), \tag{5.23}
\end{equation*}
$$

where $\lambda_{k}=k(k+N-2)$ and, for $i=1, \ldots, n_{k}$,

$$
\mu_{k}=\frac{1}{2} \int_{S^{N-1} \times S^{N-1}}\left(Y_{k}^{i}(\theta)-Y_{k}^{i}(\sigma)\right)^{2} G_{\kappa, N}(|\theta-\sigma|) d \theta d \sigma .
$$

Proof. Using Fourier decomposition in spherical harmonics, see Section 5.1, we easily deduce that

$$
\begin{aligned}
\sigma_{\gamma}(k)= & \lambda_{k}-(N-1)+\gamma \int_{S^{N-1}}(1-\theta \cdot \sigma) G_{\kappa, N}(|\theta-\sigma|) d \sigma \\
& -\frac{\gamma}{2} \int_{S^{N-1} \times S^{N-1}}\left(Y_{k}(\theta)-Y_{k}(\sigma)\right)^{2} G_{\kappa, N}(|\theta-\sigma|) d \theta d \sigma,
\end{aligned}
$$

for every $Y_{k} \in \mathcal{S}_{k}$. We have that

$$
\begin{aligned}
\int_{S^{N-1}}(1-\theta \cdot \sigma) G_{\kappa, N}(|\theta-\sigma|) d \sigma & =\frac{1}{2} \sum_{i=1}^{N} \int_{S^{N-1}}\left(\theta_{i}-\sigma_{i}\right)^{2} G_{\kappa, N}(|\theta-\sigma|) d \sigma \\
& =\frac{\left|B^{N}\right|}{2} \sum_{i=1}^{N} \int_{S^{N-1}}\left(Y_{1}^{i}(\theta)-Y_{1}^{i}(\sigma)\right)^{2} G_{\kappa, N}(|\theta-\sigma|) d \sigma \\
& =\frac{N\left|B^{N}\right|}{2} \int_{S^{N-1}}\left(Y_{1}^{i}(\theta)-Y_{1}^{i}(\sigma)\right)^{2} G_{\kappa, N}(|\theta-\sigma|) d \sigma
\end{aligned}
$$

Since the integral on the left hand side above does not depend on $\theta$, integrating the above equality and using that $\left|S^{N-1}\right|=N\left|B^{N}\right|$, we get

$$
\begin{aligned}
\int_{S^{N-1} \times S^{N-1}} & (1-\theta \cdot \sigma) G_{\kappa, N}(|\theta-\sigma|) d \sigma d \theta \\
& =\frac{1}{2} \int_{S^{N-1} \times S^{N-1}}\left(Y_{1}^{i}(\theta)-Y_{1}^{i}(\sigma)\right)^{2} G_{\kappa, N}(|\theta-\sigma|) d \sigma d \theta \\
& =\mu_{1}
\end{aligned}
$$

The following result will be useful in order to study the invertibility of the linearized operator on the space of even function.

Lemma 5.4. We let

$$
\begin{equation*}
\gamma_{N}:=\min \left(\frac{\lambda_{1}}{\mu_{1}}, \inf _{k \geq 2} \frac{\lambda_{k}-\lambda_{1}}{\mu_{k}-\mu_{1}}\right) \tag{5.24}
\end{equation*}
$$

Then for every $\gamma \in\left(0, \gamma_{N}\right)$

$$
\begin{equation*}
\sigma_{\gamma}(k)>0 \quad \text { for all } k \geq 2, \quad \sigma_{\gamma}(0)<0 \tag{5.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{\sigma_{\gamma}(k)}{k^{2}}=1 \tag{5.26}
\end{equation*}
$$

Proof. We observe that, for $N=3$,

$$
\mu_{k} \leq \mu_{k}^{0}:=\frac{1}{2} \int_{S^{N-1} \times S^{N-1}}\left(Y_{k}^{i}(\theta)-Y_{k}^{i}(\sigma)\right)^{2} G_{0, N}(|\theta-\sigma|) d \theta d \sigma,
$$

and $\mu_{k}^{0}$ is the eigenvalues of the well known hypersingular Riesz operator on the sphere ( [17, Section 7]). In this case, we have that $\lim _{k \rightarrow \infty} \frac{\lambda_{k}}{\mu_{k}^{0}}=\infty$, see e.g. [17, Section 7].

For $N=2$, we have $Y_{k}^{1}(\cos (x), \sin (x))=\frac{\cos (k x)}{\pi^{1 / 2}}$ and $Y_{k}^{2}(\cos (x), \sin (x))=\frac{\cos (k x)}{\pi^{1 / 2}}$. Moreover $K_{0}(\kappa r) \leq C|\log (\kappa r)|$. Therefore, we can estimate

$$
\begin{aligned}
\mu_{k}^{0} & \leq C \int_{0}^{2 \pi}(\cos (k x)-\cos (k y))^{2}|\log (\mid \kappa \sqrt{1-\cos (x-y) \mid})| d x d y \\
& \leq C \int_{0}^{\pi} \int_{0}^{\pi}|\log (\kappa \sqrt{2}|\sin (x-y)|)| d x d y
\end{aligned}
$$

with $C$ independent on $k$. Since $\mu_{i} \neq \mu_{j}$ for $i \neq j$, we can therefore define

$$
\gamma_{N}:=\min \left(\frac{\lambda_{1}}{\mu_{1}}, \inf _{k \geq 2} \frac{\lambda_{k}-\lambda_{1}}{\mu_{k}-\mu_{1}}\right) .
$$

It follows that for every $\gamma \in\left(0, \gamma_{N}\right)$, we have $\sigma_{\gamma}(0)<0$ and $\sigma_{\gamma}(k)>0$ for every $k \geq 2$.

We then have the following result.
Lemma 5.5. Let $\gamma \in\left(0, \gamma_{N}\right)$. Then the linear map

$$
\mathcal{L}_{e}:=\left.L\right|_{C_{e}^{2, \alpha}\left(S^{N-1}\right)}: C_{e}^{2, \alpha}\left(S^{N-1}\right) \rightarrow C_{e}^{0, \alpha}\left(S^{N-1}\right)
$$

is an isomorphism.
Proof. By Lemma 5.4 the elements of the Kernel of $L$ correspond with the eigenvalue $\sigma_{\gamma}(1)$ which is the eigenvalues corresponding to the first non-constant spherical harmonics $Y_{1}^{1}$ and $Y_{1}^{2}$ which are odd. This implies that the kernel of $\mathcal{L}_{\mathrm{e}}$ is $\{0\}$. By Lemma 5.4, for every $f \in Y \in L_{\mathbf{e}}^{2}\left(S^{N-1}\right)$, there exists a unique $w \in H_{\mathbf{e}}^{2}\left(S^{N-1}\right)$ such that $L(w)=f$. Since $2 \leq N \leq 3$, by Morrey's embedding theorem, $w \in C^{0, \alpha}\left(S^{N-1}\right)$ for every $\alpha \in(0,1)$.
We define

$$
F(\theta)=\int_{S^{N-1}}(w(\theta)-w(\sigma)) G_{\kappa, N}(|\theta-\sigma|) d \sigma
$$

so that

$$
\begin{equation*}
L(w)=-\Delta_{S^{N-1}} w(\theta)-\lambda_{1} w(\theta)+2 \gamma \mu_{1} w(\theta)-2 \gamma F(\theta)=f . \tag{5.27}
\end{equation*}
$$

