# REGULARITY ESTIMATES FOR NONLOCAL SCHRÖDINGER EQUATIONS 

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#### Abstract

We are concerned with Hölder regularity estimates for weak solutions $u$ to nonlocal Schrödinger equations subject to exterior Dirichlet conditions in an open set $\Omega \subset \mathbb{R}^{N}$. The class of nonlocal operators considered here are defined, via Dirichlet forms, by symmetric kernels $K(x, y)$ bounded from above and below by $|x-y|^{N+2 s}$, with $s \in(0,1)$. The entries in the equations are in some Morrey spaces and the underline domain $\Omega$ satisfies some mild regularity assumptions. In the particular case of the fractional Laplacian, our results are new. When $K$ defines a nonlocal operator with sufficiently regular coefficients, we obtain Hölder estimates, up to the boundary of $\Omega$, for $u$ and the ratio $u / d^{s}$, with $d(x)=\operatorname{dist}\left(x, \mathbb{R}^{N} \backslash \Omega\right)$. If the kernel $K$ defines a nonlocal operator with Hölder continuous coefficients and the entries are Hölder continuous, we obtain interior $C^{2 s+\beta}$ regularity estimates of the weak solutions $u$. Our argument is based on blow-up analysis and compact Sobolev embedding.


## 1. Introduction

We consider $s \in(0,1), N \geq 1$ and $\Omega$ an open set in $\mathbb{R}^{N}$ of class $C^{1, \gamma}$, for some $\gamma>0$. We are interested in interior and boundary Hölder regularity estimates for functions $u$ solution to the equation

$$
\begin{equation*}
\mathcal{L}_{K} u+V u=f \quad \text { in } \Omega \quad \text { and } \quad u=0 \quad \text { in } \Omega^{c} . \tag{1.1}
\end{equation*}
$$

where $\Omega^{c}:=\mathbb{R}^{N} \backslash \Omega$ and $\mathcal{L}_{K}$ is a nonolocal operator defined by a symmetric kernel $K \asymp|x-y|^{-N-2 s}$. We refer to Section 1.1 below for more details. Our model operator is $\mathcal{L}_{K}=(-\Delta)_{a}^{s}$, the so called anisotropic fractional Laplacian, up to a sign multiple. It is defined, for all $\varphi \in C_{c}^{2}\left(\mathbb{R}^{N}\right)$, by

$$
(-\Delta)_{a}^{s} \varphi(x)=P V \int_{\mathbb{R}^{N}} \frac{\varphi(x)-\varphi(x-y)}{|y|^{N+2 s}} a(y /|y|) d y
$$

with $a: S^{N-1} \rightarrow \mathbb{R}$ satisfying

$$
\begin{equation*}
a(-\theta)=a(\theta) \quad \text { and } \quad \Lambda \leq a(\theta) \leq \frac{1}{\Lambda} \quad \text { for all } \theta \in S^{N-1} \tag{1.2}
\end{equation*}
$$

for some constant $\Lambda>0$. Here, the entries $V, f$ in (1.1) belongs to some Morrey spaces.
In the recent years the study of nonlocal equations have attracted a lot of interest due to their manifestations in the modeling of real-world phenomenon and their rich structures in the mathematical point of view. In this respect, regularity theory remains central questions. Interior regularity and Harncak inequality have been intensiveley investigated in last decades, see e.g. [1, 3, 4, 9, 11, 16, 22, 33$37,40,50$ and the references therein. On the other hand, boundary regularity and Harnack inequalities was studied in [2, 5, 7, 13].

Results which are, in particular, most relevent to the content of this paper concern those dealing with nonlocal operator in "divergence form" with measurable coefficient, i.e. $K$ is symmetric on $\mathbb{R}^{N} \times \mathbb{R}^{N}$ and $K(x, y) \asymp|x-y|^{-N-2 s}$, see [39]. In this case the de Giorgi-Nash-Moser energy methods were used to obtain interior Harnack inequality and Hölder estimates, see [16, 17, 38, 42]. We note that the papers [38, 39] deal also with more general kernels than those satisfying $K(x, y) \asymp|x-y|^{-N-2 s}$ on $\mathbb{R}^{N} \times \mathbb{R}^{N}$ only. We also mention the work of Kuusi, Mingione and Sire in 42 who obtained local

[^0]pointwise behaviour of solutions to quasilinear nonlocal elliptic equations with measurable coefficients, provided the Wolff potential of the right hand side satisfies some qualitative properties.

In this paper, we are concerned in both interior and boundary regularity of nonlocal equations with "continuous coefficient". Let us recall that in the classical case of operators in divergence form with continuous coefficients, after scaling, the limit operator is given by the Laplace operator $\Delta$. The meaning of "continuous coefficient" in the nonlocal framework is not immediate due to the singularity of the kernel $K$ at the diagonal points $x=y$. However, under nonrestrictive continuity assumptions, detailed in Section 1.1 below, we find out that the limiting nonlocal operator is the anisotropic fractional Laplacian $-(-\Delta)_{a}^{s}$ in many situations.

Letting $d(x):=\operatorname{dist}\left(x, \Omega^{c}\right)$, the boundary regularity we are interested in here is the Hölder regularity estimates of $u / d^{s}$ for the nonlocal operator $\mathcal{L}_{K}$. Such regularity results for solutions to (1.1) has been studied long time ago when $\mathcal{L}_{K}=(-\Delta)^{1 / 2}$. They are of interest e.g. in fracture mechnics, see [15] and the referenes therein. The general case for $(-\Delta)^{s}, s \in(0,1)$, has been considered only in the recent years and it is by now merely well understood when $u, V, f \in L^{\infty}\left(\mathbb{R}^{N}\right)$ and $\Omega$ a domain of class $C^{1, \gamma}$, for some $\gamma>0$. Indeed, in the case of the fractional Lapalcian $(a \equiv 1)$ and $\gamma=1$, the first Hölder regularity estiamte of $u / d^{s}$ in $\bar{\Omega}$ was obtained by Ros-Oton and Serra in [49]. They sharpened and generalized this result to translation invariant operators, even to fully nonlinear equations, in their subsequent papers [45, 47, 48. We refer the reader to the recent survey paper 46] for a detailed list of existing results. In the case where $V, a, f$ and $\Omega$ are of class $C^{\infty}$, we quote the works of Grubb, 30 32, where it is proved that $u / d^{s}$ is of class $C^{\infty}$, up to the closure of $\Omega$. Especially in [30], the enteries $V, f$ are also allowed to belong to some $L^{p}$ spaces, for some large $p$. More precisely, when $V \in C^{\infty}(\bar{\Omega})$ and $f \in L^{p}(\Omega)$, for some $p>N / s$, then provided $\Omega$ and $a$ are of class $C^{\infty}$, Grubb proved in [30] that $u / d^{s}$ is of class $C^{s-N / p}(\bar{\Omega})$.
Here, we prove sharp Hölder regularity estimates of $u / d^{s}$, for $\Omega$ an open set of class $C^{1, \gamma}$ and $V, f$ are in some spaces containing the Lebesgue space $L^{p}$ for $p>N / s$. Let us now recall the Morrey space which will be considered in the following of this paper. For $\beta \in[0,2 s)$, we define the Morrey space $\mathcal{M}_{\beta}$ by the set of functions $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{N}\right)$ such that

$$
\|f\|_{\mathcal{M}_{\beta}}:=\sup _{\substack{x \in \mathbb{R}^{N} \\ r \in(0,1)}} r^{\beta-N} \int_{B_{r}(x)}|f(y)| d y<\infty
$$

with $\mathcal{M}_{0}:=L^{\infty}\left(\mathbb{R}^{N}\right)$. Such spaces introduced by Morrey in 43, are suitable for getting Hölder regularity in the study of partial differential equations.

Let us now explain in an abstract form the insight in our consideration of the Morrey space. Indeed, given a function $g \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$, we put $g_{r, x_{0}}(x):=r^{2 s} g\left(r x+x_{0}\right)$, for $x_{0} \in \mathbb{R}^{N}$ and $r>0$.
For $\beta \geq 0$, we say that $g$ satisfies a Rescaled Translated Coercivity Property (RTCP, for short) of order $\beta$, if there exists a constant $C:=C(g, N, s, \beta)>0$ such that for all $x_{0} \in \mathbb{R}^{N}$ and $r \in(0,1)$, we have

$$
\begin{equation*}
C \int_{\mathbb{R}^{N}}\left|g_{r, x_{0}}(x)\right| v^{2}(x) d x \leq r^{2 s-\beta}\|v\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \quad \text { for all } v \in H^{s}\left(\mathbb{R}^{N}\right) \tag{1.3}
\end{equation*}
$$

Then what we will prove, for solutions $u$ to (1.1) when $\mathcal{L}_{K}$ is (up to a scaling) close to $(-\Delta)_{a}^{s}$, are the following implications:

$$
\begin{array}{lll}
f, V \text { satisfy a RTCP of order } \beta \in[0,2 s) & \Longrightarrow & u \in C_{l o c}^{\min (1,2 s-\beta)-\varepsilon}(\Omega),  \tag{1.4}\\
f, V \text { satisfy a RTCP of order } \beta \in[0,2 s) & \Longrightarrow & u \in C_{l o c}^{\min (s, 2 s-\beta)-\varepsilon}(\bar{\Omega}),
\end{array}
$$

for every $\varepsilon>0$ and $\Omega$ an open set with $C^{1}$ boundary. For higher order regularity, under some regularity assumptions on $K$ and $\Omega$, quantified by some parameter $\beta^{\prime} \in[0,2 s)$, we obtain

$$
\begin{array}{ll}
f, V \text { satisfy a RTCP of order } \beta \in(0,2 s) & \Longrightarrow u \in C_{l o c}^{2 s-\max \left(\beta, \beta^{\prime}\right)}(\Omega), \\
f, V \text { satisfy a RTCP of order } \beta \in(0, s) & \Longrightarrow u / d^{s} \in C_{l o c}^{s-\max \left(\beta, \beta^{\prime}\right)}(\bar{\Omega}), \tag{1.5}
\end{array}
$$

provided $2 s-\max \left(\beta, \beta^{\prime}\right) \neq 1$. Here and in the following, it will be understood that $C^{\nu}:=C^{1, \nu-1}$ if $\nu \in(1,2)$. It is not difficult to see that functions $g$ satisfying a RTCP of order $\beta$ belongs to $\mathcal{M}_{\beta}$. On the other hand the converse, which is not trivial, also holds true, and in fact, we will prove a more general inequality for the Kato class of functions which could be of independent interest, see Lemma 2.3 below.

Since our results are already new for $(-\Delta)_{a}^{s}$, we state first simpler versions of our main results, and postponed the generalization to $\mathcal{L}_{K}$ in Section 1.1 below. To do so, we need to recall the distributional domain of the operator $(-\Delta)_{a}^{s}$. It is given by $\mathcal{L}_{s}^{1}$, the set of functions $u \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ such that $\|u\|_{\mathcal{L}_{s}^{1}}:=\int_{\mathbb{R}^{N}} \frac{|u(x)|}{1+|x|^{N+2 s}} d x<\infty$.

Theorem 1.1. Let $s \in(0,1), \beta \in(0,2 s)$ and a satisfy (1.2). Let $\Omega \subset \mathbb{R}^{N}$ be an open set of class $C^{1, \gamma}, \gamma>0$, in a neighborhood of $0 \in \partial \Omega$. Let $u \in H^{s}\left(B_{1}\right) \cap \mathcal{L}_{s}^{1}$ and $f, V \in \mathcal{M}_{\beta}$ be such that

$$
(-\Delta)_{a}^{s} u+V u=f \quad \text { in } \Omega
$$

(i) Then for every $\Omega_{1} \subset \subset \Omega \cap B_{1}$, there exists a constant $C>0$ such that

$$
\|u\|_{C^{2 s-\beta}\left(\Omega_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{1}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) .
$$

(ii) If $\beta \in(0, s)$, and $u \equiv 0$ in $\Omega^{c}$, then there exist some constants $C, \varrho>0$ such that

$$
\left\|u / d^{s}\right\|_{C^{\min (\gamma, s-\beta)}\left(B_{e} \cap \bar{\Omega}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{1}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

where $d(x)=\operatorname{dist}\left(x, \Omega^{c}\right)$. The constants $C$ and $\varrho$ above, only depend on $s, N, \beta, \gamma, \Lambda, \Omega, \Omega_{1}$ and $\|V\|_{\mathcal{M}_{\beta}}$.
Provided $u, V, f \in L^{\infty}\left(\mathbb{R}^{N}\right)$, by letting $\beta \searrow 0$, we recover the boundary regularity in 45 for $C^{1, \gamma}$ domains and partly the one in 47] for $C^{1,1}$ domains. We mention that in 47, a weaker ellipticity assumption (second condition in (1.2)) was considered.
Obviously if $f \in L^{p}\left(\mathbb{R}^{N}\right)$, with $p>1$, then $f \in \mathcal{M}_{\frac{N}{p}}$. For the strict inclusion of Lebesgue spaces in Morrey spaces, see e.g. [18. An immediate consequence of Theorem 1.1 is therefore the following result.
Corollary 1.2. Let $s \in(0,1)$ and a satisfy (1.2). Let $\Omega \subset \mathbb{R}^{N}$ be an open set of class $C^{1, \gamma}, \gamma>0$, in a neighborhood of $0 \in \partial \Omega$. Let $f, V \in L^{p}\left(B_{1}\right)$, for some $p>\frac{N}{s}$, and $u \in H^{s}\left(B_{1}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
(-\Delta)_{a}^{s} u+V u=f \quad \text { in } \Omega \quad \text { and } \quad u=0 \quad \text { in } \Omega^{c}
$$

Then there exist positive constants $C, \varrho>0$ such that

$$
\left\|u / d^{s}\right\|_{C^{\min (\gamma, s-N / p)}\left(B_{e} \cap \bar{\Omega}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{1}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{L^{p}\left(B_{1}\right)}\right)
$$

The constants $C$ and $\varrho$ depend only on $s, N, \gamma, p, \Lambda, \Omega$ and $\|V\|_{L^{p}\left(B_{1}\right)}$.
As mentioned earlier, we recall that the boundary regularity in Corollary 1.2 was known only when $a \in C^{\infty}\left(S^{N-1}\right)$ and $\Omega$ of class $C^{\infty}$, see 30 . In the classical case of the Laplace operator, the corresponding result of Corollary 1.2 is that $u$ is of class $C^{1, \min (\gamma, 1-N / p)}$ up to the boundary, see [28]. We note that interior and boundary Harnack inequalities for the operator $(-\Delta)^{s}+V$, with $V$ in the Kato class of potentials (larger than the Morrey space) and $\Omega$ a Lipschitz domain have been proven in [6, 53. In [19, we shall provide an explicit modulus of continuity for solutions to (1.1), when $V$ and $f$ belong to the Kato class of potentials.
1.1. Nonlocal operators with possibly continuous coefficients. In the following, for a function $b \in L^{\infty}\left(S^{N-1}\right)$, we define $\mu_{b}(x, y)=|x-y|^{-N-2 s} b((x-y) /|x-y|)$ ) for every $x \neq y \in \mathbb{R}^{N}$.

Let $\kappa>0$ be a positive constant and $\lambda: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow\left[0, \kappa^{-1}\right]$. We consider the class of kernels $K: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow[0,+\infty]$ satisfying the following properties:
(i) $K(x, y)=K(y, x) \quad$ for all $x \neq y \in \mathbb{R}^{N}$,
(ii) $\kappa \mu_{1}(x, y) \leq K(x, y) \leq \frac{1}{\kappa} \mu_{1}(x, y) \quad$ for all $x \neq y \in \mathbb{R}^{N}$,
(iii) $\left|K(x, y)-\mu_{b}(x, y)\right| \leq \lambda(x, y) \mu_{1}(x, y) \quad$ for all $x \neq y \in B_{2}$.