Then for every $\theta \in S^{N-1}$,

$$
\begin{equation*}
|F(\theta)| \leq C\|w\|_{C^{0, \alpha}\left(S^{N-1}\right)} \int_{S}|\theta-\sigma|^{\alpha} G_{\kappa, N}(|\theta-\sigma|) d \sigma \leq C\|w\|_{C^{0, \alpha}\left(S^{N-1}\right)} \tag{5.28}
\end{equation*}
$$

We fix $e \in S^{N-1}$ and $R_{\theta}$ be given by Example 5.1. Thanks to (5.5), a change of variable gives

$$
F(\theta)=\int_{S^{N-1}}\left(w(\theta)-w\left(R_{\theta} \sigma\right)\right) G_{\kappa, N}(|e-\sigma|) d \sigma
$$

By (5.5) and (5.6), for all $\theta_{1}, \theta_{2} \in S_{e}$ and $\sigma \in S^{N-1}$, we have

$$
\left|\left(w\left(\theta_{1}\right)-w\left(R_{\theta_{1}} \sigma\right)\right)-\left(w\left(\theta_{2}\right)-w\left(R_{\theta_{2}} \sigma\right)\right)\right| \leq C\|w\|_{C^{0, \alpha}(S)} \int_{S} \min \left(|e-\sigma|^{\alpha},\left|\theta_{1}-\theta_{2}\right|^{\alpha}\right)
$$

We assume that $\left|\theta_{1}-\theta_{2}\right| \leq 1 / 4$. Then by (2.3), we have

$$
\begin{aligned}
& \left|F\left(\theta_{1}\right)-F\left(\theta_{2}\right)\right| \leq C\|w\|_{C^{0, \alpha}\left(S^{N-1}\right)} \int_{S^{N-1}} \min \left(|e-\sigma|^{\alpha},\left|\theta_{1}-\theta_{2}\right|^{\alpha}\right)|e-\sigma|^{-1} d \sigma \\
& \leq C\|w\|_{C^{0, \alpha}\left(S^{N-1}\right)} \int_{|e-\sigma|<\left|\theta_{1}-\theta_{2}\right|}|e-\sigma|^{-1+\alpha} d \sigma \\
& \quad+C\|w\|_{C^{0, \alpha}\left(S^{N-1}\right)}\left|\theta_{1}-\theta_{2}\right|^{\alpha} \int_{1>|e-\sigma| \geq\left|\theta_{1}-\theta_{2}\right|}|e-\sigma|^{-1} d \sigma \\
& \leq C\|w\|_{C^{0, \alpha}\left(S^{N-1}\right)}\left|\theta_{1}-\theta_{2}\right|^{\alpha}\left(-\log \left(\left|\theta_{1}-\theta_{2}\right|\right)\right) .
\end{aligned}
$$

From this and (5.28), it then follows that $F \in C^{0, \beta}\left(S^{N-1}\right)$ for every $\beta \in(0,1)$. Now from (5.27), we can apply standard elliptic regularity estimates to deduce that $w \in C_{\mathrm{e}}^{2, \beta}\left(S^{N-1}\right)$.
5.2.1. Proof of Theorem 1.3 and 1.4 (completed). We recall that the function

$$
\mathcal{Q}: \mathbb{R} \times C_{\mathbf{e}}^{2, \alpha}\left(S^{N-1}\right) \cap \mathcal{D} \rightarrow C_{\mathbf{e}}^{0, \alpha}\left(S^{N-1}\right)
$$

is well-defined and smooth in a neighborhood of $(0,1) \in \mathbb{R} \times C_{\mathbf{e}}^{2, \alpha}\left(S^{N-1}\right) \cap \mathcal{D}$. Moreover, by Lemma 5.5, the linear map $D_{\varphi} \mathcal{Q}(0,1)=\mathcal{L}_{e}: X \rightarrow Y$ is an isomorphism. We can therefore apply the implicit function theorem to obtain a constant $\varepsilon_{0}>0$ and a smooth curve

$$
\begin{equation*}
\left(-\varepsilon_{0}, \varepsilon_{0}\right) \longrightarrow X, \quad \varepsilon \mapsto \omega_{\varepsilon} \tag{5.29}
\end{equation*}
$$

such that $\left\|\omega_{\varepsilon}\right\|_{C^{2, \alpha}\left(S^{N-1}\right)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$
\begin{equation*}
\mathcal{Q}\left(\varepsilon, \omega_{\varepsilon}\right)=\mathcal{Q}(0,1) \quad \text { on } C_{\mathbf{e}}^{0, \alpha}\left(S^{N-1}\right) \tag{5.30}
\end{equation*}
$$

Moreover it is easy to see from (5.18) and (5.30) that

$$
\begin{equation*}
\left\|\omega_{\varepsilon}\right\|_{C^{2, \alpha}\left(S^{N-1}\right)} \leq\left\|\mathcal{L}_{e}^{-1}\left(\mathcal{F}\left(\varepsilon, 1+\omega_{\varepsilon}\right)\right)\right\|_{C^{2, \alpha}\left(S^{N-1}\right)} \leq e^{-\frac{c}{|\varepsilon|}}, \tag{5.31}
\end{equation*}
$$

for some positive constants $c$.
5.3. Remarks on the perturbations of near-cylinders and near-balls. The aim of this section is to show that the perturbation $\varphi_{\varepsilon}$ and $\phi_{\varepsilon}$ from Theorem 1.3 and 1.4 are not constant in the respective lower dimensional lattice $M=1$ (for cylinder) and $1 \leq M \leq 2$ (for the balls). This is equivalent to show that $\mathcal{C}_{1}^{\varepsilon}$ and $\mathcal{B}_{1}^{\varepsilon}$ are not equilibrium patterns, for small $\varepsilon$.

Proposition 5.6. Let $\varphi_{\varepsilon}$ and $\phi_{\varepsilon}$ be given by Theorem 1.3 and 1.4, respectively, and let $1 \leq M \leq N$, with $N=2,3$. Then

$$
\varphi_{\varepsilon}(\theta)=1-\mathcal{L}_{e}^{-1}\left(U_{\varepsilon}\right)(\theta)+\varepsilon^{5 / 2} O\left(\sum_{p \in \mathscr{L}_{*}^{M}}|p|^{-3 / 2} e^{-\kappa \frac{|p|}{\varepsilon}}\right)
$$

and

$$
\phi_{\varepsilon}(\theta)=1-\mathcal{L}_{e}^{-1}\left(V_{\varepsilon}\right)(\theta)+\varepsilon^{2} O\left(\sum_{p \in \mathscr{L}_{*}^{M}}|p|^{-1} e^{-\kappa \frac{|p|}{\varepsilon}}\right) .
$$

Here $\left(\mathcal{L}_{e}\right)^{-1}: C_{e}^{0, \alpha}\left(S^{N-1}\right) \rightarrow C_{e}^{2, \alpha}\left(S^{N-1}\right)$ denotes the inverse of $\mathcal{L}_{e}$ (see 5.5). Moreover

$$
U_{\varepsilon}(\theta)=\sum_{p \in \mathscr{L}_{*}^{M}} \cosh \left(\kappa \frac{\theta \cdot p}{|p|}\right) \xi_{\varepsilon}(|p|), \quad V_{\varepsilon}(\theta)=\sum_{p \in \mathscr{L}_{*}^{M}} \cosh \left(\kappa \frac{\theta \cdot p}{|p|}\right) \zeta_{\varepsilon}(|p|),
$$

with

$$
\xi_{\varepsilon}(\ell)=c_{2} \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\varepsilon^{3 / 2}}{\ell^{5 / 2}} e^{-\kappa \frac{\ell}{\varepsilon}}, \quad \xi_{\varepsilon}(\ell)=c_{3} \frac{\varepsilon}{\ell} e^{-\kappa \frac{\ell}{\varepsilon}},
$$

where

$$
c_{N}:=\int_{B^{N}} e^{-\kappa y \cdot e_{1}} d y
$$

Moreover if $M=1$ then $\varphi_{\varepsilon}$ is not constant and if $M \leq 2$ then $\phi_{\varepsilon}$ is not constant.

Proof. We consider the function $f_{\varepsilon}: S^{1} \rightarrow \mathbb{R}$, given by

$$
f_{\varepsilon}(\theta)=\int_{\mathcal{C}_{1}^{\varepsilon} \backslash C_{1}} G_{\kappa}\left(\left|\theta-y-\frac{p}{\varepsilon}\right|\right) d y=2 \sum_{p \in \mathscr{L}_{*}^{M}} \int_{B_{1}^{2}} K_{0}\left(\kappa\left|\theta-z-\frac{p}{\varepsilon}\right|\right) d z .
$$

For $\varepsilon>0$, we write

$$
\left|\theta-z-\frac{p}{\varepsilon}\right|=\frac{|p|}{\varepsilon}\left(1+m_{\varepsilon}\right)^{1 / 2}=\frac{|p|}{\varepsilon}\left(1+\frac{1}{2} m_{\varepsilon}+O\left(\varepsilon^{2} /|p|\right)\right),
$$

where

$$
m_{\varepsilon}=\frac{\varepsilon^{2}|y-\theta|+2 \varepsilon(y-\theta) \cdot p}{|p|^{2}}
$$

Therefore

$$
\left|\theta-z-\frac{p}{\varepsilon}\right|=\frac{|p|}{\varepsilon}\left(1+\frac{\varepsilon(y-\theta) \cdot p}{|p|^{2}}+O\left(\varepsilon^{2} /|p|\right)\right) .
$$

By (2.2),

$$
K_{0}(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-1 / 2} e^{-r} \quad \text { as } r \rightarrow+\infty
$$

We then deduce that, as $\varepsilon \rightarrow 0$,

$$
\begin{aligned}
\frac{2 \sqrt{2}}{\sqrt{\pi}} K_{0}\left(\left|\theta-z-\frac{p}{\varepsilon}\right|\right) & =\frac{|p|^{-1 / 2}}{\varepsilon^{-1 / 2}}\left(1-\frac{\varepsilon(y-\theta) \cdot p}{2|p|^{2}}+O\left(\varepsilon^{2} /|p|\right)\right) e^{-\kappa \frac{|p|}{\varepsilon}} e^{-\kappa \frac{(y-\theta) \cdot p}{|p|}} e^{O(\varepsilon)} \\
& =\left(\frac{\varepsilon^{3 / 2}}{|p|^{5 / 2}} e^{-\kappa \frac{|p|}{\varepsilon}} e^{-\kappa \frac{(y-\theta) \cdot p}{|p|}}+O\left(\varepsilon^{5 / 2}|p|^{-3 / 2} e^{-\kappa \frac{|p|}{\varepsilon}}\right)\right) e^{O(\varepsilon)} \\
& =\left(\frac{\varepsilon^{3 / 2}}{|p|^{5 / 2}} e^{-\kappa \frac{|p|}{\varepsilon}} e^{-\kappa \frac{y \cdot p}{|p|}} e^{\kappa \frac{\theta \cdot p}{|p|}}+O\left(\varepsilon^{5 / 2}|p|^{-3 / 2} e^{-\kappa \frac{|p|}{\varepsilon}}\right)\right) .
\end{aligned}
$$

We define

$$
c_{2}:=\int_{B^{2}} e^{-\kappa \frac{y \cdot p}{|p|}} d y=\int_{B^{2}} e^{-\kappa y \cdot e_{1}} d y
$$

and

$$
\begin{equation*}
\xi_{\varepsilon}(\ell)=c_{2} \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\varepsilon^{3 / 2}}{\ell^{5 / 2}} e^{-\kappa \frac{\ell}{\varepsilon}} \tag{5.32}
\end{equation*}
$$

To see that $f_{\varepsilon}$ is not constant, it suffices to check that the map

$$
\theta \mapsto U_{\varepsilon}(\theta):=\sum_{p \in \mathscr{L}_{*}^{M}} e^{-\kappa \frac{\theta \cdot p}{|p|}} \xi_{\varepsilon}(|p|)
$$

is not constant. Since $\mathscr{L}_{*}^{M}=-\mathscr{L}_{*}^{M}$, we deduce that

$$
U_{\varepsilon}(\theta):=\sum_{p \in \mathscr{L}_{*}^{M}} \cosh \left(\kappa \frac{\theta \cdot p}{|p|}\right) \xi_{\varepsilon}(|p|) .
$$

We then conclude that

$$
\begin{equation*}
f_{\varepsilon}(\theta)=\int_{\mathcal{C}_{1}^{\varepsilon} \backslash C_{1}} G_{\kappa}\left(\left|\theta-y-\frac{p}{\varepsilon}\right|\right) d y=U_{\varepsilon}(\theta)+\varepsilon^{5 / 2} O\left(\sum_{p \in \mathscr{L}_{*}^{M}}|p|^{-3 / 2} e^{-\kappa \frac{p p \mid}{\varepsilon}}\right) . \tag{5.33}
\end{equation*}
$$

Provided $M=1$, we may assume that $\mathscr{L}^{1}=\mathbb{Z} \times\{0\} \subset \mathbb{R}^{2}$ the subspace of $\mathbb{R}^{2}=\operatorname{span}\left\{e_{1}, e_{2}\right\}$ with $e_{1}=(1,0)$ and $e_{2}=(0,1)$. Therefore we have that $e_{2} \cdot p=0$ and thus

$$
U_{\varepsilon}\left(e_{2}\right)=\sum_{p \in \mathbb{Z}_{*}} \xi_{\varepsilon}(|p|) .
$$

However

$$
U_{\varepsilon}\left(e_{1}\right)=\cosh (\kappa) \sum_{p \in \mathbb{Z}_{*}} \xi_{\varepsilon}(|p|) .
$$