The class of kernels satisfying (1.6) is denoted by $\mathscr{K}(\lambda, b, \kappa)$.
Let $\Omega \subset \mathbb{R}^{N}$ be an open set and let $f, V \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$. For $K$ satisfying (1.6)(i)-(ii), we say that $u \in H_{\text {loc }}^{s}(\Omega) \cap \mathcal{L}_{s}^{1}$ is a (weak) solution to

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } \Omega
$$

if $u V \in L_{l o c}^{1}(\Omega)$ and for all $\psi \in C_{c}^{\infty}(\Omega)$, we have

$$
\frac{1}{2} \int_{\mathbb{R}^{2 N}}(u(x)-u(x+y))(\psi(x)-\psi(x+y)) K(x, x+y) d x d y+\int_{\mathbb{R}^{N}} V(x) u(x) \psi(x) d x=\int_{\mathbb{R}^{N}} f(x) \psi(x) d x
$$

The class of operator $\mathcal{L}_{K}$ corresponding to the kernels $K$ satisfying (1.6)(i)-(ii) can be seen as the nonlocal version of operators in divergence form in the classical case. Here we obtain regularity estimates for $K \in \mathscr{K}(\lambda, a, \kappa)$ provided $\lambda$ is small and $a$ satisfies 1.2. We thus include, in particular, nonlocal operators with "continuous" coefficients. The meaning of continuous coefficients for nonlocal operators might be awkward, since one is dealing with kernels which are not finite at the points $x=y$. Using polar coordinates, we can depict an encoded limiting operator which is nothing but the anisotropic fractional Laplacian.
In view of Remark 2.1 below, all results stated below remains valid if we consider kernels $\widetilde{K}: \mathbb{R}^{N} \times$ $\mathbb{R}^{N} \rightarrow[0, \infty]$ (with possibly compact support) satisfying

$$
\begin{array}{cc}
(i) \widetilde{K}(x, y)=\widetilde{K}(y, x) & \text { for all } x \neq y \in \mathbb{R}^{N} \\
\left(i i^{\prime}\right) \kappa \mu_{1}(x, y) \leq \widetilde{K}(x, y) & \text { for all } x \neq y \in B_{2}  \tag{1.7}\\
\left(i i^{\prime \prime}\right) \widetilde{K}(x, y) \leq \frac{1}{\kappa} \mu_{1}(x, y) & \text { for all } x \neq y \in \mathbb{R}^{N}
\end{array}
$$

This is due to the fact that the regularity theory of the operators $\mathcal{L}_{\widetilde{K}}$ is included in those of the form $\mathcal{L}_{K}+V$, with $K$ satisfying (2.2) $(i)-(i i)$, for some potential $V$ of class $C^{\infty}$.

We introduce $\widetilde{\mathscr{K}}(\kappa)$, the class of kernels $K$, satisfying (1.7) and such that the map

$$
\mathbb{R}^{N} \times(0, \infty) \times S^{N-1} \rightarrow \mathbb{R}, \quad(x, r, \theta) \mapsto r^{N+2 s} K(x, x+r \theta)
$$

has an extension $\widetilde{\lambda}_{K}: \mathbb{R}^{N} \times[0, \infty) \times S^{N-1} \rightarrow \mathbb{R}$ that is continuous in the variables $x, r$ on $B_{2} \times\{0\}$. That is, for every $x_{0} \in B_{2}$, we have $\left|\widetilde{\lambda}_{K}(x, r, \theta)-\widetilde{\lambda}_{K}\left(x_{0}, 0, \theta\right)\right| \rightarrow 0$ as $\left|x-x_{0}\right|+r \rightarrow 0$.

Now for $K \in \widetilde{\mathscr{K}}(\kappa)$ and $x_{0} \in B_{2}$ let us suppose that

$$
\sup _{\theta \in S^{N-1}}\left|\widetilde{\lambda}_{K}(x, r, \theta)-\widetilde{\lambda}_{K}\left(x_{0}, 0, \theta\right)\right| \rightarrow 0 \quad \text { as }\left|x-x_{0}\right|+r \rightarrow 0
$$

and consider the rescaled kernel around $x_{0}$, given by $K_{\rho, x_{0}}(x, y):=\rho^{N+2 s} K\left(\rho x+x_{0}, \rho y+x_{0}\right)$. Then we can show that, provided $\rho$ is small, $K_{\rho, x_{0}}$ satisfies (1.6) (iii) with $b(\theta)=\widetilde{\lambda}_{K}\left(x_{0}, 0, \theta\right)$, for some function function $\lambda_{\rho}$, satisfying $\left\|\lambda_{\rho}\right\|_{L^{\infty}\left(B_{1} \times B_{1}\right)} \rightarrow 0$ as $\rho \rightarrow 0$. Obviously, to expect the limiting kernel to be symmetric, we need to require that $\widetilde{\lambda}_{K}\left(x_{0}, 0, \theta\right)=\widetilde{\lambda}_{K}\left(x_{0}, 0,-\theta\right)$, for all $\theta \in S^{N-1}$. From this, it is natural to expect that $\mathcal{L}_{K}$ inherits certain regularity properties of $(-\Delta)_{a}^{s}$ whenever $K \in \mathscr{K}(\lambda, a, \kappa)$, provided $\lambda$ is small, in the spirit of Caffarelli [12] and Caffarelli-Silvestre [10]. This is the purpose of the next results, under mild regularity assumptions on $K$ and $\Omega$.

It is worth to mention that the kernels in $\widetilde{\mathscr{K}}(\kappa)$ appear for instance in the study of nonlocal mean curvature operator about a smooth hypersurface, see e.g. 3, 20. More generally, a typical example is when considering a $C^{1}$-change of coordinates $\Phi$ e.g. in the kernel $|x-y|^{-N-2 s}$ (could be defined on the product of hypersurfaces $\mathcal{M} \times \mathcal{M})$. The singular part of the new kernel is then given by $K_{\Phi}(x, y)=|\Phi(x)-\Phi(y)|^{-N-2 s}$, for some local diffeomorphism $\Phi \in C^{1}\left(B_{4} ; \mathbb{R}^{N}\right)$. In this case,

$$
\tilde{\lambda}_{K_{\Phi}}(x, r, \theta)=\left|\int_{0}^{1} D \Phi(x+r \tau \theta) \theta d \tau\right|^{-N-2 s}
$$

and thus $K_{\Phi} \in \widetilde{\mathscr{K}}(\kappa)$, for some $\kappa>0$. As a consequence, $\widetilde{\lambda}_{K_{\Phi}}(x, 0, \theta)=|D \Phi(x) \theta|^{-N-2 s}$, which is even in $\theta$. We refer to Section 7.1 below in a more general setting. Thanks to the à priori estimates that we are about to state below, we shall prove in 20 optimal regularity results paralleling the regularity theory for elliptic equations in divergence form, with regular coefficients, and provide applications in the study of nonlocal geometric problems.

To obtain interior regularity in (1.5), we need to care on the kernels $K$ for which the action of $\mathcal{L}_{K}$ on affine functions can be quantified. In this respect, some regularity on $K$ is required. More precisely, for $K \in \mathscr{K}(\lambda, a, \kappa)$ and $x \in \mathbb{R}^{N}$, we define

$$
\begin{equation*}
j_{o, K}(x):=(2 s-1)_{+} P V \int_{\mathbb{R}^{N}} y\{K(x, x+y)-K(x, x-y)\} d y \tag{1.8}
\end{equation*}
$$

where $\ell_{+}:=\max (\ell, 0)$, for $\ell \in \mathbb{R}$. Letting

$$
\begin{equation*}
\widetilde{\lambda}_{o, K}(x, r, \theta)=\frac{1}{2}\left\{\widetilde{\lambda}_{K}(x, r, \theta)-\widetilde{\lambda}_{K}(x, r,-\theta)\right\} \tag{1.9}
\end{equation*}
$$

we see that

$$
j_{o, K}(x)=2(2 s-1)_{+} \int_{S^{N-1}} \theta\left\{P V \int_{0}^{\infty} r^{-2 s} \widetilde{\lambda}_{o, K}(x, r, \theta) d r\right\} d \theta
$$

The main regularity assumption we make on $K$ is that $j_{o, K}$ is locally in $\mathcal{M}_{\beta^{\prime}}$, in the sense that $\varphi_{2} j_{o, K} \in \mathcal{M}_{\beta^{\prime}}$, for some $\beta^{\prime}=\beta^{\prime}(K) \in[0,2 s)$. Here and in the following, $\varphi_{R} \in C_{c}^{\infty}\left(B_{2 R}\right)$, with $\varphi_{R} \equiv 1$ on $B_{R}$. Since, $\beta^{\prime}$ depends on $K$, the main point will be to obtain regularity estimate by constants independent in $\beta^{\prime}$. We observe that, for example, for a kernel $K$ such that $\widetilde{\lambda}_{o, K}(x, r, \theta) \leq r^{\alpha+(2 s-1)_{+}}$, for $r \in(0,1)$ and for some $\alpha>0$, then $\varphi_{2} j_{o, K} \in L^{\infty}\left(\mathbb{R}^{N}\right)=\mathcal{M}_{0}$. Of course if $s \in(0,1 / 2]$ such additional regularity assumption on $K$ is unnecessary. This is also the case if $\mathcal{L}_{K}$ is a translation invariant, i.e. $K(x, y)=k(x-y)$, for some even function $k: \mathbb{R}^{N} \rightarrow \mathbb{R}$.

Our main result for interior regularity reads as follows.
Theorem 1.3. Let $s \in(0,1)$ and $\beta, \delta \in(0,2 s)$. Let a satisfy (1.2). Let $K \in \mathscr{K}(\lambda, a, \kappa)$ and assume that $j_{o, K}$ defined in (1.8), satisfies

$$
\left\|\varphi_{2} j_{o, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}
$$

for some $c_{0}$ and $\beta^{\prime}=\beta^{\prime}(K) \in[0,2 s-\delta)$. Let $f, V \in \mathcal{M}_{\beta}$ and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ be such that

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} .
$$

Then, provided $2 s-\max \left(\beta, \beta^{\prime}\right) \neq 1$, there exist $C, \varepsilon_{0}>0$, only depending only on $N, s, \beta, \Lambda, \kappa, c_{0}, \delta$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)} \leq \varepsilon_{0}$, we have

$$
\|u\|_{C^{2 s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Moreover if $2 s \leq 1$, then we can let $\beta^{\prime}=0$.
As a consequence, we have the following result.
Corollary 1.4. Let $s \in(0,1), \beta, \delta \in(0,2 s)$ and $\kappa>0$. Let $K \in \widetilde{\mathscr{K}}(\kappa)$ satisfy: for every $x_{1}, x_{2} \in B_{2}$, $r \in(0,2), \theta \in S^{N-1}$,

- $\left|\widetilde{\lambda}_{K}\left(x_{1}, r, \theta\right)-\widetilde{\lambda}_{K}\left(x_{2}, 0, \theta\right)\right| \leq \tau\left(\left|x_{1}-x_{2}\right|+r\right)$;
- $\widetilde{\lambda}_{K}\left(x_{1}, 0, \theta\right)=\widetilde{\lambda}_{K}\left(x_{1}, 0,-\theta\right)$,
for some modulus of continuity $\tau \in L^{\infty}\left(\mathbb{R}_{+}\right)$, with $\tau(t) \rightarrow 0$ as $t \rightarrow 0$. Assume that $j_{o, K}$ defined in (1.8), satisfies

$$
\left\|\varphi_{2} j_{o, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}
$$

for some $c_{0}$ and $\beta^{\prime}=\beta^{\prime}(K) \in[0,2 s-\delta)$. Let $f, V \in \mathcal{M}_{\beta}$ and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ be such that

$$
\begin{equation*}
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} . \tag{1.10}
\end{equation*}
$$

Then, provided $2 s-\max \left(\beta, \beta^{\prime}\right) \neq 1$, there exists $C>0$, only depending on $N, s, \beta, \delta, \kappa, c_{0}, \tau$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\|u\|_{C^{2 s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Moreover if $2 s \leq 1$, then we can let $\beta^{\prime}=0$.
It is natural to expect that under some Hölder regularity assumption on $\widetilde{\lambda}_{K}$ and on the entries, solutions are in fact classical. Indeed we have.
Theorem 1.5. Let $s \in(0,1), \kappa>0$ and let $K \in \widetilde{\mathscr{K}}(\kappa)$ satisfy:

- for every $x_{1}, x_{2} \in \mathbb{R}^{N}, r_{1}, r_{2} \in[0,2), \theta \in S^{N-1}$,

$$
\left|\widetilde{\lambda}_{K}\left(x_{1}, r_{1}, \theta\right)-\widetilde{\lambda}_{K}\left(x_{2}, r_{2}, \theta\right)\right| \leq c_{0}\left(\left|x_{1}-x_{2}\right|^{\alpha}+\left|r_{1}-r_{2}\right|^{\alpha}\right)
$$

- for every $x_{1}, x_{2} \in B_{2}, r \in[0,2), \theta \in S^{N-1}$, we have $\widetilde{\lambda}_{o, K}\left(x_{1}, 0, \theta\right)=0$ and

$$
\left|\widetilde{\lambda}_{o, K}\left(x_{1}, r, \theta\right)-\widetilde{\lambda}_{o, K}\left(x_{2}, r, \theta\right)\right| \leq c_{0} \min \left(\left|x_{1}-x_{2}\right|^{\alpha+(2 s-1)_{+}}, r^{\alpha+(2 s-1)_{+}}\right)
$$

for some constants $\alpha, c_{0}>0$, where $\widetilde{\lambda}_{o, K}$ is given by (1.9). Let $f \in C^{\alpha}\left(B_{2}\right)$ and $v \in H^{s}\left(B_{2}\right) \cap C^{\alpha}\left(\mathbb{R}^{N}\right)$ be such that

$$
\mathcal{L}_{K} v=f \quad \text { in } B_{2}
$$

Then there exists $\bar{\alpha}>0$ only depending on s, $N, c_{0}, \kappa$ and $\alpha$, such that for all $\beta \in(0, \bar{\alpha})$, with $2 s+\beta \notin \mathbb{N}$,

$$
\begin{equation*}
\|v\|_{C^{2 s+\beta}\left(B_{1}\right)} \leq C\left(\|v\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}+\|f\|_{C^{\beta}\left(B_{2}\right)}\right) \tag{1.11}
\end{equation*}
$$

for some constant $C$ depending only on $s, N, c_{0}, \kappa, \alpha$ and $\beta$.
We note that the $C^{\beta}\left(\mathbb{R}^{N}\right)$-norm of $v$ in (1.11) can be replaced with $\|v\|_{L^{2}\left(B_{2}\right)}+\|v\|_{\mathcal{L}_{s}^{1}}$, provided, we require Hölder regularity of $\widetilde{\lambda}_{K}$ in the variable $\theta$ i.e. $\left\|\widetilde{\lambda}_{K}\right\|_{C^{\alpha}\left(\mathbb{R}^{N} \times[0,2) \times S^{N-1}\right)} \leq c_{0}$.

We now turn to our boundary regularity estimates in (1.5). In this case, it is important to consider those kernels $K$ for which $\mathcal{L}_{K} d^{s}$ can be quantified. Here our assumption is that $\mathcal{L}_{K} d^{s}$ is given by a function in $\mathcal{M}_{\beta^{\prime}}$, for some $\beta^{\prime} \in[0, s)$. To be more precise, we consider all kernel $K$ for which, there exist $\beta^{\prime}=\beta^{\prime}(\Omega, K) \in[0, s)$ and a function $g_{\Omega, K} \in \mathcal{M}_{\beta^{\prime}}$ such that

$$
\begin{equation*}
\mathcal{L}_{K}\left(\varphi_{2} d^{s}\right)=g_{\Omega, K} \quad \text { in the weak sense in } B_{r_{0}} \cap \Omega \tag{1.12}
\end{equation*}
$$

where $r_{0}>0$, only depends on $\Omega$, is such that $\varphi_{2} d^{s} \in H^{s}\left(B_{r_{0}}\right) \cap \mathcal{L}_{s}^{1}$. We note that $g_{\Omega, K}$ might be singular near the boundary, since we are considering only domains of class $C^{1, \gamma}$. In fact, see [45], for $\gamma \neq s$ then $\left|(-\Delta)_{a}^{s} d^{s}(x)\right| \leq C d^{(\gamma-s)}+(x)$ for every $x \in B_{r_{0}} \cap \Omega$, for some $r_{0}$, only depending on $\Omega$. This, in particular, shows that there exists a $g_{\Omega, \mu_{a}} \in \mathcal{M}_{(s-\gamma)_{+}}$satisfying (1.12). We note that (1.12) encode both the regularity of $K$ and of $\Omega$.