This implies that $U_{\varepsilon}\left(e_{1}\right)>U_{\varepsilon}\left(e_{2}\right)$. That is $\mathcal{H}_{\mathscr{C}_{1}^{\varepsilon}}\left(e_{1}\right) \neq \mathcal{H}_{\mathscr{C}_{1}^{\varepsilon}}\left(e_{2}\right)$, for $\varepsilon>0$ small.
We now turn to the sphere lattices and provide Taylor expansion of $\phi_{\varepsilon}$ in Theorem 1.4, and prove that it is not constant for $M \leq 2$. We adopt the same strategy as a above. We consider the function $F_{\varepsilon}: S^{2} \rightarrow \mathbb{R}$, given by

$$
F_{\varepsilon}(\theta)=\int_{\mathcal{B}_{1}^{\varepsilon} \backslash B_{1}^{2}} G_{\kappa}(|\theta-y|) d y=\sum_{p \in \mathscr{L}_{*}^{M}} \int_{B_{1}^{2}} G_{\kappa}\left(\left|\theta-z-\frac{p}{\varepsilon}\right|\right) d z .
$$

Using the Taylor expansions as above, we get

$$
\begin{aligned}
G_{\kappa}\left(\left|\theta-z-\frac{p}{\varepsilon}\right|\right) & =\frac{\varepsilon}{|p|}\left(1-\frac{\varepsilon(y-\theta) \cdot p}{|p|^{2}}+O\left(\varepsilon^{2} /|p|\right)\right) e^{-\kappa \frac{|p|}{\varepsilon}\left(1+\frac{\varepsilon(y-\theta \cdot p}{|p|^{2}}+O\left(\varepsilon^{2} /|p|\right)\right)} \\
& =\frac{\varepsilon}{|p|} e^{-\kappa \frac{|p|}{\varepsilon}} e^{O(\varepsilon)} e^{-\kappa(y-\theta) \cdot \frac{p}{|p|}}\left(1+O\left(\varepsilon^{2} /|p|\right)\right) \\
& =\frac{\varepsilon}{|p|} e^{-\kappa \frac{|p|}{\varepsilon}} e^{-\kappa(y-\theta) \cdot \frac{p}{|p|}}+O\left(\frac{\varepsilon^{2}}{|p|} e^{-\kappa \frac{|p|}{\varepsilon}}\right) .
\end{aligned}
$$

We define

$$
c_{2}:=\int_{B^{3}} e^{-\kappa y \cdot \frac{p}{\mid p p}} d y=\int_{B^{3}} e^{-\kappa y \cdot e_{1}} d y
$$

and

$$
\begin{equation*}
\zeta_{\varepsilon}(\ell):=c_{1} \frac{\varepsilon}{\ell} e^{-\kappa \frac{\ell}{\varepsilon}} . \tag{5.34}
\end{equation*}
$$

Letting

$$
V_{\varepsilon}(\theta):=\sum_{p \in \mathscr{L}_{*}} e^{\frac{\theta \cdot p}{p} \kappa} \zeta_{\varepsilon}(|p|),
$$

it follows that

$$
\begin{equation*}
F_{\varepsilon}(\theta)=\sum_{p \in \mathscr{L}_{*}^{M}} G_{\kappa}\left(\left|\theta-z-\frac{p}{\varepsilon}\right|\right)=V_{\varepsilon}(\theta)+\varepsilon^{2} O\left(\sum_{p \in \mathscr{L}_{*}^{M}}|p|^{-1} e^{-\kappa \frac{|p|}{\varepsilon}}\right) . \tag{5.35}
\end{equation*}
$$

Since $\mathscr{L}_{*}^{M}=-\mathscr{L}_{*}^{M}$, we deduce that

$$
V_{\varepsilon}(\theta):=\sum_{p \in \mathscr{L}_{*}^{M}} \cosh \left(\kappa \frac{\theta \cdot p}{|p|}\right) \zeta_{\varepsilon}(|p|)=\sum_{\ell \in \mathbb{N}_{*}} \sum_{p \in E_{\ell}} \cosh \left(\kappa \theta \cdot \frac{p}{\ell}\right) \zeta_{\varepsilon}(\ell),
$$

where $E_{\ell}:=\left\{p \in \mathscr{L}_{*}^{M}:|p|=\ell\right\}$. We are therefore reduced to prove that $V_{\varepsilon}$ is not constant. Now if $M \leq 2$, then $e_{3} \cdot p=0$ and thus

$$
V_{\varepsilon}\left(e_{3}\right)=\sum_{\ell \in \mathbb{N}_{*}}\left|E_{\ell}\right| \zeta_{\varepsilon}(\ell) .
$$

We have

$$
V_{\varepsilon}\left(e_{1}\right):=\sum_{\ell \in \mathbb{N}_{*}} \sum_{p \in E_{\ell}} \cosh \left(\kappa e_{1} \cdot \frac{p}{\ell}\right) \zeta_{\varepsilon}(\ell) .
$$

It is clear that for some $\ell \in \mathbb{N}_{*}$, we must have $\sum_{p \in E_{\ell}} \cosh \left(\kappa e_{1} \cdot \frac{p}{\ell}\right)>\left|E_{\ell}\right|$, since $e_{1}$ cannot be perpendicular to both $\mathbf{a}_{1}$ and $\mathbf{a}_{2}$.

It follows from the construction, via the Implicit Function Theorem, of near-cylinder and the near- sphere lattices that, $\varphi_{\varepsilon}$ in Theorem 1.3 has the following Taylor expansion

$$
\varphi_{\varepsilon}=1-\mathcal{L}_{\mathrm{e}}^{-1}\left(U_{\varepsilon}\right)+\varepsilon^{5 / 2} O\left(\sum_{p \in \mathscr{L}_{*}}|p|^{-3 / 2} e^{-\kappa \frac{|p|}{\varepsilon}}\right) .
$$

and $\phi_{\varepsilon}$ in Theorem 1.4 Taylor expands as

$$
\phi_{\varepsilon}=1-\mathcal{L}_{\mathrm{e}}^{-1}\left(V_{\varepsilon}\right)+\varepsilon^{2} O\left(\sum_{p \in \mathscr{L}_{*}}|p|^{-1} e^{-\kappa \frac{|p|}{\varepsilon}}\right) .
$$

Here $\mathcal{L}_{\mathrm{e}}$ is given by Lemma 5.5 (with the corresponding dimensions $N=2,3$ ) and

$$
U_{\varepsilon}(\theta)=\sum_{p \in \mathscr{L}_{*}^{M}} \cosh \left(\kappa \frac{\theta \cdot p}{|p|}\right) \xi_{\varepsilon}(|p|), \quad V_{\varepsilon}(\theta)=\sum_{p \in \mathscr{L}_{*}^{M}} \cosh \left(\kappa \frac{\theta \cdot p}{|p|}\right) \zeta_{\varepsilon}(|p|),
$$

with $\xi_{\varepsilon}$ and $\zeta$ defined in (5.32) and (5.34) respectively. The above expressions of $\varphi_{\varepsilon}$ and $\phi_{\varepsilon}$ follow from (5.33) and (5.35) without any restriction on $M$, the dimension of the lattices.