Our next main result is the following.
Theorem 1.6. Let $s \in(0,1), \beta, \delta \in(0, s)$ and $\Omega$ an open set of class $C^{1, \gamma}, \gamma>0$, near $0 \in \partial \Omega$. Let a satisfy (1.2), for some $\Lambda>0$. Let $K \in \mathscr{K}(\lambda, a, \kappa)$ satisfy (1.12), with

$$
\left\|g_{\Omega, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}
$$

for some $\beta^{\prime}=\beta^{\prime}(\Omega, K) \in[0, s-\delta)$ and $c_{0}>0$. Let $f, V \in \mathcal{M}_{\beta}$, and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ be such that

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } \Omega \quad \text { and } \quad u=0 \quad \text { in } \Omega^{c} .
$$

Then there exist $C, \varrho>0$, only depending only on $N, s, \beta, \Lambda, \kappa, \Omega, c_{0}, \delta$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)} \leq \varepsilon_{0}$, we have

$$
\left\|u / d^{s}\right\|_{C^{s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{e} \cap \bar{\Omega}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) .
$$

In the case of uniformly continuous coefficient also, we have the following boundary regularity estimates.

Corollary 1.7. Let $s \in(0,1), \beta, \delta \in(0, s)$ and $\Omega$ an open set of class $C^{1, \gamma}, \gamma>0$, near $0 \in \partial \Omega$. For $\kappa>0$, let $K \in \widetilde{\mathscr{K}}(\kappa)$ satisfy: for every $x_{1}, x_{2} \in B_{2}, r \in(0,2), \theta \in S^{N-1}$,

- $\left|\tilde{\lambda}_{K}\left(x_{1}, r, \theta\right)-\tilde{\lambda}_{K}\left(x_{2}, 0, \theta\right)\right| \leq \tau\left(\left|x_{1}-x_{2}\right|+r\right) ;$
- $\tilde{\lambda}_{K}\left(x_{1}, 0, \theta\right)=\tilde{\lambda}_{K}\left(x_{1}, 0,-\theta\right)$,
for some function $\tau \in L^{\infty}\left(\mathbb{R}_{+}\right)$, with $\tau(t) \rightarrow 0$ as $t \rightarrow 0$. Suppose also that

$$
\left\|g_{\Omega, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}
$$

for some $\beta^{\prime}=\beta^{\prime}(\Omega, K) \in[0, s-\delta)$ and $c_{0}>0$. Let $f, V \in \mathcal{M}_{\beta}$, and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ be such that

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } \Omega \quad \text { and } \quad u=0 \quad \text { in } \Omega^{c} .
$$

Then there exist $C, \varrho>0$, only depending on $N, s, \beta, \tau, \kappa, \Omega, c_{0}, \delta$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\left\|u / d^{s}\right\|_{C^{s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{e} \cap \bar{\Omega}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) .
$$

As an application of the above result together with a global diffeomorphism that locally flatten the boundary $\partial \Omega$ near 0 , we get the following

Theorem 1.8. Let $s \in(0,1), \beta \in(0, s)$ and $\Omega$ an open set of class $C^{1,1}$, near $0 \in \partial \Omega$. For $\kappa>0$, let $K \in \widetilde{\mathscr{K}}(\kappa)$ satisfy:

- $\left\|\widetilde{\lambda}_{K}\right\|_{C^{s+\delta}\left(B_{2} \times[0,2) \times S^{N-1}\right)} \leq c_{0}$;
- $\widetilde{\lambda}_{K}(x, 0, \theta)=\widetilde{\lambda}_{K}(x, 0,-\theta)$, for every $x \in B_{2}$, and $\theta \in S^{N-1}$,
for some $\delta, c_{0}>0$. Let $f, V \in \mathcal{M}_{\beta}$, and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ be such that

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } \Omega \quad \text { and } \quad u=0 \quad \text { in } \Omega^{c} .
$$

Then there exist $C, \varrho>0$, only depending on $N, s, \beta, \tau, \kappa, \Omega, c_{0}, \delta$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\left\|u / d^{s}\right\|_{C^{s-\beta}\left(B_{e} \cap \bar{\Omega}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) .
$$

The proof of Theorem 1.6 and Theorem 1.3 are based on some blow-up analysis argument, where normalized, rescaled and translated sequence of a solution to a PDE satisfy certain growth control and converges to a solution on a symmetric space, so that Liouville-type results allow to calssify the limiting solutions. Here, we are inpired by the work of Serra in [52, see also [45,47, 48, 51 for boundary regularity estimates for translation invariant nonlocal operators. Note that in the aforementioned papers, since entries and solutions are in $L^{\infty}$, the use of barriers to get à priori pointwise estimates and Arzelà-Ascoli compactness theorems were the main tools to carry out their blow-up analysis. In our situation, it is clear that there is no hope of using such tools. Our argument will be based on the estimate of the $L^{2}$-average mean oscillation of $u$ to get à priori pointwise estimates. Indeed, to prove (1.4), we show the growth estimates

$$
\begin{equation*}
\sup _{z \in B_{1}}\left\|u-(u)_{B_{r}(z)}\right\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+\min (1,2 s-\beta)-\varepsilon} \tag{1.13}
\end{equation*}
$$

with $(u)_{B_{r}(z)}:=\frac{1}{\left|B_{r}\right|} \int_{B_{r}(z)} u(x) d x$, and

$$
\begin{equation*}
\sup _{z \in B_{1} \cap \partial \Omega}\|u\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+\min (s, 2 s-\beta)-\varepsilon}, \tag{1.14}
\end{equation*}
$$

for interior regularity and boundary regularity, respectively. The use of Caccioppoli-type estimates, the rescaled-translated-coercivity condition (1.3) and Liouville-type theorems for semi-bounded nonlocal operators are crucial to carry out the argument. Note that (1.13) always implies $C^{\min (1,2 s-\beta)-\varepsilon}$ estimates. On the other hand coupling (1.14) with interior estimates yield $C^{\min (s, 2 s-\beta)-\varepsilon}$ estimate up to the boundary.
To prove Theorem 1.3 we show an expansion of the form

$$
\begin{equation*}
\left|u(x)-u(z)-(2 s-1)_{+} T(z) \cdot(x-z)\right| \leq C|x-z|^{2 s-\max \left(\beta, \beta^{\prime}\right)} \quad \text { for every } x, z \in B_{1} \tag{1.15}
\end{equation*}
$$

with $\|T\|_{L^{\infty}\left(B_{1}\right)} \leq C$, while for Theorem 1.6

$$
\begin{equation*}
\left\|u-\psi(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+2 s-\max \left(\beta, \beta^{\prime}\right)} \quad \text { for every } z \in B_{1} \cap \partial \Omega \text { and } r \in\left(0, r_{0}\right) \tag{1.16}
\end{equation*}
$$

with $\|\psi\|_{L^{\infty}\left(B_{1} \cap \partial \Omega\right)} \leq C$ and the constant $C$ does not depend on $\beta^{\prime}$. Recall that $\ell_{+}:=\max (\ell, 0)$. Now using appropriate interior regularity estimates (1.4) is enough), we translate the $L^{2}$ estimates in (1.16) to a pointwise estimate which yields the conclusion of the theorem. The proof of (1.16) uses blow-up argument that allows to estimate the growth, in $r>0$, of the difference between $u$ and its $L^{2}\left(B_{r}(z)\right)$-projection on $\mathbb{R} d^{s}$, the one-dimensional space generated by $d^{s}$. Similarly the proof of (1.15) is achieved by estimating the growth, in $r>0$, of the difference between $u$ and its $L^{2}\left(B_{r}(z)\right)$-projection on the finite dimensional space of affine functions $\left\{t+(2 s-1)_{+} T \cdot(x-z): t \in \mathbb{R}\right.$ and $\left.T \in \mathbb{R}^{N}\right\}$.
We obtain Theorem 1.5 by freezing the radial variable $r$ at $r=0$ and by using the Shauder estimates for nontranslation invariant nonlocal operators of Serra 51. For that, we use our lower order term estimates Corollary 1.4 together with some approximation procedure and boundary regularity.
Related to this work is the one of Monneau in [44] where blow-up arguments were used to estimate the modulus of mean oscillation (in $L^{p}$ average) for solutions to the Laplace equation with Dini-continuous right hand sides.
Sharp boundary regularity in $C^{1, \gamma}$ domains and refined Harnack inequalities in $C^{1}$ domains are useful tools to obtain sharp regularity of the free boundaries in the study of nonlocal obstacle problems, see e.g. [8]. We believe that our result and arguments might be of interest in the study of obstacle problems with non smooth obstacles and for parabolic problems.

For the organization of the paper, we put in Section 2 some notations and preliminary results related to Kato class of potentials. Section 3 is devoted to interior and boundary $L^{2}$-growth estimates of solutions to (1.1) in $C^{1}$ domains. Statement (1.4), is proved in Section 4 Higher order boundary and interior regularity are proved in Section 5 and Section 6) respectively. The proof of the main results (in particular (1.5)) are gathered in Section 7 . Finally, we prove the Liouville theorems in Appendix 8 and we put some useful technical results in Appendix 9

## 2. Notations and Preliminary Results

In this paper, the ball centered at $z \in \mathbb{R}^{N}$ with radius $r>0$ is denoted by $B(z, r)$ and $B_{r}:=B_{r}(0)$. Here and in the following, we let $\varphi_{1} \in C_{c}^{\infty}\left(B_{2}\right)$ such that $\varphi_{1} \equiv 1$ on $B_{1}$ and $0 \leq \varphi_{1} \leq 1$ on $\mathbb{R}^{N}$. We put $\varphi_{R}(x):=\varphi(x / R)$. For $b \in L^{\infty}\left(S^{N-1}\right)$, we define $\mu_{b}(x, y)=|x-y|^{-N-2 s} b\left(\frac{x-y}{|x-y|}\right)$.

Recall that (see e.g. [21), if $b$ is even, there exists $C=C\left(N, s,\|b\|_{L^{\infty}\left(S^{N-1}\right)}\right)$, such that for all $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ and for every $x \in \mathbb{R}^{N}$, we have

$$
\begin{equation*}
\left|P V \int_{\mathbb{R}^{N}}(\psi(x)-\psi(y)) \mu_{b}(x, y) d y\right| \leq C \frac{\|\psi\|_{C^{2}\left(\mathbb{R}^{N}\right)}}{1+|x|^{N+2 s}} \tag{2.1}
\end{equation*}
$$

where $P V$ means that the integral is understood in the principle value sense. Throughout this paper, for the seminorm of the fractional Sobolev spaces, we adopt the notation

$$
[u]_{H^{s}(\Omega)}:=\left(\int_{\Omega \times \Omega}|u(x)-u(y)|^{2} \mu_{1}(x, y) d x d y\right)^{1 / 2}
$$

and for the Hölder seminorm, we write

$$
[u]_{C^{\alpha}(\Omega)}:=\sup _{x \neq y \in \Omega} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}},
$$

for $\alpha \in(0,1)$. Letting $u \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$, the mean value of $u$ in $B_{r}(z)$ is denoted by

$$
u_{B_{r}(z)}=(u)_{B_{r}(z)}:=\frac{1}{\left|B_{r}\right|} \int_{B_{r}(z)} u(x) d x
$$

2.1. The class of operators. In the following, it will be crucial to consider certain class of operators which we describe next.
2.1.1. Symmetric operators with bounded measurable coefficients. Firstly we will consider kernels $K$ : $\mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow(0, \infty]$ satisfying the following properties:
(i) $K(x, y)=K(y, x) \quad$ for all $x \neq y \in \mathbb{R}^{N}$,
(ii) $\kappa \mu_{1}(x, y) \leq K(x, y) \leq \kappa^{-1} \mu_{1}(x, y) \quad$ for all $x \neq y \in \mathbb{R}^{N}$, for some constant $\kappa>0$.

Let $\Omega \subset \mathbb{R}^{N}$ be an open set and let $f, V \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$. For $K$ satisfying (2.2), we say that $u \in$ $H_{l o c}^{s}(\Omega) \cap \mathcal{L}_{s}^{1}$ is a (weak) solution to

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } \Omega
$$

if $u V \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ and for all $\psi \in C_{c}^{\infty}(\Omega)$, we have

$$
\begin{equation*}
\frac{1}{2} \int_{\mathbb{R}^{2 N}}(u(x)-u(y))\left(\psi(x)-\psi(y) K(x, y) d x d y+\int_{\mathbb{R}^{N}} V(x) u(x) \psi(x) d x=\int_{\mathbb{R}^{N}} f(x) \psi(x) d x\right. \tag{2.3}
\end{equation*}
$$

Note, in fact, that for the first term in (2.3) to be finite, it is enough that $K$ satisfies only the upper bound in (2.2) (ii).

Remark 2.1. [Kernels with possible compact support] In many applications, it is important to consider kernels $K^{\prime}$ with possible compact support. This allows to treat kernels which are only locally symmetric and locally elliptic ( (2.4) below). As a matter of fact, we note that the regularity theory of the operators $\mathcal{L}_{K^{\prime}}$ is included in those of the form $\mathcal{L}_{K}+V$, with $K$ satisfying (2.2), for some potential $V$ of class $C^{\infty}$. Indeed, consider a kernel $\widetilde{K}: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow[0,+\infty]$ satisfying
(i) $\widetilde{K}(x, y)=\widetilde{K}(y, x) \quad$ for all $x \neq y \in \mathbb{R}^{N}$,

$$
\begin{array}{ll}
\left(i i^{\prime}\right) \kappa \mu_{1}(x, y) \leq \widetilde{K}(x, y) & \text { for all } x \neq y \in B_{2}  \tag{2.4}\\
\left(i i^{\prime \prime}\right) \widetilde{K}(x, y) \leq \frac{1}{\kappa} \mu_{1}(x, y) & \text { for all } x \neq y \in \mathbb{R}^{N}
\end{array}
$$

We define $\eta_{1}(x):=1-\varphi_{1}(x)$ and $\eta(x, y)=\eta_{1}(x)+\eta_{1}(y)$, which satisfies

$$
\eta(x, y) \geq \begin{cases}1 & \text { if }(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} \backslash\left(B_{2} \times B_{2}\right) \\ 0 & \text { if }(x, y) \in B_{2} \times B_{2}\end{cases}
$$

Then letting $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ be a weak solution (in the sense of (2.3)) to the equation

$$
\mathcal{L}_{\widetilde{K}} u+\widetilde{V} u=\widetilde{f} \quad \text { in } B_{2}
$$

we then have that

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{1 / 2},
$$

where $K(x, y)=\widetilde{K}(x, y)+\eta(x, y) \mu_{1}(x, y), V(x)=\widetilde{V}(x)-\int_{|y| \geq 1} \eta_{1}(y) \mu_{1}(x, y) d y$ and $f(x)=\widetilde{f}(x)-$ $\int_{|y| \geq 1} u(y) \eta_{1}(y) \mu_{1}(x, y) d y$. It is clear that $K$ satisfies (2.2), for a new constant $\kappa>0$. In addition $\|V-\widetilde{V}\|_{C^{k}\left(B_{1 / 2}\right)} \leq C(N, s, k)$ and $\|f-\widetilde{f}\|_{C^{k}\left(B_{1 / 2}\right)} \leq C(N, s, k)\|u\|_{\mathcal{L}_{s}^{1}}$, for all $k \in \mathbb{N}$.
2.1.2. Symmetric translation invariant operators with semi-bounded measurable coefficients. The class of operators we will consider next appears as limit of rescaled operators $\mathcal{L}_{K}$, for $K \in \mathscr{K}(\lambda, b, \kappa)$ (see Section 1.1). Let $\left(a_{n}\right)_{n}$ be a sequence of functions, satisfying (1.2). Then, up to a subsequence, it converges, in the weak-star topology of $L^{\infty}\left(S^{N-1}\right)$, to some $b \in L^{\infty}\left(S^{N-1}\right)$. It follows that $b$ is even on $S^{N-1}$ and satisfies

$$
\begin{equation*}
0<\Lambda \int_{S^{N-1}}\left|e_{1} \cdot \theta\right|^{2 s} d \theta \leq \inf _{\eta \in S^{N-1}} \int_{S^{N-1}}|\eta \cdot \theta|^{2 s} b(\theta) d \theta \quad \text { and } \quad\|b\|_{L^{\infty}\left(S^{N-1}\right)} \leq \frac{1}{\Lambda} \tag{2.5}
\end{equation*}
$$

For such function $b$, we denote by $L_{b}$ the corresponding operator, which is given by

$$
\begin{equation*}
L_{b} \psi(x):=P V \int_{\mathbb{R}^{N}}(\psi(x)-\psi(y)) \mu_{b}(x, y) d y \quad \text { for every } \psi \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right) \tag{2.6}
\end{equation*}
$$

where $P V$ means that the integral is in the principle value sense. Here also solutions $u \in H_{l o c}^{s}(\Omega) \cap \mathcal{L}_{s}^{1}$ to the equation $L_{b} u+V u=f$ in an open set $\Omega$ are functions satisfying (2.3) -replacing $K$ with $\mu_{b}$. The following result is concerned with limiting of a sequence of operators which are close to a translation invariant operator.