## 6. Appendix: Proof of Lemma 5.2

We prove the regularity of the $\operatorname{map} \mathcal{Q}: \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right)$. We note that $\mathcal{H}$ and $\mathcal{F}$ are smooth in $\mathcal{D}$ and $\left(-\varepsilon_{0}, \varepsilon_{0}\right) \times \mathcal{D}$, for small $\varepsilon_{0}$, respectively. We will only consider $\mathcal{F}_{0, N}$ in the following. Recalling the notations in Section 5.1] we have the following result.

Lemma 6.1. The map $\mathcal{F}_{0, N}: \mathcal{D} \rightarrow C^{0, \alpha}\left(S^{N-1}\right)$ is smooth. Moreover for every $\phi \in \mathcal{D}$ and $w \in C^{2, \alpha}\left(S^{N-1}\right)$,

$$
\begin{equation*}
D \mathcal{F}_{0, N}(\phi)[w](\theta)=\int_{S^{N-1}}(w(\sigma) \sigma-w(\theta) \theta) \cdot \nu_{\phi}(\sigma) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \Gamma_{\phi}(\sigma) d \sigma \tag{6.1}
\end{equation*}
$$

where $F_{\phi}:=F_{\phi}(1, \cdot)$.
We first make a formal proof of (6.1). Then we prove regularity estimates for the map $\phi \mapsto D \mathcal{F}_{0, N}(\phi)[w]$, for every fixed $w$. This will end the proof of the lemma. For every $\delta>0$, we define

$$
\begin{equation*}
\mathcal{D}_{\delta}:=\left\{\phi \in C^{2, \alpha}\left(S^{N-1}\right): \phi>\delta\right\}, \tag{6.2}
\end{equation*}
$$

Let $\phi \in \mathcal{D}_{\delta}$ and $w \in C^{2, \alpha}\left(S^{N-1}\right)$ such that $\|w\|_{C^{2, \alpha}\left(S^{N-1}\right)}<\delta / 2$. We recall our definition of $F_{\varphi}$ in (5.7) and $S_{\phi}^{N-1}:=\partial B_{\phi}^{N}$ for every $\phi \in \mathcal{D}_{\delta}$. Using polar coordinates, (5.9) and (5.10),
we have

$$
\begin{aligned}
D \mathcal{F}_{0, N}(\phi)[w](\theta) & =\left.\frac{d}{d t}\right|_{t=0} \int_{S^{N-1}} \int_{0}^{\phi(\sigma)+t w(\sigma)} G_{\kappa, N}\left(\left|F_{\phi+t w}(\theta)-r \sigma\right|\right) r^{N-1} d r d \sigma \\
& =\int_{S^{N-1}} G_{\kappa, N}\left(\left|F_{\phi}(\theta)-\sigma \phi(\sigma)\right|\right) \phi^{N-1} w(\sigma) d \sigma \\
& \left.+\int_{S^{N-1}} \int_{0}^{\phi(\sigma)} G_{\kappa, N}^{\prime}\left(\left|F_{\phi}(\theta)-r \sigma\right|\right) w(\theta) \theta \cdot(\theta \phi(\theta)-\sigma r)\right) r^{N-1} d r \\
& =\int_{S^{N-1}} \sigma w(\sigma) \cdot \nu_{\phi}(\sigma) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \Gamma_{\phi}(\sigma) d \sigma \\
& -\int_{B_{\phi}^{N}} \nabla_{y} G_{\kappa, N}\left(\left|F_{\phi}(\theta)-y\right|\right) \cdot \theta w(\theta) d y
\end{aligned}
$$

where $\Gamma_{\phi}, \nu_{\phi}$ are defined in Section 5.1. Using the divergence theorem in the last integral, we obtain

$$
D \mathcal{F}_{0, N}(\phi)[w](\theta)=\int_{S^{N-1}}(w(\sigma) \sigma-w(\theta) \theta) \cdot \nu_{\phi}(\sigma) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \Gamma_{\phi}(\sigma) d \sigma
$$

which is the expression in (6.1). We will now prove that $\mathcal{F}_{0, N}$ is smooth by deriving regularity estimates for $D \mathcal{F}_{0, N}(\phi)[w]$ computed above. By (5.9) and (5.10), we have

$$
(\theta-\sigma) \cdot \nu_{\phi}\left(F_{\phi}(\sigma)\right) \Gamma_{\phi}(\sigma)=-(\theta-\sigma) \cdot \nabla \phi(\sigma) \phi^{N-2}(\sigma)+(\theta-\sigma) \cdot \sigma \phi^{N-1}(\sigma)
$$

so that

$$
\begin{align*}
D \mathcal{F}_{0, N}(\phi)[w](\theta) & =\int_{S^{N-1}}(w(\sigma)-w(\theta)) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \phi^{N-1}(\sigma) d \sigma \\
& +w(\theta) \int_{S^{N-1}}(\theta-\sigma) \cdot \nabla \phi(\sigma) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \phi^{N-2}(\sigma) d \sigma  \tag{6.3}\\
& +\frac{w(\theta)}{2} \int_{S^{N-1}}|\theta-\sigma|^{2} G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \phi^{N-1}(\sigma) d \sigma,
\end{align*}
$$

where we have used $2(\theta-\sigma) \cdot \sigma=-|\theta-\sigma|^{2}$. To prove the regularity of $\phi \mapsto D \mathcal{F}_{0, N}(\phi)[w](\theta)$, for a fixed $w \in C^{2, \alpha}\left(S^{1}\right)$, it is enough to consider the first term. In fact, we will prove a stronger result than needed.

For $\beta \in(0,1)$ and $\delta>0$, we consider the open sets

$$
\mathcal{A}:=\left\{\phi \in C^{1, \beta}\left(S^{N-1}\right): \phi>0\right\}
$$

and

$$
\mathcal{A}_{\delta}:=\left\{\phi \in C^{1, \beta}\left(S^{N-1}\right): \phi>\delta\right\} .
$$

We define the function (the first term in the expression of $D \mathcal{F}_{0, N}(\phi)[w](\theta)$ ) given by

$$
\begin{equation*}
\mathcal{A}_{\delta} \rightarrow L^{\infty}\left(S^{N-1}\right), \quad \phi \mapsto h(\phi):=\int_{S^{N-1}}(w(\sigma)-w(\theta)) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|\right) \phi^{N-1}(\sigma) d \sigma . \tag{6.4}
\end{equation*}
$$

Recalling the notations in Section [5.1, by a change of variable, for every $\theta \in S_{e}$, we get

$$
\begin{equation*}
h(\phi)(\theta)=\int_{S^{N-1}}\left(w\left(R_{\theta} \sigma\right)-w(\theta)\right) G_{\kappa, N}\left(\left|F_{\phi}(\theta)-F_{\phi}\left(R_{\theta} \sigma\right)\right|\right) \phi^{N-1}(\sigma) d \sigma \tag{6.5}
\end{equation*}
$$

Once the following proposition is proved, Lemma 6.1 follows immediately.
Proposition 6.2. For every $\varepsilon \in(0, \beta)$, the map $h: \mathcal{A} \subset C^{1, \beta}\left(S^{N-1}\right) \rightarrow C^{0, \beta-\varepsilon}\left(S^{N-1}\right)$ is of class $C^{\infty}$. Moreover

$$
\left\|D^{k} h(\phi)\right\| \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{N-1}\right)}\right)^{c}\|w\|_{C^{\beta}\left(S^{N-1}\right)}
$$

for some positive constant $c$ depending only on $\kappa, \beta$ and $\varepsilon$. In addition if $N=3$, we can take $\varepsilon=0$. In particular $\mathcal{F}_{0, N}$ is of class $C^{\infty}$ in $\mathcal{D}$.