Lemma 2.2. Let $\left(a_{n}\right)_{n}$ be a sequence of functions, satisfying (1.2) and converging in the weak-star sense to some $b \in L^{\infty}\left(S^{N-1}\right)$. Let $\lambda_{n}: \mathbb{R}^{2 N} \rightarrow\left[0, \kappa^{-1}\right]$, with $\lambda_{n} \rightarrow 0$ pointwise on $\mathbb{R}^{N} \times \mathbb{R}^{N}$. Let $K_{n}$ be a symmetric kernel satisfying

$$
\left|K_{n}(x, y)-\mu_{a_{n}}(x, y)\right| \leq \lambda_{n}(x, y) \mu_{1}(x, y) \quad \text { for all } x \neq y \in \mathbb{R}^{N} \text { and for all } n \in \mathbb{N}
$$

If $\left(v_{n}\right)_{n}$ is a bounded sequence in $\mathcal{L}_{s}^{1} \cap H_{\text {loc }}^{s}\left(\mathbb{R}^{N}\right)$ such that $v_{n} \rightarrow v$ in $\mathcal{L}_{s}^{1}$, then
$\int_{\mathbb{R}^{N}} v(x) L_{b} \psi(x) d x=\frac{1}{2} \lim _{n \rightarrow \infty} \int_{\mathbb{R}^{2 N}}\left(v_{n}(x)-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \quad$ for all $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$.
Proof. Letting $w_{n}=v_{n}-v$, then direct computations give

$$
\begin{align*}
& \int_{\mathbb{R}^{N}} v(x) L_{b} \psi(x) d x-\frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(v_{n}(x)-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \\
& =\int_{\mathbb{R}^{N}} v(x)\left(L_{b}-L_{a_{n}}\right) \psi(x) d x+\frac{1}{2} \int_{\mathbb{R}^{2 N}}(v(x)-v(y))(\psi(x)-\psi(y))\left(\mu_{a_{n}}(x, y)-K_{n}(x, y)\right) d x d y \\
& -\frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \tag{2.7}
\end{align*}
$$

By eveness of $a_{n}$ and $b$, Fubini's theorem and a change of variable, we can write

$$
\left(L_{b}-L_{a_{n}}\right) \psi(x)=\int_{S^{N-1}}\left[\int_{0}^{\infty}(\psi(x-t \theta)+\psi(x+t \theta)-2 \psi(x)) t^{-1-2 s} d t\right]\left(b(\theta)-a_{n}(\theta)\right) d \theta
$$

Clearly, the function $\theta \mapsto \int_{0}^{\infty}(\psi(x-t \theta)+\psi(x+t \theta)-2 \psi(x)) t^{-1-2 s} d t$ is bounded on $S^{N-1}$ and thus belongs to $L^{1}\left(S^{N-1}\right)$. Therefore the sequence of functions $h_{n}(x):=\left(L_{b}-L_{a_{n}}\right) \psi(x)$ converges pointwise to zero on $\mathbb{R}^{N}$. Moreover by (2.1), we have that $\left|h_{n}(x)\right| \leq C_{\psi}(1+|x|)^{-N-2 s}$. Since $v \in \mathcal{L}_{s}^{1}$, it follows from the dominated convergence theorem that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} v(x)\left(L_{b}-L_{a_{n}}\right) \psi(x) d x=o(1) \quad \text { as } n \rightarrow \infty \tag{2.8}
\end{equation*}
$$

Next, we put $V_{n}(x, y):=\left|v(x)-v(y)\|\psi(x)-\psi(y)\| \mu_{a_{n}}(x, y)-K_{n}(x, y)\right|$. Pick $R>0$ such that $\operatorname{Supp} \psi \subset B_{R / 2}$. For $n$ large enough so that $B_{R} \subset B_{1 /\left(2 r_{n}\right)}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{2 N}} V_{n}(x, y) d x d y & \leq \int_{B_{R} \times B_{R}}\left|v(x)-v(y)\|\psi(x)-\psi(y)\| \mu_{a_{n}}(x, y)-K_{n}(x, y)\right| d x d y \\
& +2 \int_{B_{R}}|\psi(y)|\left[\int_{\mathbb{R}^{N} \backslash B_{R}}\left|v(x)-v(y) \| \mu_{a_{n}}(x, y)-K_{n}(x, y)\right| d x\right] d y
\end{aligned}
$$

Note that $y \mapsto|\psi(y)| \int_{\mathbb{R}^{N} \backslash B_{R}}|v(x)-v(y)|\left|\mu_{a_{n}}(x, y)-K_{n}(x, y)\right| d x$ is bounded and converges pointwise to zero, as $n \rightarrow \infty$. By the dominated convergence theorem, we then have that $\int_{\mathbb{R}^{2 N}} V_{n}(x, y) d x d y=$ $o(1)$, as $n \rightarrow \infty$, so that

$$
\begin{equation*}
\int_{\mathbb{R}^{2 N}}(v(x)-v(y))(\psi(x)-\psi(y))\left(\mu_{a_{n}}(x, y)-K_{n}(x, y)\right) d x d y=o(1) \quad \text { as } n \rightarrow \infty \tag{2.9}
\end{equation*}
$$

Since $w_{n}=v_{n}-v$ is bounded in $H_{l o c}^{s} \cap \mathcal{L}_{s}^{1}$ and $w_{n} \rightarrow 0$ in $\mathcal{L}_{s}^{1}$, by similar arguments as above, we get

$$
\int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)(\psi(x)-\psi(y))\left(\mu_{a_{n}}(x, y)-K_{n}(x, y)\right) d x d y=o(1) \quad \text { as } n \rightarrow \infty
$$

In addition, since $w_{n} \rightarrow 0$ in $\mathcal{L}_{s}^{1}$, by (2.1),

$$
\frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)(\psi(x)-\psi(y)) \mu_{a_{n}}(x, y) d x d y=\int_{\mathbb{R}^{N}} w_{n}(x) L_{a_{n}} \psi(x) d x=o(1) \quad \text { as } n \rightarrow \infty
$$

Combining the two estimates above, we conclude that

$$
\int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y=o(1) \quad \text { as } n \rightarrow \infty
$$

Using this, (2.9) and (2.8) in (2.7), we get the conclusion in the lemma.
2.2. Coercivity and Caccioppoli type inequality with Kato class potentials. For $s>0$, we let $\Gamma_{s}:=(-\Delta)^{-s}$ be the Riesz potential, which satisfy $(-\Delta)^{s} \Gamma_{s}=\delta_{0}$ in $\mathbb{R}^{N}$. Recall that for $N \neq 2 s$, $\Gamma_{s}(z)=c_{N, s}|z|^{2 s-N}$ and for $N=2 s, \Gamma_{s}(z)=c_{N, s} \log (|z|)$, for some normalization constant $c_{N, s}$. We consider the Kato class of functions given by

$$
\begin{equation*}
\mathcal{K}_{s}:=\left\{V \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right): \sup _{x \in \mathbb{R}^{N}} \int_{B_{1}(x)}|V(y)| \omega_{s}(|x-y|) d y<\infty\right\} \tag{2.10}
\end{equation*}
$$

where for $N \geq 2 s, \omega_{s}(|z|)=\left|\Gamma_{s}(z)\right|$ and if $2 s>N$, we set $\omega_{s} \equiv 1$. Here and in the following, for every $V \in \mathcal{K}_{s}$ and $r \in(0,1]$, we define

$$
\begin{equation*}
\eta_{V}(r):=\sup _{x \in \mathbb{R}^{N}} \int_{B_{r}(x)}|V(y)| \omega_{s}(|x-y|) d y \tag{2.11}
\end{equation*}
$$

The following compactness result will be useful in the following. We also note that it holds for all $s>0$, and in this case $[u]_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}:=\int_{\mathbb{R}^{N}}|\xi|^{2 s}|\widehat{u}(\xi)|^{2} d \xi$, for $u \in H^{s}\left(\mathbb{R}^{N}\right)$ with Fourier transform $\widehat{u}$.

Lemma 2.3. Let $s>0$ and $V \in \mathcal{K}_{s}$. Then, there exists a constant $c=c(N, s)>0$ such that for every $\delta \in(0,1]$, there exists an other constant $c_{\delta}=c(N, s, \delta)>0$ such that for every $u \in H^{s}\left(\mathbb{R}^{N}\right)$,

$$
\begin{equation*}
\left\||V|^{1 / 2} u\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq \eta_{V}(\delta)\left(c[u]_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+c_{\delta}\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right) \tag{2.12}
\end{equation*}
$$

Proof. For $r>0$, we consider the Bessel potential $G_{s, r}=\left(-\Delta+r^{-2}\right)^{-s / 2}$. See e.g. [29, Section 6.1.2], there exists a constant $c=c(N, s)>0$ such that

$$
\begin{equation*}
G_{s, 1}(x) \leq c \omega_{s}(|x|) \quad \text { for }|x| \leq 1 / 2 \quad \text { and } \quad G_{s, 1}(x) \leq c|x|^{-N-1+s} \exp (-|x| / 2) \quad \text { for }|x| \geq 1 / 2 \tag{2.13}
\end{equation*}
$$

where $\omega_{s}$ is defined in the beginning of this section.
Step 1: We assume that $V \in L^{\infty}\left(\mathbb{R}^{N}\right)$.
For $\delta>0$, we consider the operator $L: L^{2}\left(\mathbb{R}^{N}\right) \rightarrow L^{2}\left(\mathbb{R}^{N}\right)$ given by

$$
L v=|V|^{1 / 2}\left(-\Delta+\delta^{-2}\right)^{-s / 2} v
$$

We note that the adjoint of $L$ is given by $L^{*}=\left(-\Delta+\delta^{-2}\right)^{-s / 2}|V|^{1 / 2}$.
Claim: There exists $c=c(N, s)>0$ such that for every $\delta \in(0,1 / 2]$,

$$
\begin{equation*}
\left\|L L^{*} v\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq c \eta_{V}(\delta)^{2}\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \tag{2.14}
\end{equation*}
$$

By Hölder's inequality and using the fact that $G_{s, r}(x)=G_{s, 1}(x / r)$, we obtain

$$
\begin{aligned}
\left\|L L^{*} v\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} & =\left\||V|^{1 / 2}\left(-\Delta+\delta^{-2}\right)^{-s}|V|^{1 / 2} v\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}=\left\||V|^{1 / 2} G_{s, \delta} \star\left(V^{1 / 2} v\right)\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \\
& \leq \int_{\mathbb{R}^{N}}|V(x)|\left(\int_{\mathbb{R}^{N}}|V|^{1 / 2}(y)|v(y)| G_{s, \delta}((x-y) / \delta) d y\right)^{2} d x \\
& \leq \int_{\mathbb{R}^{N}}|V(x)|\left(\int_{\mathbb{R}^{N}}|V(y)| G_{s, \delta}((x-y) / \delta) d y \int_{\mathbb{R}^{N}}|v(z)|^{2} G_{s, \delta}((x-z) / \delta) d z\right) d x
\end{aligned}
$$

Using a change of variable and (2.13), for $x \in \mathbb{R}^{N}$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}|V(y)| G_{s, \delta}(x-y) d y & \leq c \eta_{V}(\delta)+\int_{\delta \leq|x-y|}|V(y)| G_{s, 1}((x-y) / \delta) d y \\
& \leq c \eta_{V}(\delta)+\sum_{i=1}^{\infty} \int_{i \delta \leq|x-y| \leq(i+1) \delta}|V(y)| G_{s, 1}((x-y) / \delta) d y \\
& \leq c \eta_{V}(\delta)+\sum_{i=1}^{\infty} \delta^{N} \int_{i \leq|z| \leq i+1}|V(\delta z+x)| G_{s, 1}(z) d z \\
& \leq c \eta_{V}(\delta)+C \sum_{i=1}^{\infty} i^{-N-1+s} \exp (-i / 2) \delta^{N} \int_{i \leq|z| \leq i+1}|V(\delta z+x)| d z
\end{aligned}
$$

For every fixed $i$, we cover the annulus $A_{i}:=\{i \leq|y| \leq i+1\}$ by $n(i)$ balls $B_{1}\left(z_{j}\right)$, with $z_{j} \in A_{i}$, $n(i) \leq C i^{N-1}$ and $C$ a positive constant only depending on $N$. Letting $\rho_{i}:=i^{-N-1+s} \exp (-i / 2)$, for every $x \in \mathbb{R}^{N}$ and $\delta \in(0,1 / 2]$, we then have

$$
\begin{aligned}
\int_{\mathbb{R}^{N}}|V(y)| G_{s, \delta}(x-y) d y & \leq c \eta_{V}(\delta)+C \sum_{i=1}^{\infty} \rho_{i} \sum_{j=1}^{n(i)} \delta^{N} \int_{\left|z-z_{j}\right| \leq 1}|V(\delta z+x)| d z \\
& =c \eta_{V}(\delta)+C \sum_{i=1}^{\infty} \rho_{i} \sum_{j=1}^{n(i)} \int_{\left|y-x-\delta z_{j}\right| \leq \delta}|V(y)| d z \\
& \leq c \eta_{V}(\delta)+C \sum_{i=1}^{\infty} \rho_{i} \sum_{j=1}^{n(i)} \omega_{s}(\delta)^{-1} \int_{\left|y-x-\delta z_{j}\right| \leq \delta}|V(y)| \omega_{s}\left(\left|y-x-\delta z_{j}\right|\right) d z \\
& \leq c \eta_{V}(\delta)\left(1+C \sum_{i=1}^{\infty} i^{-2+s} \exp (-i / 2)\right) \omega_{s}(\delta)^{-1} \leq c \eta_{V}(\delta)
\end{aligned}
$$

We then get, for every $\delta \in(0,1 / 2]$

$$
\left\|L L^{*} v\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq c \eta_{V}(\delta) \int_{\mathbb{R}^{N}}|V(x)| \int_{\mathbb{R}^{N}}|v(z)|^{2} G_{s, 1}((x-z) / \delta) d z d x \leq c^{2} \eta_{V}(\delta)^{2} \int_{\mathbb{R}^{N}}|v(z)|^{2} d z
$$

That is (2.14) as claimed.
Since $\left\|L L^{*}\right\|=\|L\|^{2}$, it then follows that

$$
\begin{equation*}
\|L v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq c \eta_{V}(\delta)\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \tag{2.15}
\end{equation*}
$$

Now given $u \in H^{s}\left(\mathbb{R}^{N}\right)$ and $\delta \in(0,1]$, we plug $v=\left(-\Delta+(\delta / 2)^{-2}\right)^{s / 2} u \in L^{2}\left(\mathbb{R}^{N}\right)$ in (2.15), and noting that $\|v\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq c\left([u]_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+(\delta / 2)^{-2}\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}\right)$, for some positive constant $c=c(N, s)$. This with the fact that that $\eta_{V}(\delta / 2) \leq \eta_{V}(\delta)$ give (2.12), if $V \in L^{\infty}\left(\mathbb{R}^{N}\right)$.

Step 2: For $V \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$, we consider $V_{k}=\min (|V|, k)$, for $k \in \mathbb{N}$. Thence since $\eta_{V_{k}} \leq \eta_{V}$, we get (2.12) by Fatou lemma.