We now distinguish the two case $N=2$ and $N=3$. We start with the
Case $N=2$.
Here, $h$ takes the form (recall (2.4))

$$
h(\phi)(\theta)=\int_{\mathbb{R}} \int_{S^{1}}\left(w\left(R_{\theta} \sigma\right)-w(\theta)\right) G_{\kappa}\left(\left|\left(F_{\phi}(\theta), 0\right)-\left(F_{\phi}\left(R_{\theta} \sigma\right), 0\right)+t^{2} e_{3}\right|\right) \phi(\sigma) d \sigma d t
$$

We have

$$
\left|F_{\phi}(\theta)-F_{\phi}(\sigma)\right|^{2}+t^{2}=\Lambda_{0}(\phi, \theta, \sigma)^{2}+\phi(\sigma) \phi(\theta)|\theta-\sigma|^{2}+t^{2} .
$$

Then with the change of variable $\rho=\frac{t}{|\theta-\sigma|}$, we get

$$
h(\phi)(\theta)=\int_{\mathbb{R}} \int_{S^{1}}\left(w\left(R_{\theta} \sigma\right)-w(\theta)\right) \mathcal{K}\left(\phi, \theta, R_{\theta} \sigma, \rho\right) \phi\left(R_{\theta} \sigma\right) d \sigma d \rho,
$$

where $\mathcal{K}: \mathcal{A}_{\delta} \times S^{1} \times S^{1} \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$
\mathcal{K}(\phi, \theta, \sigma, \rho):=G_{|\theta-\sigma| \kappa}\left(\Lambda_{0}(\phi, \theta, \sigma)^{2}+\phi(\sigma) \phi(\theta)+\rho^{2}\right),
$$

with

$$
\Lambda_{0}(\phi, \theta, \sigma)=\frac{\phi(\theta)-\phi(\sigma)}{|\theta-\sigma|} .
$$

See e.g. [5], there exists a constant $C>0$ such that for all $\sigma \in S$, all $\theta, \theta_{1}, \theta_{2} \in S_{e}$ and all $\phi_{1}, \phi_{2} \in C^{1, \beta}\left(S^{1}\right)$ we have

$$
\begin{equation*}
\left|\Lambda_{0}\left(\phi, R_{\theta} \sigma, \theta\right)\right| \leq C\|\phi\|_{C^{1, \beta}(S)}\|\phi\|_{C^{1, \beta}\left(S^{1}\right)} . \tag{6.6}
\end{equation*}
$$

and

$$
\begin{align*}
\mid \Lambda_{0}\left(\phi, R_{\theta_{1}} \sigma, \theta_{1}\right) & -\Lambda_{2}\left(\phi, R_{\theta_{2}} \sigma, \theta_{2}\right) \mid  \tag{6.7}\\
& \leq C\|\phi\|_{C^{1, \beta}(S)}\left|\theta_{1}-\theta_{2}\right|^{\beta} .
\end{align*}
$$

We follow the arguments as in [5], where a more singular kernel was considered.
Lemma 6.3. For every $e \in S^{1}, k \in \mathbb{N}, \delta>0$, there exists a constant $c=c(\kappa, k, \delta, \alpha)>0$ such that for $\phi \in \mathcal{A}_{\delta}, \rho \in \mathbb{R}, \theta, \theta_{1}, \theta_{2} \in S_{e}$ and $\sigma \in S^{1}$, we have

$$
\left\|D_{\phi}^{k} \mathcal{K}\left(\phi, \theta, R_{\theta} \sigma, \rho\right)\right\| \leq c\left(1+\|\phi\|_{C^{1, \beta}(S)}\right)^{c} \exp \left(-\kappa|\sigma-e|\left(\delta^{2}+\rho^{2}\right)\right)
$$

and

$$
\begin{aligned}
\| D_{\phi}^{k} \mathcal{K}\left(\phi, \theta_{1}, R_{\theta_{1}} \sigma, \rho\right) & -D_{\phi}^{k} \mathcal{K}\left(\phi, \theta_{2}, R_{\theta_{2}} \sigma, \rho\right) \| \\
& \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{1}\right)}\right)^{c}\left|\theta_{1}-\theta_{2}\right|^{\beta} \exp \left(-\kappa|\sigma-e|\left(\delta^{2}+\rho^{2}\right)\right)
\end{aligned}
$$

Proof. We have

$$
\mathcal{K}\left(\phi, \theta, R_{\theta} \sigma, \rho\right):=G_{|\sigma-e| \kappa}\left(\Lambda_{0}\left(\phi, \phi, \theta, R_{\theta} \sigma\right)^{2}+\phi\left(R_{\theta} \sigma\right) \phi(\theta)+\rho^{2}\right) .
$$

In view of the arguments in [5], it suffices to have some estimates related

$$
x \mapsto G_{a}\left(\left(x+\rho^{2}\right)^{1 / 2}\right)=\frac{\exp \left(-a\left(x+\rho^{2}\right)^{1 / 2}\right)}{\left(x+\rho^{2}\right)^{1 / 2}}
$$

for every $x>\delta^{2}, \rho \in \mathbb{R}$ and $a \in(0,2 \kappa)$. Now elementary computations show that

$$
\begin{equation*}
\partial_{x}^{m} G_{a}\left(\left(x+\rho^{2}\right)^{1 / 2}\right) \leq c_{\kappa, \delta, m} \exp \left(-a\left(x+\rho^{2}\right)^{1 / 2}\right) \quad \text { for all } x>\delta^{2}, \rho \in \mathbb{R}, a \in(0,2 \kappa) . \tag{6.8}
\end{equation*}
$$

As in [5], combining this estimate together with (6.6), (6.7), and eventually with the Faá de Bruno formula (see e.g. [26]), we get the estimates in the lemma.