The following energy estimate is a consequence of the above coercivity result and a nonlocal Caccioppoli-type inequality proved in an appendix, Section 9

Lemma 2.4. We consider $\Omega$ an open set with $0 \in \partial \Omega$ and $K$ satisfying (1.2). Let $v \in H^{s}\left(\mathbb{R}^{N}\right)$ and $V, f \in \mathcal{K}_{s}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} v+V v=f \quad \text { in } B_{2 R} \cap \Omega \quad \text { and } \quad v=0 \quad \text { in } B_{2 R} \cap \Omega^{c} . \tag{2.16}
\end{equation*}
$$

Then there exists $\bar{C}=\bar{C}(N, s, \kappa)>0$ such that for every $\varepsilon>0$ and every $\delta \in(0,1]$, there exists $C=C(\varepsilon, \delta, s, N, \kappa)$ such that

$$
\begin{aligned}
& \left\{\kappa-\varepsilon \bar{C}\left(1+\eta_{f}(\delta)\right)-\bar{C} \eta_{V}(\delta)\right\} \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y \leq C \eta_{f}(1)\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \\
& \quad+C\left(\eta_{V}(1)+\eta_{f}(1)+1\right) \int_{\mathbb{R}^{N}} \frac{R^{N}|v(x)|^{2}}{R^{N+2 s}+|x|^{N+2 s}} d x+C\left(\eta_{V}(1)+\eta_{f}(1)\right)\left\|v \varphi_{R}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}
\end{aligned}
$$

Proof. By Lemma 9.1 and (2.1), we get

$$
\begin{align*}
&(\kappa-\varepsilon) \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y=\int_{\mathbb{R}^{N}}\left|V(x) \| v(x) \varphi_{R}(x)\right|^{2} d x \\
&+\int_{\mathbb{R}^{N}}|f(x) \| v(x)| \varphi_{R}^{2}(x) d x+C_{\varepsilon} \int_{\mathbb{R}^{N}} \frac{R^{N}|v(x)|^{2}}{R^{N+2 s}+|x|^{N+2 s}} d x \tag{2.17}
\end{align*}
$$

Thanks to Lemma 2.3 (2.1) and the fact that $\int_{\mathbb{R}^{N}}\left(\varphi_{1}(x)-\varphi_{1}(y)\right)^{2} \mu_{1}(x, y) d y \leq C\left(1+|x|^{N-2 s}\right)$ for every $x \in \mathbb{R}^{N}$, we have

$$
\begin{align*}
\int_{\mathbb{R}^{N}}\left|V(x) \| v(x) \varphi_{R}(x)\right|^{2} d x \leq & \bar{C} \eta_{V}(\delta) \int_{\mathbb{R}^{2 N}}\left(\left(v \varphi_{R}\right)(x)-\left(v \varphi_{R}\right)(y)\right)^{2} \mu_{1}(x, y) d x d y \\
& +C \eta_{V}(\delta)\left\|v \varphi_{R}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \\
\leq & \bar{C} \eta_{V}(\delta) \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y \\
& +C \eta_{V}(\delta) \int_{\mathbb{R}^{N}} \frac{R^{N}|v(x)|^{2}}{R^{N+2 s}+|x|^{N+2 s}} d x+C \eta_{V}(\delta)\left\|v \varphi_{R}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \tag{2.18}
\end{align*}
$$

Similarly, by Young's inequality, (2.12) and (2.1), we get

$$
\begin{aligned}
& \int_{\mathbb{R}^{N}}\left|f ( x ) \left\|\left.\varphi_{R}(x)\right|^{2}|v(x)| d x \leq \varepsilon \int_{\mathbb{R}^{N}}\left|f(x) \| v(x) \varphi_{R}(x)\right|^{2} d x+C_{\varepsilon} \int_{\mathbb{R}^{N}}|f(x)| \varphi_{R}(x)^{2} d x\right.\right. \\
& \leq \varepsilon \bar{C} \eta_{f}(\delta) \int_{\mathbb{R}^{2 N}}\left(\left(v \varphi_{R}\right)(x)-\left(v \varphi_{R}\right)(y)\right)^{2} \mu_{1}(x, y) d x d y+C_{\varepsilon} \eta_{f}(\delta)\left\|v \varphi_{R}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \\
& \quad+C_{\varepsilon} \eta_{f}(\delta)\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \\
& \leq \varepsilon \bar{C} \eta_{f}(\delta) \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi_{R}^{2}(y) \mu_{1}(x, y) d x d y+C \eta_{f}(\delta) \int_{\mathbb{R}^{N}} \frac{R^{N}|v(x)|^{2}}{R^{N+2 s}+|x|^{N+2 s}} d x \\
& \quad+C \eta_{f}(\delta)\left\|v \varphi_{R}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2}+C_{\varepsilon} \eta_{f}(\delta)\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} .
\end{aligned}
$$

Using the above estimate and (2.18) in (2.17) and using the monotonicity of $\eta_{V}$ and $\eta_{f}$, we get the result.

We close this section with the following result.
Lemma 2.5. We consider $\Omega$ an open set with $0 \in \partial \Omega$ and $K$ satisfying (2.2). Let $v \in H_{l o c}^{s}\left(B_{2 R}\right) \cap \mathcal{L}_{s}^{1}$ and $V, f \in \mathcal{K}_{s}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} v+V v=f \quad \text { in } B_{2 R} \cap \Omega \quad \text { and } \quad v=0 \quad \text { in } B_{2 R} \cap \Omega^{c} . \tag{2.19}
\end{equation*}
$$

Then for every $\psi \in C_{c}^{\infty}\left(B_{R} \cap \Omega\right)$, we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{2 N}}(v(x)-v(y))(\psi(x)-\psi(y)) K(x, y) d x d y\right| & \leq C \eta_{V}(1)\left(\left\|v \varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right) \\
& +C \eta_{f}(1)\left(\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right)
\end{aligned}
$$

where $C>0$ is a constant, only depending on $N$ and $s$.

Proof. Testing the equation (2.19) with $\psi \in C_{c}^{\infty}\left(B_{R} \cap \Omega\right)$ and using Young's inequality, we get

$$
\begin{aligned}
& \frac{1}{2}\left|\int_{\mathbb{R}^{2 N}}(v(x)-v(y))(\psi(x)-\psi(y)) K(x, y) d x d y\right| \\
& \leq \int_{\mathbb{R}^{N}}\left|V(x)\left\|v(x) \psi(x)\left|\varphi_{R}(x) d x+\int_{\mathbb{R}^{N}}\right| f(x)\right\| \psi(x)\right| \varphi_{R}(x) d x \\
& \leq 2 \int_{\mathbb{R}^{N}}\left|V(x) \| v(x) \varphi_{R}(x)\right|^{2} d x+2 \int_{\mathbb{R}^{N}}|V(x)| \psi^{2}(x) d x+2 \int_{\mathbb{R}^{N}}|f(x)| \psi^{2}(x) d x+2 \int_{\mathbb{R}^{N}}|f(x)| \varphi_{R}^{2}(x) d x
\end{aligned}
$$

Hence using Lemma 2.3, we conclude that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^{2 N}}(v(x)-v(y))(\psi(x)-\psi(y)) K(x, y) d x d y\right| & \leq C \eta_{V}(1)\left(\left\|v \varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right) \\
& +C \eta_{f}(1)\left(\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\left\|\varphi_{R}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right)
\end{aligned}
$$

which finishes the proof.

## 3. Interior and boundary growth estimates

We recall the Morrey space already introduced in the first section, for $\beta \in[0,2 s)$, defined as

$$
\mathcal{M}_{\beta}:=\left\{f \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right):\|f\|_{\mathcal{M}_{\beta}}:=\sup _{r \in(0,1), x \in \mathbb{R}^{N}} r^{\beta-N} \int_{B_{r}(x)}|f(y)| d y<\infty\right\}
$$

Let $f \in \mathcal{M}_{\beta}$ and define $f_{r, x_{0}}(x)=r^{2 s} f\left(r x+x_{0}\right)$ for $x_{0} \in \mathbb{R}^{N}$ and $r>0$. Recalling (2.11), an important property of $\eta_{f}$ we will use frequently in the following is that, for every $x_{0} \in \mathbb{R}^{N}$ and $r \in(0,1]$, we have

$$
\begin{equation*}
\eta_{f_{r, x_{0}}}(1) \leq C\left\|f_{r, x_{0}}\right\|_{\mathcal{M}_{\beta}} \leq C r^{2 s-\beta}\|f\|_{\mathcal{M}_{\beta}} \tag{3.1}
\end{equation*}
$$

with $C$ a positive constant, only depending on $N, s$, and $\beta$. The first inequality in (3.1) can be easily checked by change of variables and using summations over annuli with small thickness. We note that (3.1) and Lemma 2.3) show that functions $V, f \in \mathcal{M}_{\beta}$ satisfy RTCP of order $\beta$ (see (1.3)) as mentioned in the first section.
3.1. Interior growth estimates for solutions to Schrödinger equations. The next result is merely classical but we add the proof for the sake of completeness.

Lemma 3.1. Let $u \in L_{l o c}^{2}\left(\mathbb{R}^{N}\right)$ and $\alpha>0$.
(i) Suppose that

$$
\begin{equation*}
\left\|u-u_{B_{\rho}}\right\|_{L^{2}\left(B_{\rho}\right)} \leq \bar{C} \rho^{N / 2+\alpha} \quad \text { for every } \rho \in[1, \infty) \tag{3.2}
\end{equation*}
$$

Then

$$
\left\|u-u_{B_{1}}\right\|_{L^{2}\left(B_{\rho}\right)} \leq C \bar{C} \rho^{N / 2+\alpha} \quad \text { for every } \rho \in[1, \infty)
$$

with $C$ depends only on $N$ and $\alpha$.
(ii) Suppose that 0 is a Lebesgue point of $u$ and

$$
\begin{equation*}
\left\|u-u_{B_{\rho}}\right\|_{L^{2}\left(B_{\rho}\right)} \leq \bar{C} \rho^{N / 2+\alpha} \quad \text { for every } \rho \in(0,1) \tag{3.3}
\end{equation*}
$$

Then

$$
\|u-u(0)\|_{L^{2}\left(B_{\rho}\right)} \leq C \bar{C} \rho^{N / 2+\alpha} \quad \text { for every } \rho \in(0,1)
$$

with $C$ depends only on $N$ and $\alpha$.

Proof. First, to prove ( $i$, we note that, for every $\rho \geq 1$,

$$
\left|B_{\rho}\right|^{1 / 2}\left|u_{B_{\rho}}-u_{B_{2 \rho}}\right| \leq\left\|u-u_{B_{\rho}}\right\|_{L^{2}\left(B_{\rho}\right)}+\left\|u-u_{B_{2 \rho}}\right\|_{L^{2}\left(B_{2 \rho}\right)} \leq 2 \bar{C} \rho^{N / 2+\alpha}
$$

Therefore, for $\rho=2^{m}$, with $m \geq 1$, we get

$$
\left|u_{B_{\rho}}-u_{B_{1}}\right| \leq \sum_{i=0}^{m-1}\left|u_{B_{2^{i}}}-u_{B_{2^{i+1}}}\right| \leq 2 \bar{C} \sum_{i=0}^{m-1} 2^{i \alpha} \leq C \bar{C} \rho^{\alpha}
$$

where $C$ is independent on $m, \rho$ and $u$. Next, if $m$ is the smallest integer for which, $2^{m-1} \leq \rho \leq 2^{m}$, then using (3.2) and the above estimate, we conclude that

$$
\left\|u-u_{B_{1}}\right\|_{L^{2}\left(B_{\rho}\right)} \leq\left\|u-u_{B_{2^{m}}}\right\|_{L^{2}\left(B_{2^{m}}\right)}+\left|B_{2^{m}}\right|^{1 / 2}\left|u_{B_{2^{m}}}-u_{B_{1}}\right| \leq C \bar{C} \rho^{N / 2+\alpha}
$$

For (ii), by assumption, we have

$$
\left|B_{\rho / 2}\right|^{1 / 2}\left|u_{B_{\rho}}-u_{B_{\rho / 2}}\right| \leq\left\|u-u_{B_{\rho}}\right\|_{L^{2}\left(B_{\rho}\right)}+\left\|u-u_{B_{\rho / 2}}\right\|_{L^{2}\left(B_{\rho / 2}\right)} \leq 2 \bar{C} \rho^{N / 2+\alpha}
$$

Therefore

$$
\left|u_{B_{\rho}}-u(0)\right| \leq \sum_{i=0}^{\infty}\left|u_{B_{2}-i_{\rho}}-u_{B_{2}-i_{\rho / 2}}\right| \leq 2 \bar{C} \sum_{i=1}^{\infty}\left(2^{-i} \rho\right)^{\alpha} \leq C \bar{C} \rho^{\alpha}
$$

Using this and (3.3), we obtain.

$$
\|u-u(0)\|_{L^{2}\left(B_{\rho}\right)} \leq\left\|u-u_{B_{\rho}}\right\|_{L^{2}\left(B_{\rho}\right)}+\left|B_{\rho}\right|^{1 / 2}\left|u_{B_{\rho}}-u(0)\right| \leq C \bar{C} \rho^{N / 2+\alpha}
$$

where $C$ depends only on $N$ and $\alpha$.
Let $a$ satisfy (1.2) and $K \in \mathscr{K}(\lambda, a, \kappa)$ (satisfy (2.2)) $V, f \in \mathcal{M}_{\beta}$, we define the set of solutions to the Schrödinger equations with entries $V$ and $f$ by

$$
\mathcal{S}_{K, V, f}:=\left\{u \in H^{s}\left(B_{2}\right) \cap L^{2}\left(\mathbb{R}^{N}\right): \mathcal{L}_{K} u+V u=f \quad \text { in } B_{2}\right\}
$$

and we note that this set is nonempty thanks to Lemma 2.3 and a direct minimization argument. In fact this set is nonempty for all $f, V \in \mathcal{K}_{s}$ for the same reason.
We consider the class of (normalized) potentials

$$
\begin{equation*}
\mathcal{V}_{\beta}:=\left\{V \in \mathcal{M}_{\beta}:\|V\|_{\mathcal{M}_{\beta}} \leq 1\right\} \tag{3.4}
\end{equation*}
$$

Having these notations in mind, we now state the following result.
Proposition 3.2. Let $s \in(0,1), \beta \in[0,2 s), \alpha \in(0, \min (1,2 s-\beta))$ and $\Lambda, \kappa>0$. Then there exists $\varepsilon_{0}>0$ and $C>0$ such that for every $\lambda: \mathbb{R}^{2 N} \rightarrow\left[0, \kappa^{-1}\right]$ satisfying $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{0}$, a satisfying (1.2), $K \in \mathscr{K}(\lambda, a, \kappa), V \in \mathcal{V}_{\beta}, f \in \mathcal{M}_{\beta}, u \in \mathcal{S}_{K, V, f}$ and for every $r>0$, we have

$$
\begin{equation*}
\sup _{x \in B_{1}}\left\|u-u_{B_{r}(x)}\right\|_{L^{2}\left(B_{r}(x)\right)}^{2} \leq C r^{N+2 \alpha}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)^{2} \tag{3.5}
\end{equation*}
$$

Proof. The proof of (3.5) will be divided into two steps. Due to the presence of the potential $V$, the set $\mathcal{S}_{K, V, f}$ might not be invariant when adding constants to its elements. As a way out to this difficulty, we prove first a uniform estimate of the form $\left|u_{B_{r}(x)}\right| \leq C r^{-\varrho}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)$, for all $\varrho>0$. Once we get this, we complete the proof of (3.5) in the second step.

Step 1: We claim that for every $\varrho \in(0,1 / 2)$, there exist $C>0$ and a small number $\varepsilon_{0}>0$ such that for every $\lambda: \mathbb{R}^{2 N} \rightarrow\left[0, \kappa^{-1}\right]$ satisfying $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{0}$, every function $a$ satisfying (1.2), $K \in \mathscr{K}(\lambda, a, \kappa), V \in \mathcal{V}_{\beta}, f \in \mathcal{M}_{\beta}, u \in \mathcal{S}_{K, V, f}$, and $r>0$, we have

$$
\begin{equation*}
\sup _{x \in B_{1}}\|u\|_{L^{2}\left(B_{r}(x)\right)}^{2} \leq C r^{N-2 \varrho}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)^{2} \tag{3.6}
\end{equation*}
$$

Assume that (3.6) does not hold, then there exists $\varrho \in(0,1 / 2)$ such that for every $n \in \mathbb{N}$, we can find $\lambda_{n}: \mathbb{R}^{2 N} \rightarrow\left[0, \kappa^{-1}\right]$ satisfying $\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\frac{1}{n}, a_{n}$ satisfying (1.2), $K_{a_{n}} \in \mathscr{K}\left(\lambda_{n}, a_{n}, \kappa\right), V_{n} \in \mathcal{V}_{\beta}$, $f_{n} \in \mathcal{M}_{\beta}, u_{n} \in \mathcal{S}_{K_{a_{n}}, V_{n}, f_{n}}$, with $\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}} \leq 1$ and $\bar{r}_{n}>0$, such that

$$
\begin{equation*}
\bar{r}_{n}^{-N+2 \varrho} \sup _{x \in B_{1}}\left\|u_{n}\right\|_{L^{2}\left(B_{\bar{T}_{n}}(x)\right)}^{2}>n \tag{3.7}
\end{equation*}
$$

We consider the (well defined, because $\left.\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)} \leq 1\right)$ nonincreasing function $\Theta_{n}:(0, \infty) \rightarrow[0, \infty)$ given by

$$
\begin{equation*}
\Theta_{n}(\bar{r})=\sup _{r \in[\bar{r}, \infty)} r^{-N+2 \varrho} \sup _{x \in B_{1}}\left\|u_{n}\right\|_{L^{2}\left(B_{r}(x)\right)}^{2} \tag{3.8}
\end{equation*}
$$

Obviously by (3.7),

$$
\begin{equation*}
\Theta_{n}\left(\bar{r}_{n}\right)>n . \tag{3.9}
\end{equation*}
$$

Clearly, there exists $r_{n} \in\left[\bar{r}_{n}, \infty\right)$ such that

$$
\Theta_{n}\left(r_{n}\right) \geq r_{n}^{-N+2 \varrho} \sup _{x \in B_{1}}\left\|u_{n}\right\|_{L^{2}\left(B_{r_{n}}(x)\right)}^{2} \geq(1-1 / n) \Theta_{n}\left(\bar{r}_{n}\right) \geq(1-1 / n) \Theta_{n}\left(r_{n}\right)
$$

where we used the monotonicity of $\Theta_{n}$ for the last inequality, while the first inequality comes from the definition of $\Theta_{n}$. In particular, thanks to (3.9), $\Theta_{n}\left(r_{n}\right) \geq(1-1 / n) n$. Now since $\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)} \leq 1$, we have that $r_{n}^{-N+2 \varrho} \geq(1-1 / n) n$, so that $r_{n} \rightarrow 0$ as $n \rightarrow \infty$. Moreover by (3.9), it is clear that, there exists $x_{n} \in B_{1}$ such that

$$
\begin{equation*}
r_{n}^{-N+2 \varrho}\left\|u_{n}\right\|_{L^{2}\left(B_{r_{n}}\left(x_{n}\right)\right)}^{2} \geq(1-1 / n-1 / 2) \Theta_{n}\left(r_{n}\right) . \tag{3.10}
\end{equation*}
$$

We now define the blow-up sequence of functions

$$
w_{n}(x)=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{\varrho} u_{n}\left(r_{n} x+x_{n}\right)
$$

which, by (3.10), satisfy

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{3}{4} \quad \text { for every } n \geq 2 \tag{3.11}
\end{equation*}
$$

In view of (3.8), we have that

$$
\begin{aligned}
\left\|w_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} & =\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N+2 \varrho}\left\|u_{n}\right\|_{L^{2}\left(B_{r_{n} R}\left(x_{n}\right)\right)}^{2} \\
& \leq \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N+2 \varrho} \Theta_{n}\left(r_{n} R\right)\left(r_{n} R\right)^{N-2 \varrho} \leq R^{N-2 \varrho}
\end{aligned}
$$

where we have used the monotonicity of $\Theta_{n}$ for the last inequality. Consequently,

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq R^{N-2 \varrho} \quad \text { for every } R \geq 1 \text { and } n \geq 2 \tag{3.12}
\end{equation*}
$$

We define

$$
\bar{f}_{n}(x):=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{2 s} f_{n}\left(r_{n} x+x_{n}\right) \quad \text { and } \quad \bar{V}_{n}(x):=r_{n}^{2 s} V_{n}\left(r_{n} x+x_{n}\right)
$$

Because $u_{n} \in \mathcal{S}_{K_{a_{n}}, V_{n}, f_{n}}$, it is plain that

$$
\mathcal{L}_{K_{n}} w_{n}+\bar{V}_{n} w_{n}=r_{n}^{\varrho} \bar{f}_{n} \quad \text { in } B_{1 / 2 r_{n}}
$$

where

$$
\begin{equation*}
K_{n}(x, y)=r_{n}^{N+2 s} K_{a_{n}}\left(r_{n} x+x_{n}, r_{n} y+x_{n}\right) \tag{3.13}
\end{equation*}
$$

Clearly $K_{n}$ satisfies (2.2). Therefore applying Lemma 2.4 and using (3.12), for every $1<M<\frac{1}{2 r_{n}}$, we get

$$
\begin{aligned}
& \left\{\kappa-\varepsilon \bar{C}\left(1+r_{n}^{\varrho} \eta_{\bar{f}_{n}}(1)\right)-\bar{C} \eta_{\bar{V}_{n}}(1)\right\}\left[w_{n}\right]_{H^{s}\left(B_{M}\right)}^{2} \leq C r_{n}^{\varrho} \eta_{\bar{f}_{n}}(1)\left\|\varphi_{M}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \\
& +C\left(\eta_{\bar{V}_{n}}(1)+r_{n}^{\varrho} \eta_{\bar{f}_{n}}(1)+1\right) \int_{\mathbb{R}^{N}} \frac{M^{N}\left|w_{n}(x)\right|^{2}}{M^{N+2 s}+|x|^{N+2 s}} d x+C\left(\eta_{\bar{V}_{n}}(1)+r_{n}^{\varrho} \eta_{\bar{f}_{n}}(1)\right)\left\|w_{n}\right\|_{L^{2}\left(B_{2 M}\right)}^{2}
\end{aligned}
$$

By (3.1), we have that $\eta_{\bar{V}_{n}}(1)+\eta_{\bar{f}_{n}}(1) \leq C r_{n}^{2 s-\beta}$ (recalling that $\Theta_{n}\left(r_{n}\right)^{-1} \leq 1$ ). Hence, there exists a constant $C(M)$ independent on $n \geq 2$ such that

$$
\begin{equation*}
\left(\kappa-\varepsilon \bar{C}\left(1+r_{n}^{2 s-\beta+\varrho}\right)-\bar{C} r_{n}^{2 s-\beta}\right)\left[w_{n}\right]_{H^{s}\left(B_{M}\right)}^{2} \leq C(M) \tag{3.14}
\end{equation*}
$$

Therefore provided $\varepsilon$ is small and $n$ is large enough, we deduce that $w_{n}$ is bounded in $H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Hence by Sobolev embedding, up to a subsequence, $w_{n}$ converges strongly, in $L_{\text {loc }}^{2}\left(\mathbb{R}^{N}\right)$, to some $w \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. In addition by (3.12), we have that $v_{n} \rightarrow v$ in $\mathcal{L}_{s}^{1}$. Moreover, by (3.11) and (3.12), we deduce that

$$
\begin{equation*}
\|w\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{3}{4} \quad \text { and } \quad\|w\|_{L^{2}\left(B_{R}\right)}^{2} \leq R^{N-2 \varrho} \quad \text { for every } R \geq 1 \tag{3.15}
\end{equation*}
$$

We let $\psi \in C_{c}^{\infty}\left(B_{M}\right)$, with $M<\frac{1}{2 r_{n}}$. By Lemma (2.5) (3.14) and (3.12), we get

$$
\begin{align*}
& \left|\int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y\right| \\
& \leq C r_{n}^{2 s-\beta}\left(\left\|w_{n} \varphi_{M}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right)+r_{n}^{2 s-\beta+\varrho}\left(\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\left\|\varphi_{M}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right) \\
& \leq r_{n}^{2 s-\beta} C(M) . \tag{3.16}
\end{align*}
$$

Next, we observe that $K_{n} \in \mathscr{K}\left(\bar{\lambda}_{n}, a_{n}, \kappa\right)$, with $\bar{\lambda}_{n}(x, y)=\lambda_{n}\left(r_{n} x+x_{n}, r_{n} y+x_{n}\right)$ (see (2.2)). On the other hand

$$
\left\|\bar{\lambda}_{n}\right\|_{L^{\infty}\left(B_{1 / r_{n}} \times B_{\left.1 / r_{n}\right)}\right)}=\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{1}\left(x_{n}\right) \times B_{1}\left(x_{n}\right)\right)} \leq\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)} \leq \frac{1}{n} .
$$

In view of this and (3.16), by Lemma 2.2, as $n \rightarrow \infty$, we have that

$$
\frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(w_{n}(x)-w_{n}(y)\right)\left(\psi(x)-\psi(y) K_{n}(x, y) d x d y \rightarrow \int_{\mathbb{R}^{N}} w L_{b} \psi(x) d x=0\right.
$$

where $b$ is the weak-star limit of $a_{n}$, which satisfy (2.5). We then conclude that $L_{b} w=0 \quad$ in $\mathbb{R}^{N}$. Now Lemma 8.3 implies that $w$ is an affine function. This is clearly in contradiction with (3.15) since $\varrho>0$.

Step 2: Assuming that (3.5) does not hold true, then as in the first step, we can find sequences $\lambda_{n}: \mathbb{R}^{2 N} \rightarrow\left[0, \kappa^{-1}\right]$ satisfying $\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\frac{1}{n}, a_{n}$ satisfying (1.2), $K_{a_{n}} \in \mathscr{K}\left(\lambda_{n}, a_{n}, \kappa\right), x_{n} \in B_{1}$, $V_{n} \in \mathcal{V}_{\beta}, f_{n} \in \mathcal{M}_{\beta}, u_{n} \in \mathcal{S}_{K_{a_{n}}, V_{n}, f_{n}}$, with $\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}} \leq 1$ and $r_{n} \rightarrow 0$, such that

$$
\begin{equation*}
r_{n}^{-N-2 \alpha}\left\|u_{n}-\left(u_{n}\right)_{B_{r_{n}}\left(x_{n}\right)}\right\|_{L^{2}\left(B_{r_{n}}\left(x_{n}\right)\right)}^{2} \geq \frac{1}{8} \Theta_{n}\left(r_{n}\right) \tag{3.17}
\end{equation*}
$$

Here, for every $n \geq 2, \Theta_{n}:(0, \infty) \rightarrow[0, \infty)$ is a nonincreasing function satisfying

$$
\begin{equation*}
\Theta_{n}(r) r^{N+2 \alpha} \geq\left\|u_{n}-\left(u_{n}\right)_{B_{r}\left(x_{n}\right)}\right\|_{L^{2}\left(B_{r}\left(x_{n}\right)\right)}^{2} \quad \text { for every } r>0 \tag{3.18}
\end{equation*}
$$

and $\Theta_{n}\left(r_{n}\right) \geq n / 2$. We define

$$
v_{n}(x)=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\alpha} u_{n}\left(r_{n} x+x_{n}\right)-\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\alpha} \frac{1}{\left|B_{1}\right|} \int_{B_{1}} u_{n}\left(r_{n} x+x_{n}\right) d x
$$

so that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{8} \quad \text { and } \quad \int_{B_{1}} v_{n}(x) d x=0 \tag{3.19}
\end{equation*}
$$

Claim: There exists $C=C(s, N, \alpha)>0$ such that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \alpha} \quad \text { for every } R \geq 1 \text { and } n \geq 2 \tag{3.20}
\end{equation*}
$$

To prove this claim, we note that by a change of variable, we have

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2}=\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha}\left\|u_{n}-\left(u_{n}\right)_{B_{r_{n}}\left(x_{n}\right)}\right\|_{L^{2}\left(B_{r_{n} R}\left(x_{n}\right)\right)}^{2} \tag{3.21}
\end{equation*}
$$

Since, by (3.18) and the monotonicity of $\Theta_{n}$,
$\left\|u_{n}-\left(u_{n}\right)_{B_{R r_{n}}\left(x_{n}\right)}\right\|_{L^{2}\left(B_{R r_{n}}\left(x_{n}\right)\right)}^{2} \leq\left(r_{n} R\right)^{N+2 \alpha} \Theta_{n}\left(R r_{n}\right) \leq r_{n}^{N+2 \alpha} \Theta_{n}\left(r_{n}\right) R^{N+2 \alpha} \quad$ for every $R \geq 1$, it follows from Lemma $3.1(i)$ that

$$
\left\|u_{n}-\left(u_{n}\right)_{B_{r_{n}}\left(x_{n}\right)}\right\|_{L^{2}\left(B_{R r_{n}}\left(x_{n}\right)\right)}^{2} \leq C r_{n}^{N+2 \alpha} \Theta_{n}\left(r_{n}\right) R^{N+2 \alpha}
$$

Using this in (3.21), we get (3.20) as claimed.
Thanks to the choice of $\alpha<2 s$, by (3.20), we get

$$
\begin{equation*}
\left\|v_{n}\right\|_{\mathcal{L}_{s}^{1}} \leq C \tag{3.22}
\end{equation*}
$$

Using the same notations as in Step 1 for $\bar{V}_{n}, \bar{f}_{n}$ and $K_{n} \in \mathscr{K}\left(\bar{\lambda}_{n}, a_{n}, \kappa\right)$, we see that

$$
\mathcal{L}_{K_{n}} v_{n}+\bar{V}_{n} v_{n}=-\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\alpha} \bar{V}_{n} A_{n}\left(r_{n}\right)+r_{n}^{-\alpha} \bar{f}_{n} \quad \text { in } B_{1 / 2 r_{n}}
$$

where $A_{n}\left(r_{n}\right):=\frac{1}{\left|B_{r_{n}}\right|} \int_{B_{r_{n}}\left(x_{n}\right)} u_{n}(x) d x$. Note that from Step 1 and Hölder's inequality, for every $\varrho \in(0,(2 s-\beta-\alpha) / 2)$, we can find a constant $C>0$ such that for every $n \geq 2$,

$$
\begin{equation*}
\left|A_{n}\left(r_{n}\right)\right| \leq \frac{1}{\left|B_{r_{n}}\right|} \int_{B_{r_{n}}\left(x_{n}\right)}\left|u_{n}(x)\right| d x \leq C r_{n}^{-\varrho} \tag{3.23}
\end{equation*}
$$

We then define

$$
F_{n}(x):=-\Theta_{n}\left(r_{n}\right)^{-1 / 2} A_{n}\left(r_{n}\right) \bar{V}_{n}(x)+\bar{f}_{n}(x)
$$

As above, we observe that $\eta_{\bar{V}_{n}}(1) \leq C r_{n}^{2 s-\beta}$, while by (3.23), we have $\eta_{F_{n}}(1) \leq C r_{n}^{2 s-\beta-\varrho}$. On the other hand

$$
\begin{equation*}
\mathcal{L}_{K_{n}} v_{n}+\bar{V}_{n} v_{n}=r_{n}^{-\alpha} F_{n} \quad \text { in } B_{1 / 2 r_{n}} \tag{3.24}
\end{equation*}
$$

By Lemma 9.2, for $1<M<\frac{1}{2 r_{n}}$, we have

$$
\mathcal{L}_{K_{n}}\left(\varphi_{M} v_{n}\right)+\bar{V}_{n}\left(\varphi_{M} v_{n}\right)=r_{n}^{-\alpha} \varphi_{M} F_{n}+G_{v_{n}, M} \quad \text { in } B_{M / 2}
$$

where $\left\|G_{v_{n}, M}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C\left\|v_{n}\right\|_{\mathcal{L}_{s}^{1}} \leq C$, by (3.22). In view of (3.20), (3.22) and Lemma 2.4, we then get

$$
\left(\kappa-\varepsilon \bar{C}\left(1+r_{n}^{2 s-\beta-\varrho-\alpha}\right)-\bar{C} r_{n}^{2 s-\beta}\right)\left[\varphi_{M} v_{n}\right]_{H^{s}\left(B_{M / 4}\right)}^{2} \leq C(M) \quad \text { whenever } 1<M<\frac{1}{2 r_{n}}
$$

Consequently, provided $n$ is large enough and $\varepsilon$ small, we obtain

$$
\begin{equation*}
\left[v_{n}\right]_{H^{s}\left(B_{M / 4}\right)}^{2} \leq C(M) \tag{3.25}
\end{equation*}
$$

This with (3.20) imply that $v_{n}$ is bounded in $H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$ and, up to a subsequence, converges strongly, in $L_{l o c}^{2}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$, to some $v \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Since $v_{n}$ satisfy (3.24), by Lemma (2.5, (3.22) and (3.25), we have

$$
\begin{aligned}
\mid \int_{\mathbb{R}^{2 N}}\left(v_{n}(x)\right. & \left.-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \mid \\
& \leq C r_{n}^{2 s-\beta-\alpha-\varrho}\left(\left\|v_{n} \varphi_{M}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\left\|\varphi_{M}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right) \\
& \leq r_{n}^{2 s-\beta-\alpha-\varrho} C(M)
\end{aligned}
$$

for every $\psi \in C_{c}^{\infty}\left(B_{M}\right)$, with $M<\frac{1}{2 r_{n}}$. Letting $n \rightarrow \infty$ in the above inequality and using Lemma 2.2, we find that $L_{b} v=0$ in $\mathbb{R}^{N}$, with $b$ the limit of $a_{n}$ in the weak-star topology of $L^{\infty}\left(S^{N-1}\right)$. Moreover, from (3.20), we get

$$
\int_{B_{R}}|v(x)|^{2} d x \leq C R^{N+2 \alpha}
$$

By Lemma 8.3, $v$ is a constant function (because $\alpha<1$ ), which leads to a contradiction after passing to the limit in (3.19).
3.2. Uniform growth estimates at the boundary for solutions to Schrödinger equations. Let $\Omega$ be an open subset of $\mathbb{R}^{N}$ such that $\partial \Omega \cap B_{2}$ is a $C^{1}$ hypersurface. We will assume that $0 \in \partial \Omega$ and that $\partial \Omega$ separates $B_{2}$ into two domains. As before, for $K \in \mathscr{K}(\lambda, a, \kappa), V \in \mathcal{V}_{\beta}$ and $f \in \mathcal{M}_{\beta}$, we consider the (nonempty) set of solutions:

$$
\begin{equation*}
\mathcal{S}_{K, V, f ; \Omega}:=\left\{u \in H^{s}\left(B_{2}\right) \cap L^{2}\left(\mathbb{R}^{N}\right): \mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} \cap \Omega, \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c}\right\} . \tag{3.26}
\end{equation*}
$$