In the following, for a function $f: S^{1} \rightarrow \mathbb{R}$, we use the notation

$$
\left[f ; \theta_{1}, \theta_{2}\right]:=f\left(\theta_{1}\right)-f\left(\theta_{2}\right) \quad \text { for } \theta_{1}, \theta_{2} \in S^{1}
$$

and we note the obvious equality

$$
\begin{equation*}
\left[f g ; \theta_{1}, \theta_{2}\right]=\left[f ; \theta_{1}, \theta_{2}\right] g\left(\theta_{1}\right)+f\left(\theta_{2}\right)\left[g ; \theta_{1}, \theta_{2}\right] \quad \text { for } f, g: S^{1} \rightarrow \mathbb{R}, \theta_{1}, \theta_{2} \in S^{1} . \tag{6.9}
\end{equation*}
$$

The next step is to have estimates of all possible candidates for the derivatives of $h$. This is the purpose of the following

Lemma 6.4. For $\beta \in(0,1)$ and $\delta>0$, we let $\phi \in \mathcal{A}_{\delta}$, with $\delta>0$. Let $k \in \mathbb{N}$. Let also $W, w \in C^{\beta}\left(S^{1}\right)$ and $\psi_{1}, \ldots, \psi_{k} \in C^{1, \beta}\left(S^{1}\right)$. Define the functions $\widetilde{h}: S^{1} \rightarrow \mathbb{R}$ by

$$
\widetilde{h}(\theta)=\int_{S^{1}}\left(w\left(R_{\theta} \sigma\right)-w(\theta)\right) W\left(R_{\theta} \sigma\right) D_{\phi}^{k} \mathcal{K}\left(\phi, R_{\theta} \sigma, \theta, \rho\right)\left[\psi_{1}, \ldots, \psi_{k}\right] d \sigma d \rho,
$$

Then for every $\varepsilon \in(0, \beta)$ there exists a constant $c=c(N, \beta, k, \delta, \varepsilon)$ such that

$$
\begin{equation*}
\|\widetilde{h}(\theta)\|_{C^{\beta-\varepsilon}\left(S^{1}\right)} \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{1}\right)}\right)^{c} \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{C^{1, \beta}}\|w\|_{C^{\beta}\left(S^{1}\right)}\|W\|_{C^{\beta}\left(S^{1}\right)} \tag{6.10}
\end{equation*}
$$

Proof. Let $\theta_{1}, \theta_{2} \in S_{e}^{1}$ and $\sigma \in S^{1}$. We first note that

$$
\begin{gather*}
\left|W\left(R_{\theta_{1}} \sigma\right)-W\left(R_{\theta_{2}} \sigma\right)\right| \leq C\|W\|_{C^{\beta}\left(S^{1}\right)}\left|R_{\theta_{1}} \sigma-R_{\theta_{2}} \sigma\right|^{\beta} \leq C\|W\|_{C^{\beta}(S)}\left|\theta_{1}-\theta_{2}\right|^{\beta},  \tag{6.11}\\
\left.\mid\left(w\left(R_{\theta_{1}} \sigma\right)\right)-w\left(\theta_{1}\right)\right)-\left(w\left(R_{\theta_{2}} \sigma\right)-w\left(\theta_{1}\right)\right) \mid \leq C\|w\|_{C^{\beta}\left(S^{1}\right)} \min \left(|\sigma-e|^{\beta},\left|\theta_{1}-\theta_{2}\right|^{\beta}\right) \tag{6.12}
\end{gather*}
$$

and

$$
\begin{equation*}
\left.\mid\left(w\left(R_{\theta} \sigma\right)\right)-w(\theta)\right)\left|\leq C\|w\|_{C^{\beta}\left(S^{1}\right)}\right| \sigma-\left.e\right|^{\beta} . \tag{6.13}
\end{equation*}
$$

Using inductively (6.9) together with Lemma (6.3) (6.11), (6.13) and (6.12), we get the estimate

$$
\begin{align*}
& \left|\left[\widetilde{h} ; \theta_{1}, \theta_{2}\right]\right| \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{1}\right)}^{c}\right)\|w\|_{C^{\beta}\left(S^{1}\right)}\|W\|_{C^{\beta}\left(S^{1}\right)} \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{C^{1, \beta}\left(S^{1}\right)} \\
& \quad \times\left(\left|\theta_{1}-\theta_{2}\right|^{\beta} \int_{\mathbb{R}} \int_{S^{1}}|\sigma-e|^{\beta} \exp (-\kappa|\sigma-e| \rho) d \sigma d \rho\right. \\
& \left.\quad+\int_{\mathbb{R}} \int_{S^{1}} \min \left(|\sigma-e|^{\beta},\left|\theta_{1}-\theta_{2}\right|^{\beta}\right) \exp (-\kappa|\sigma-e| \rho) d \sigma d \rho\right) \\
& =c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{1}\right)}^{c}\right)\|w\|_{C^{\beta}\left(S^{1}\right)}\|W\|_{C^{\beta}\left(S^{1}\right)} \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{C^{1, \beta}\left(S^{1}\right)} \times \\
& \left(\left|\theta_{1}-\theta_{2}\right|^{\beta} \int_{\mathbb{R}} \int_{S^{1}} \frac{\exp (-\kappa r)}{|\sigma-e|^{1-\beta}} d \sigma d r+\int_{\mathbb{R}} \int_{S^{1}} \frac{\min \left(|\sigma-e|,\left|\theta_{1}-\theta_{2}\right|\right) \exp (-\kappa r)}{|\sigma-e|} d \sigma d r\right) . \tag{6.14}
\end{align*}
$$

For all $\theta_{1}, \theta_{2} \in S$, we then have

$$
\begin{aligned}
\int_{S^{1}} \frac{\min \left(|\sigma-e|^{\beta},\left|\theta_{1}-\theta_{2}\right|^{\beta}\right)}{|\sigma-e|} d \sigma \leq & \int_{|e-\sigma| \leq\left|\theta_{1}-\theta_{2}\right|}|\sigma-e|^{\beta-1} d \sigma \\
& +\left|\theta_{1}-\theta_{2}\right|^{\beta} \int_{\left|\theta_{1}-\theta_{2}\right| \leq|e-\sigma| \leq 2}|\sigma-\theta|^{-1} d \sigma \\
\leq & c\left|\theta_{1}-\theta_{2}\right|^{\beta}+c\left|\theta_{1}-\theta_{2}\right|^{\beta}\left|\log \left(\left|\theta_{1}-\theta_{2}\right|\right)\right| .
\end{aligned}
$$

We thus deduce from (6.14) that for all $\varepsilon \in(0,1)$

$$
\left|\left[\widetilde{h} ; \theta_{1}, \theta_{2}\right]\right| \leq c\left|\theta_{1}-\theta_{2}\right|^{\beta-\varepsilon}\left(1+\|\phi\|_{C^{1, \beta}\left(S^{1}\right)}^{c}\right)\|w\|_{C^{\beta}\left(S^{1}\right)}\|W\|_{C^{\beta}\left(S^{1}\right)} \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{C^{1, \beta}\left(S^{1}\right)}
$$

6.0.1. Completion of the proof of Proposition 6.2 for $N=2$. We consider

$$
\begin{aligned}
\bar{h}(\theta) & =\sum_{i=1}^{k} \int_{\mathbb{R}} \int_{S^{1}}\left(w\left(R_{\theta} \sigma\right)-w(\theta)\right) \phi_{i}\left(R_{\theta} \sigma\right) D_{\phi}^{k} \mathcal{K}\left(\phi, \theta, R_{\theta} \sigma, \rho\right)\left[\psi_{1}, \ldots, \psi_{k}\right] d \sigma d \rho \\
& +\sum_{i=1}^{k} \int_{\mathbb{R}} \int_{S^{1}}\left(w\left(R_{\theta} \sigma\right)-w(\theta)\right) \psi_{i}\left(R_{\theta} \sigma\right) D_{\phi}^{k-1} \mathcal{K}\left(\phi, \theta, R_{\theta} \sigma, \rho\right)\left[\psi_{j}\right]_{\substack{j=1 \ldots, k \\
j \neq i}} d \sigma d \rho .
\end{aligned}
$$