We have the following result.
Proposition 3.3. Let $s \in(0,1), \beta \in[0,2 s), \alpha \in(0, \min (s, 2 s-\beta))$ and $\Lambda, \kappa>0$. Then there exist $\varepsilon_{1}>0$ small and $C>0$ such that for every $\lambda: \mathbb{R}^{2 N} \rightarrow\left[0, \kappa^{-1}\right]$ satisfying $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{1}$, a satisfying (1.2), $K \in \mathscr{K}(\lambda, a, \kappa), V \in \mathcal{V}_{\beta}, f \in \mathcal{M}_{\beta}, u \in \mathcal{S}_{K, V, f ; \Omega}$ and for every $r>0$, we have

$$
\begin{equation*}
\sup _{z \in B_{1} \cap \partial \Omega}\|u\|_{L^{2}\left(B_{r}(z)\right)}^{2} \leq C r^{N+2 \alpha}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)^{2} \tag{3.27}
\end{equation*}
$$

Proof. As in the proof of Proposition 3.2, if (3.27) does not hold, then we can a find sequence of real numbers $r_{n} \rightarrow 0$, sequence of points $z_{n} \in B_{1} \cap \partial \Omega$, sequences of functions $\lambda_{n}$ satisfying $\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\frac{1}{n}, a_{n}$ satisfying (1.2), $K_{a_{n}} \in \mathscr{K}\left(\lambda_{n}, a_{n}, \kappa\right), V_{n} \in \mathcal{V}_{\beta}, f_{n} \in \mathcal{M}_{\beta} u_{n} \in \mathcal{S}_{K_{a_{n}}, V_{n}, f_{n} ; \Omega}$, with $\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}} \leq 1$, such that

$$
r_{n}^{-N-2 \alpha}\left\|u_{n}\right\|_{L^{2}\left(B_{r_{n}}\left(z_{n}\right)\right)}^{2} \geq \frac{1}{8} \Theta_{n}\left(r_{n}\right),
$$

where, $\Theta_{n}:(0, \infty) \rightarrow[0, \infty)$, is a nonincreasing function satisfying

$$
\begin{equation*}
\Theta_{n}(r) r^{N+2 \alpha} \geq\left\|u_{n}\right\|_{L^{2}\left(B_{r}\left(z_{n}\right)\right)}^{2} \quad \text { for every } r>0 \text { and } n \geq 2 \tag{3.28}
\end{equation*}
$$

and $\Theta_{n}\left(r_{n}\right) \geq n / 2$ for all integer $n \geq 2$. We define

$$
v_{n}(x)=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\alpha} u_{n}\left(r_{n} x+z_{n}\right)
$$

so that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{8} \tag{3.29}
\end{equation*}
$$

We also let

$$
\begin{gathered}
\Omega_{n}:=\frac{1}{r_{n}}\left(\Omega-z_{n}\right) \\
\bar{f}_{n}(x):=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{2 s} f_{n}\left(r_{n} x+z_{n}\right) \quad \text { and } \quad \bar{V}_{n}(x):=r_{n}^{2 s} V_{n}\left(r_{n} x+z_{n}\right)
\end{gathered}
$$

By (3.1), we get

$$
\begin{equation*}
\eta_{\bar{f}_{n}}(1)+\eta_{\bar{V}_{n}}(1) \leq C r_{n}^{2 s-\beta} \tag{3.30}
\end{equation*}
$$

with a constant $C=C(N, s, \beta, \alpha)$. It is clear that

$$
\begin{cases}\mathcal{L}_{K_{n}} v_{n}+\bar{V}_{n} v_{n}=r_{n}^{-\alpha} \bar{f}_{n} & \text { in } B_{1 / 2 r_{n}}\left(-z_{n}\right) \cap \Omega_{n}  \tag{3.31}\\ v_{n}=0 & \text { in } B_{1 / 2 r_{n}}\left(-z_{n}\right) \cap \Omega_{n}^{c}\end{cases}
$$

where

$$
K_{n}(x, y)=r_{n}^{N+2 s} K_{a_{n}}\left(r_{n} x+z_{n}, r_{n} y+z_{n}\right)
$$

Next, by the monotonicity of $\Theta_{n}$ and (3.28), we get

$$
\begin{aligned}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} & =\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha}\left\|u_{n}\right\|_{L^{2}\left(B_{r_{n} R}\left(z_{n}\right)\right)}^{2} \\
& \leq \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha} \Theta_{n}\left(R r_{n}\right)\left(R r_{n}\right)^{N+2 \alpha}
\end{aligned}
$$

Hence, for every $n \geq 2$,

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq R^{N+2 \alpha} \quad \text { for every } R \geq 1 \tag{3.32}
\end{equation*}
$$

This with Hölder's inequality imply that

$$
\begin{equation*}
\int_{B_{R}}\left|v_{n}(x)\right| d x \leq C R^{N+\alpha} \quad \text { for every } R \geq 1 \tag{3.33}
\end{equation*}
$$

From now on, we let $n_{0}$ large, so that $B_{1 /\left(2 r_{n}\right)} \subset B_{1 / r_{n}}\left(-z_{n}\right)$ for every $n \geq n_{0}$. Since $v_{n}$ satisfies (3.31) and $K_{n}$ satisfies (2.2) $(i)$-(ii), by Lemma 2.4, (3.30) and (3.32), there exists a constant $C(M)$ independent on $n \geq n_{0}$ such that

$$
\begin{equation*}
\left[v_{n}\right]_{H^{s}\left(B_{M}\right)}^{2} \leq C(M) \tag{3.34}
\end{equation*}
$$

whenever $1 \leq M \leq \frac{1}{2 r_{n}}$. We then deduce that $v_{n}$ is bounded in $H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Hence by Sobolev embedding, (3.33) and since $\alpha<2 s$, we may assume that the sequence $v_{n}$ converges strongly, in $L_{l o c}^{2}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$, to some $v \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Moreover, by (3.29), we deduce that

$$
\begin{equation*}
\|v\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{8} . \tag{3.35}
\end{equation*}
$$

Next, we note that $1_{\Omega_{n} \cap B_{1 /\left(2 r_{n}\right)}} \rightarrow 1_{H}$ in $L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ as $n \rightarrow \infty$, where $H$ is a half-space, with $0 \in H$. In fact $H=\left\{x \in \mathbb{R}^{N}:(x-\bar{z}) \cdot \nu(\bar{z})>0\right\}$, where $\bar{z}=\lim _{n \rightarrow \infty} z_{n} \in \partial H$ and $\nu$ is the unit interior normal vector of $\partial H$. Now, we pick $\psi \in C_{c}^{\infty}\left(H \cap B_{M}\right)$, with $M<\frac{1}{2 r_{n}}$. Since $\Omega$ is of class $C^{1}$, provided $n$ is large enough, we have that $\psi \in C_{c}^{\infty}\left(\Omega_{n}\right)$. Therefore by Lemma 2.5, (3.34), (3.32) and (3.30), we obtain

$$
\left|\int_{\mathbb{R}^{2 N}}\left(v_{n}(x)-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y\right| \leq r_{n}^{2 s-\beta-\alpha} C(M)
$$

with $C(M)$ a constant not depending on $n \geq n_{0}$ and large. Denoting by $b$ the weak-star limit of $a_{n}$, then by Lemma 2.2, we get

$$
L_{b} v=0 \quad \text { in } H \quad \text { and } \quad v=0 \quad \text { on } \mathbb{R}^{N} \backslash H
$$

Furthermore by (3.32), $\|v\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \alpha}$, for every $R \geq 1$. Since $\alpha<s$, it follows from Lemma 8.3 that $v=0$, which is impossible by (3.35).

## 4. Interior and boundary Hölder regularity estimates

4.1. Interior Hölder regularity. We have the following regularity estimates.

Corollary 4.1. Let $s \in(0,1), \beta \in[0,2 s), \alpha \in(0, \min (1,2 s-\beta)), \kappa, \Lambda>0$. Let a satisfy (1.2) and $K \in \mathscr{K}(\lambda, a, \kappa)$. Let $f, V \in \mathcal{M}_{\beta}$ and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2}
$$

Then there exists $\varepsilon_{0}, C>0$, only depending on $N, s, \beta, \kappa, \alpha,\|V\|_{\mathcal{M}_{\beta}}$ and $\Lambda$ such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<$ $\varepsilon_{0}$, then

$$
\|u\|_{C^{\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Proof. We let $x_{0} \in B_{3 / 2}$ and $\delta \in(0,1 / 8)$ and we define $\lambda_{\delta}(x, y)=\lambda\left(\delta x+x_{0}, \delta y+x_{0}\right)$ and $K_{\delta}(x, y)=$ $\delta^{N+2 s} K\left(\delta x+x_{0}, \delta y+x_{0}\right)$. Then $K_{\delta} \in \mathscr{K}\left(\lambda_{\delta}, a, \kappa\right)$. For $x \in B_{2}$, we define $u_{\delta}(x)=u\left(\delta x+x_{0}\right)$, $f_{\delta}(x)=\delta^{2 s} f\left(\delta x+x_{0}\right)$ and $V_{\delta}(x)=\delta^{2 s} V\left(\delta x+x_{0}\right)$. Since $\delta \in(0,1 / 8)$, by direct computations, we get

$$
\begin{equation*}
\mathcal{L}_{K_{\delta}} u_{\delta}+V_{\delta} u_{\delta}=f_{\delta} \quad \text { in } B_{8} \tag{4.1}
\end{equation*}
$$

By Lemma 9.2, letting $v_{\delta}:=\varphi_{4} u_{\delta}$ we have

$$
\begin{equation*}
\mathcal{L}_{K_{\delta}} v_{\delta}+V_{\delta}(x) v_{\delta}=\tilde{f}_{\delta} \quad \text { in } B_{2} \tag{4.2}
\end{equation*}
$$

with $\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}} \leq\left\|f_{\delta}\right\|_{\mathcal{M}_{\beta}}+C_{0}\left\|u_{\delta}\right\|_{\mathcal{L}_{s}^{1}}$ and

$$
\left\|V_{\delta}\right\|_{\mathcal{M}_{\beta}} \leq C \delta^{2 s-\beta}\|V\|_{\mathcal{M}_{\beta}}
$$

with $C_{0}$ depending only on $N, s, \kappa, \Lambda$ and $\beta$. Hence, there exists $\delta_{0} \in(0,1 / 8)$, only depending on $N, s, \kappa, \Lambda, \beta$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that $\left\|V_{\delta}\right\|_{\mathcal{M}_{\beta}} \leq 1$ for every $\delta \in\left(0, \delta_{0}\right)$. Obviously $\left\|\lambda_{\delta}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)} \leq$ $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}$ for every $\delta \in\left(0, \delta_{0}\right)$. Hence by Proposition 3.2, there exists $\varepsilon_{0}>0$ and $C>0$ such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{0}$, then for every $x \in B_{1}, r>0$ and $\delta \in\left(0, \delta_{0}\right)$, we have

$$
\left\|v_{\delta}-\left(v_{\delta}\right)_{B_{r}(x)}\right\|_{L^{2}\left(B_{r}(x)\right)}^{2} \leq C r^{N+2 \alpha}\left(\left\|v_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}\right)^{2}
$$

where, the constant $C$ and $\varepsilon_{0}$ only depend on $N, s, \beta, \alpha, \Lambda,\|V\|_{\mathcal{M}_{\beta}}$ and $\kappa$. By Lemma 3.1(ii), for almost all $x \in B_{1}$ and for all $r \in(0,1]$, we have

$$
\begin{equation*}
\left\|v_{\delta}-v_{\delta}(x)\right\|_{L^{2}\left(B_{r}(x)\right)}^{2} \leq C r^{N+2 \alpha}\left(\left\|v_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}\right)^{2} \tag{4.3}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\left\|v_{\delta}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C\left(\left\|v_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}\right) \tag{4.4}
\end{equation*}
$$

Let $x, y \in B_{1 / 4}$ be two Lebesgue points of $u$, and take $\rho=|x-y| / 2$. Then $B_{\rho}(x) \subset B_{3 \rho}(y) \subset B_{1}$. Therefore, by (4.3), we get

$$
\begin{aligned}
\left|B_{\rho}\right|^{N / 2}\left|v_{\delta}(x)-v_{\delta}(y)\right|=\left\|v_{\delta}(x)-v_{\delta}(y)\right\|_{L^{2}\left(B_{\rho}(x)\right)} & \leq\left\|v_{\delta}(x)-v_{\delta}\right\|_{L^{2}\left(B_{\rho}(x)\right)}+\left\|v_{\delta}(y)-v_{\delta}\right\|_{L^{2}\left(B_{3 \rho}(y)\right)} \\
& \leq C \rho^{N / 2+\alpha}\left(\left\|v_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}\right)
\end{aligned}
$$

That is

$$
\left|v_{\delta}(x)-v_{\delta}(y)\right| \leq C|x-y|^{\alpha}\left(\left\|v_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}\right) \quad \text { for almost every } x, y \in B_{1 / 4}
$$

We then conclude, from (4.4), that

$$
\left\|v_{\delta}\right\|_{C^{\alpha}\left(B_{1 / 4}\right)} \leq C\left(\left\|v_{\delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}\right)
$$

It follows that

$$
\left\|u_{\delta}\right\|_{C^{\alpha}\left(B_{1 / 4}\right)} \leq C\left(\left\|u_{\delta}\right\|_{L^{2}\left(B_{8}\right)}+\left\|\tilde{f}_{\delta}\right\|_{\mathcal{M}_{\beta}}+\left\|u_{\delta}\right\|_{\mathcal{L}_{s}^{1}}\right)
$$

Scaling and translating back, we get

$$
\|u\|_{C^{\alpha}\left(B_{\delta / 4}\left(x_{0}\right)\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|f\|_{\mathcal{M}_{\beta}}+\|u\|_{\mathcal{L}_{s}^{1}}\right)
$$

where $C$ depends only on $N, s, \beta, \alpha, \Lambda, \kappa, \delta$ and $\|V\|_{\mathcal{M}_{\beta}}$. Since $B_{1}$ can be covered by a finite number of such balls $B_{\delta / 4}\left(x_{0}\right)$, with $x_{0} \in B_{3 / 4}$, we get the desired estimate.