Then by Lemma 6.4, we have

$$
\begin{equation*}
\|\bar{h}\|_{C^{\beta-\varepsilon}\left(S^{1}\right)} \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{1}\right)}\right)^{c} \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{C^{1, \beta}}\|w\|_{C^{\beta}\left(S^{1}\right)} \tag{6.15}
\end{equation*}
$$

We can now follow step by step the arguments in 5] to deduce that $h: \mathcal{A}_{\delta} \subset C^{1, \beta}\left(S^{1}\right) \rightarrow$ $C^{0, \beta-\varepsilon}\left(S^{1}\right)$ is of class $C^{\infty}$. Moreover, by (2.9), we get

$$
D^{k} h(\phi)\left[\psi_{1}, \ldots, \psi_{k}\right](\theta)=\bar{h}(\theta)
$$

Now, we pick a $w \in C^{1, \beta}(\mathbb{R})$, with $\|w\|_{C^{1, \beta}(\mathbb{R})}<\delta / 2$. Then for $\phi \in \mathcal{A}_{\delta / 2}$, we have

$$
\begin{aligned}
{\left[\mathcal{F}_{0, N}(\phi+w)-\mathcal{F}_{0, N}(w)-h(\phi)\right](\theta) } & =\int_{0}^{1}[h(\phi+w)-h(\phi)](\theta) d \varrho \\
& =\int_{0}^{1} \varrho \int_{0}^{1} D h(\phi+\rho \varrho w)[w](\theta) d \rho d \varrho
\end{aligned}
$$

recall the expression of $h$ in (6.5). Since $\phi+\rho \varrho w \in \mathcal{A}_{\delta}$ for $\rho, \varrho \in(0,1)$, it follows from Lemma 6.2 that

$$
\begin{gathered}
\left\|\mathcal{F}_{0, N}(\phi+w)-\mathcal{F}_{0, N}(\phi)-h(\phi)\right\|_{C^{0, \alpha}\left(S^{N-1}\right)} \leq \int_{0}^{1} \int_{0}^{1}\|h(\phi+\rho \varrho w)[w]\|_{C^{0, \beta}\left(S^{N-1}\right)} d \rho d \varrho \\
\leq c\|w\|_{C^{0, \beta}(\mathbb{R})}^{2}\left(1+\|\phi\|_{C^{1, \beta}\left(S^{N-1}\right)}+\|w\|_{C^{0, \beta}\left(S^{N-1}\right)}\right)^{c}
\end{gathered}
$$

From this, we conclude that $\mathcal{F}_{0, N}$ is differentiable on $\mathcal{A}_{\delta / 2}$ with

$$
\mathcal{F}_{0, N}(\varphi)[w]=h(\varphi) .
$$

The $C^{\infty}$-character of $\mathcal{F}_{0, N}$ on $\mathcal{A}$ now follows from the one of $h$ and the fact that $\delta$ is an arbitrary positive number.
6.0.2. Completion of the proof of Proposition 6.2 for $N=3$. As we will see, this case is much easier because

$$
\begin{equation*}
\int_{S^{2}}|\sigma-e|^{-1} d \sigma<\infty \tag{6.16}
\end{equation*}
$$

Here, $h$ takes the form (recall (2.4))

$$
\begin{equation*}
h(\phi)(\theta)=\int_{S^{2}} \frac{w\left(R_{\theta} \sigma\right)-w(\theta)}{|\sigma-e|} \mathcal{K}_{3}\left(\phi, \theta, R_{\theta} \sigma\right) \phi^{2}\left(R_{\theta} \sigma\right) d \sigma \tag{6.17}
\end{equation*}
$$

where $\mathcal{K}_{3}: \mathcal{A}_{\delta} \times S^{2} \times S^{2} \rightarrow \mathbb{R}$ is given by

$$
\mathcal{K}_{3}(\phi, \theta, \sigma):=G_{|\theta-\sigma| \kappa}\left(\Lambda_{0}(\phi, \theta, \sigma)^{2}+\phi(\sigma) \phi(\theta)\right),
$$

with, as above,

$$
\Lambda_{0}(\phi, \theta, \sigma)=\frac{\phi(\theta)-\phi(\sigma)}{|\theta-\sigma|} .
$$

Moreover by (6.8), (6.6) and (6.7), we have the estimates

$$
\left\|D_{\phi}^{k} \mathcal{K}_{3}\left(\phi, \theta, R_{\theta} \sigma\right)\right\| \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{2}\right)}\right)^{c}
$$

and

$$
\| D_{\phi}^{k} \mathcal{K}_{3}\left(\phi, \theta_{1}, R_{\theta_{1}} \sigma\right)-D_{\phi}^{k} \mathcal{K}\left(\phi, \theta_{2}, R_{\theta_{2}} \sigma\right) \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{2}\right)}\right)^{c}\left|\theta_{1}-\theta_{2}\right|^{\beta}
$$

which holds for every $e \in S^{2}, k \in \mathbb{N}, \delta>0, \phi \in \mathcal{A}_{\delta}, \rho \in \mathbb{R}, \theta, \theta_{1}, \theta_{2} \in S_{e}$ and $\sigma \in S^{2}$. We define

$$
\begin{aligned}
\hat{h}(\theta) & =\sum_{i=1}^{k} \int_{S^{2}} \frac{w\left(R_{\theta} \sigma\right)-w(\theta)}{|\sigma-e|} \phi\left(R_{\theta} \sigma\right) D_{\phi}^{k} \mathcal{K}_{3}\left(\phi, \theta, R_{\theta} \sigma\right)\left[\psi_{1}, \ldots, \psi_{k}\right] d \sigma \\
& +\sum_{i=1}^{k} \int_{S^{2}} \frac{w\left(R_{\theta} \sigma\right)-w(\theta)}{|\sigma-e|} \psi_{i}\left(R_{\theta} \sigma\right) D_{\phi}^{k-1} \mathcal{K}_{3}\left(\phi, \theta, R_{\theta} \sigma\right)\left[\psi_{j}\right]_{\substack{j=1 \ldots, k \\
j \neq i}} d \sigma .
\end{aligned}
$$

Then, from the estimates of the kernel $\mathcal{K}_{3}$ above and (6.16), we can deduce that

$$
\begin{equation*}
\|\hat{h}\|_{C^{\beta}\left(S^{2}\right)} \leq c\left(1+\|\phi\|_{C^{1, \beta}\left(S^{2}\right)}\right)^{c} \prod_{i=1}^{k}\left\|\psi_{i}\right\|_{C^{1, \beta}}\|w\|_{C^{\beta}\left(S^{2}\right)} . \tag{6.18}
\end{equation*}
$$

Now similar arguments as in the case $N=2$, show that $h: \mathcal{A}_{\delta} \rightarrow C^{0, \beta}\left(S^{2}\right)$ is of class $C^{\infty}$ and $D^{k} h\left[\psi_{1}, \ldots, \psi_{k}\right](\phi)(\theta)=\hat{h}(\theta)$. The proof of Proposition 6.2 is then completed.

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[^0]:    Key words and phrases. Screened Coulomb potential, short-range, long-range, Polymer melt, equilibrium periodic patterns.