As a consequence of Corollary 4.1 we obtain regularity estimates for nonlocal operators with "uniformly continuous" coefficient. For $K \in \widetilde{\mathscr{K}}(\kappa)$, we define the functions

$$
\begin{align*}
& \tilde{\lambda}_{e, K}(x, r, \theta)=\frac{1}{2}\left\{\lambda_{K}(x, r, \theta)+\lambda_{K}(x, r,-\theta)\right\} \\
& \widetilde{\lambda}_{o, K}(x, r, \theta)=\frac{1}{2}\left\{\lambda_{K}(x, r, \theta)-\lambda_{K}(x, r,-\theta)\right\} . \tag{4.5}
\end{align*}
$$

If there is no ambiguity, we will simply write $\widetilde{\lambda}_{e}$ and $\widetilde{\lambda}_{o}$ in the place of $\widetilde{\lambda}_{e, K}$ and $\widetilde{\lambda}_{o, K}$, respectively.
Theorem 4.2. Let $s \in(0,1), \beta \in[0,2 s)$ and $\alpha \in(0, \min (1,2 s-\beta))$. Let $K \in \widetilde{K}(\kappa)$ and suppose that $\widetilde{\lambda}_{e}$ and $\widetilde{\lambda}_{o}$ (defined in (4.5)) satisfy

- for every $x_{1}, x_{2} \in B_{2}, r \in(0,2), \theta \in S^{N-1}$,

$$
\left|\widetilde{\lambda}_{e}\left(x_{1}, r, \theta\right)-\widetilde{\lambda}_{e}\left(x_{2}, 0, \theta\right)\right| \leq \tau\left(\left|x_{1}-x_{2}\right|+r\right)
$$

- for every $x \in B_{2}, r \in(0,2), \theta \in S^{N-1}$,

$$
\left|\tilde{\lambda}_{o}(x, r, \theta)\right| \leq \tau(r)
$$

for some function $\tau \in L^{\infty}\left(\mathbb{R}_{+}\right)$and $\tau(t) \rightarrow 0$ as $t \rightarrow 0$. Let $f, V \in \mathcal{M}_{\beta}$ and $u \in H_{\text {loc }}^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2}
$$

Then there exists $C>0$, only depending on $N, s, \beta, \alpha, \kappa, \tau$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\|u\|_{C^{\alpha}\left(B_{1}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Proof. Pick $x_{0} \in B_{3 / 2}$. By assumption, for every $x \in B_{2}, r \in(0,2)$ and $\theta \in S^{N-1}$,

$$
\left|\widetilde{\lambda}_{o}(x, r, \theta)\right|+\left|\widetilde{\lambda}_{e}(x, r, \theta)-\widetilde{\lambda}_{e}\left(x_{0}, 0, \theta\right)\right| \leq \tau(r)+\tau\left(\left|x-x_{0}\right|+r\right)
$$

Then for every $\varepsilon>0$ there exists $\delta=\delta_{x_{0}, \varepsilon} \in(0,1 / 100)$ such that, for every $x \in B_{4 \delta}\left(x_{0}\right)$ and $r \in(0,4 \delta)$, we have

$$
\left|K(x, x+r \theta)-\widetilde{\lambda}_{e}\left(x_{0}, 0, \theta\right) r^{-N-2 s}\right| \leq \varepsilon r^{-N-2 s}
$$

Therefore, for every $x \in B_{4 \delta}\left(x_{0}\right)$ and $0<|z|<4 \delta$,

$$
\left.\left.\left|K(x, x+z)-\widetilde{\lambda}_{e}\left(x_{0}, 0, z /|z|\right)\right| z\right|^{-N-2 s}|\leq \varepsilon| z\right|^{-N-2 s}
$$

and thus, for every $x, y \in B_{2 \delta}\left(x_{0}\right)$, with $x \neq y$,

$$
\left|K(x, y)-\mu_{a}(x, y)\right| \leq \varepsilon \mu_{1}(x, y)
$$

where $a(\theta):=\widetilde{\lambda}_{e}\left(x_{0}, 0, \theta\right)$. It is clear that $a$ is even on $S^{N-1}$. By changing variables, we find that for every $x, y \in B_{2}$, with $x \neq y$,

$$
\begin{equation*}
\left|\delta^{N+2 s} K\left(\delta x+x_{0}, \delta y+x_{0}\right)-\mu_{a}(x, y)\right| \leq \varepsilon \mu_{1}(x, y) \tag{4.6}
\end{equation*}
$$

In addition,

$$
\kappa \leq a(\theta) \leq \kappa^{-1} \quad \text { for every } \theta \in S^{N-1}
$$

We define

$$
\begin{cases}\lambda(x, y)=\varepsilon, & \text { for } x, y \in B_{2} \\ \lambda(x, y)=\kappa^{-1} & \text { elsewhere }\end{cases}
$$

We now let $K_{\delta}(x, y)=\delta^{N+2 s} K\left(\delta x+x_{0}, \delta y+x_{0}\right)$, which, by (4.6), clearly satisfies $K_{\delta} \in \mathscr{K}(\lambda, a, \kappa)$.
For $x \in B_{2}$, we define $u_{\delta}(x)=u\left(\delta x+x_{0}\right), f_{\delta}(x)=\delta^{2 s} f\left(\delta x+x_{0}\right)$ and $V_{\delta}(x)=\delta^{2 s} V\left(\delta x+x_{0}\right)$. Since $\delta \in(0,1 / 16)$, by direct computations, we get

$$
\begin{equation*}
\mathcal{L}_{K_{\delta}} u_{\delta}+V_{\delta} u_{\delta}=f_{\delta} \quad \text { in } B_{8} \tag{4.7}
\end{equation*}
$$

Recall that $\left\|V_{\delta}\right\|_{\mathcal{M}_{\beta}} \leq C \delta^{2 s-\beta}\|V\|_{\mathcal{M}_{\beta}}$ and thus, decreasing $\delta$ if necessary, we get $\left\|V_{\delta}\right\|_{\mathcal{M}_{\beta}} \leq 1$. Since $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}=\varepsilon$, then provided $\varepsilon>0$ small, by Corollary 4.1 and a change of variable, we get

$$
\|u\|_{C^{\alpha}\left(B_{\delta_{x_{0}, \varepsilon}}\left(x_{0}\right)\right)} \leq C\left(x_{0}\right)\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|f\|_{\mathcal{M}_{\beta}}+\|u\|_{\mathcal{L}_{s}^{1}}\right)
$$

where $C\left(x_{0}\right)$ is a constant, only depending on $N, s, c_{0}, \delta, \kappa, \tau, x_{0}$ and $\|V\|_{\mathcal{M}_{\beta}}$. Next, we cover $\bar{B}_{1}$ by a finite number of balls $B_{\frac{1}{2} \delta_{x_{i}, \varepsilon}}\left(x_{i}\right)$, for $i=1, \ldots, n$, with $x_{i} \in \bar{B}_{1}$. Put $C:=\max _{1 \leq i \leq n} C\left(x_{i}\right)$ and $\varrho=\frac{1}{2} \min _{1 \leq i \leq n} \delta_{x_{i}, \varepsilon}$. Then on any ball $B_{\varrho}(\bar{x})$, with $\bar{x} \in \bar{B}_{1}$, we have the estimate

$$
\|u\|_{C^{\alpha}\left(B_{\varrho}(\bar{x})\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|f\|_{\mathcal{M}_{\beta}}+\|u\|_{\mathcal{L}_{s}^{1}}\right)
$$

where $\varrho$ and $C$ depend only on $N, s, c_{0}, \kappa, \tau$ and $\|V\|_{\mathcal{M}_{\beta}}$. Since $\bar{B}_{1}$ can be covered by a finite number of balls $B_{\varrho}(\bar{x})$, with $\bar{x} \in \bar{B}_{1}$, we get the result.

Remark 4.3. We note that the conclusion of Theorem 4.2 remains unchanged if we considered, say, a "better" modulus of continuity $\tau$. More precisely, in Theorem 4.2, we could choose $\tau_{\rho}(r)=\tau(\rho r)$ for some $\rho \in(0,1)$ and $\tau$ as in the theorem. In this case, the constant $C$ in the theorem will not depend on $\rho$.
4.2. Hölder regularity estimates up to the boundary. Hölder regularity up to the boundary for the linear second order partial differential equations with coefficients in Morrey space was obtained in [14. Coupling the interior regularity in Corollary 4.1 and the uniform $L^{2}$ growth estimates up to the boundary given by Proposition 3.3 together with some scaling arguments, we get the following result.

Theorem 4.4. Let $\Omega$ be an open set such that $\partial \Omega \cap B_{2}$ is a $C^{1}$ hypersurface. Suppose that $0 \in \partial \Omega$ and that $\partial \Omega$ separates $B_{2}$ into two domains. Let $s \in(0,1), \beta \in[0,2 s), \alpha \in(0, \min (s, 2 s-\beta))$ and a satisfy (1.2). Let $K \in \mathscr{K}(\lambda, a, \kappa)$. Let $V, f \in \mathcal{M}_{\beta}$ and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c} .
$$

Then there exists $C, \bar{\varepsilon}_{0}, r_{0}>0$ such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\bar{\varepsilon}_{0}$, we have

$$
\|u\|_{C^{\alpha}\left(B_{r_{0}}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

with $C, \bar{\varepsilon}_{0}, r_{0}$ depending only on $N, s, \beta, \alpha, \Omega,\|V\|_{\mathcal{M}_{\beta}}, \kappa$ and $\Lambda$.
Proof. By similar scaling and cut-off argument as in the proof of Corollary 4.1 and using Proposition 3.3, we get

$$
\begin{equation*}
\sup _{z \in B_{1} \cap \partial \Omega}\|u\|_{L^{2}\left(B_{r}(z)\right)}^{2} \leq C r^{N+2 \alpha}\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) \quad \text { for every } r>0 \tag{4.8}
\end{equation*}
$$

with $C$ a constant depending only on $N, s, \beta, \kappa, \Lambda, \alpha, \Omega,\|V\|_{\mathcal{M}_{\beta}}$. We assume in the following that $\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}} \leq 1$, up to dividing the equation by this quantity. Moreover by Corollary 4.1. $u$ is continuous in $\Omega$, provided $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}$ is small.

Let $r_{0}>0$, only depending on $\Omega$, be such that every point $x_{0} \in \Omega \cap B_{r_{0}}$, with $d\left(x_{0}\right) \leq r_{0}$, has a unique projection $z$ on $\partial \Omega \cap B_{1}$. For such $x_{0} \in \Omega \cap B_{r_{0}}$, we let $z \in \partial \Omega \cap B_{1}$ be such that $\left|x_{0}-z\right|=d\left(x_{0}\right)$. Put $\rho=\frac{d\left(x_{0}\right)}{2}$, so that $B_{\rho}\left(x_{0}\right) \subset B_{3 \rho}(z) \cap \Omega$. Next, we define $v_{\rho}(x):=u\left(\rho x+x_{0}\right), V_{\rho}(x):=\rho^{2 s} V\left(\rho x+x_{0}\right)$ and $f_{\rho}(x):=\rho^{2 s} f\left(\rho x+x_{0}\right)$. It is plain that $v \in H_{l o c}^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ and

$$
\begin{equation*}
\mathcal{L}_{K_{\rho, x_{0}}} v_{\rho}+V_{\rho} v_{\rho}=f_{\rho} \quad \text { in } B_{1}, \tag{4.9}
\end{equation*}
$$

with

$$
K_{\rho, x_{0}}(x, y)=\rho^{N+2 s} K\left(\rho x+x_{0}, \rho y+x_{0}\right)
$$

Recall from (3.1) that $\left\|f_{\rho}\right\|_{\mathcal{M}_{\beta}} \leq \rho^{2 s-\beta}\|f\|_{\mathcal{M}_{\beta}}$ and $\left\|V_{\rho}\right\|_{\mathcal{M}_{\beta}} \leq \rho^{2 s-\beta}\|V\|_{\mathcal{M}_{\beta}}$. Hence decreasing $r_{0}$ if necessary, we may assume that $\left\|V_{\rho}\right\|_{\mathcal{M}_{\beta}} \leq 1$. We note that $K_{\rho, x_{0}} \in \mathscr{K}\left(\lambda_{\rho, x_{0}}, a, \kappa\right)$, where $\lambda_{\rho, x_{0}}(x, y)=$ $\lambda\left(\rho x+x_{0}, \rho y+y_{0}\right)$. In particular, decreasing $r_{0}$ if necessary,

$$
\left\|\lambda_{\rho, x_{0}}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)} \leq\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}
$$

Therefore, by Theorem4.2, we can find $\bar{\varepsilon}_{0}>0$ small, independent on $\rho$, such that, if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<$ $\bar{\varepsilon}_{0}$, we have

$$
\begin{equation*}
\left\|v_{\rho}\right\|_{L^{\infty}\left(B_{1 / 2}\right)} \leq C\left(\left\|v_{\rho}\right\|_{L^{2}\left(B_{1}\right)}+\left\|v_{\rho}\right\|_{\mathcal{L}_{s}^{1}}+\left\|f_{\rho}\right\|_{\mathcal{M}_{\beta}}\right) . \tag{4.10}
\end{equation*}
$$

By (4.8) and Hölder's inequality, we have

$$
\begin{equation*}
\left\|v_{\rho}\right\|_{L^{2}\left(B_{1}\right)} \leq \rho^{-N / 2}\|u\|_{L^{2}\left(B_{3 \rho}(z)\right)} \leq C \rho^{\alpha} \quad \text { and } \quad\|u\|_{L^{1}\left(B_{r}(z)\right)} \leq C r^{N+\alpha} \quad \text { for all } r>0 \tag{4.11}
\end{equation*}
$$

Using the second estimate in (4.11), we get

$$
\begin{aligned}
\int_{|x| \geq 1}|x|^{-N-2 s}\left|v_{\rho}(x)\right| d x & =\rho^{2 s} \int_{\left|y-x_{0}\right| \geq \rho}\left|y-x_{0}\right|^{-N-2 s}|u(y)| d y \\
& \leq \rho^{2 s} \sum_{k=0}^{\infty} \int_{\rho 2^{k+1} \geq\left|y-x_{0}\right| \geq \rho 2^{k}}\left|y-x_{0}\right|^{-N-2 s}|u(y)| d y \\
& \leq \rho^{2 s} \sum_{k=0}^{\infty}\left(2^{k} \rho\right)^{-N-2 s} \int_{\rho 2^{k+1} \geq\left|y-x_{0}\right|}|u(y)| d y \\
& \leq \rho^{2 s} \sum_{k=0}^{\infty}\left(2^{k} \rho\right)^{-N-2 s}\|u\|_{L^{1}\left(B_{\rho 2^{k+3}}(z)\right)} \\
& \leq C \rho^{2 s} \sum_{k=0}^{\infty}\left(2^{k} \rho\right)^{-N-2 s}\left(\rho 2^{k}\right)^{N+\alpha} \leq C \rho^{\alpha} \sum_{k=0}^{\infty} 2^{-k(2 s-\alpha)} \leq C \rho^{\alpha}
\end{aligned}
$$

We then conclude that $\left\|v_{\rho}\right\|_{\mathcal{L}_{s}^{1}} \leq C \rho^{\alpha}$. It follows from (4.9), (4.11) and (4.10), that

$$
\left\|v_{\rho}\right\|_{L^{\infty}\left(B_{1 / 2}\right)} \leq C\left(\left\|v_{\rho}\right\|_{L^{2}\left(B_{1}\right)}+\left\|v_{\rho}\right\|_{\mathcal{L}_{s}^{1}}+\left\|f_{\rho}\right\|_{\mathcal{M}_{\beta}}\right) \leq C\left(\rho^{\alpha}+\rho^{2 s-\beta}\right)
$$

Scaling back, we get

$$
\|u\|_{L^{\infty}\left(B_{\rho / 2}\left(x_{0}\right)\right)} \leq C\left(\rho^{\alpha}+\rho^{2 s-\beta}\right)
$$

which, in particular, yields

$$
\left|u\left(x_{0}\right)\right| \leq C \rho^{\alpha} \leq C d\left(x_{0}\right)^{\alpha} \quad \text { for every } x_{0} \in B_{r_{0}} \cap \Omega
$$

Now by a classical scaling argument as above and using the interior regularity estimates in Theorem 4.2, we get $u \in C^{\alpha}\left(B_{r_{0}} \cap \bar{\Omega}\right)$, with $\|u\|_{C^{\alpha}\left(B_{r_{0} / 2}\right)} \leq C$.

As a consequence, we have
Theorem 4.5. Let $s \in(0,1), \beta \in[0,2 s)$ and $\alpha \in(0, \min (s, 2 s-\beta))$. Let $K \in \widetilde{\mathscr{K}}(\kappa)$ and suppose that $\widetilde{\lambda}_{e}$ and $\widetilde{\lambda}_{o}$ (defined in (4.5)) satisfy

- for every $x_{1}, x_{2} \in B_{2}, r \in(0,2), \theta \in S^{N-1}$,

$$
\left|\widetilde{\lambda}_{e}\left(x_{1}, r, \theta\right)-\widetilde{\lambda}_{e}\left(x_{2}, 0, \theta\right)\right| \leq \tau\left(\left|x_{1}-x_{2}\right|+r\right)
$$

- for every $x \in B_{2}, r \in(0,2), \theta \in S^{N-1}$,

$$
\left|\widetilde{\lambda}_{o}(x, r, \theta)\right| \leq \tau(r)
$$

for some function $\tau \in L^{\infty}\left(\mathbb{R}_{+}\right)$and $\tau(t) \rightarrow 0$ as $t \rightarrow 0$. Let $f, V \in \mathcal{M}_{\beta}$ and $u \in H_{\text {loc }}^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
\mathcal{L}_{K} u+V u=f \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c} .
$$

Then there exist $C, r_{0}>0$, only depending only on $N, s, \beta, \alpha, \Lambda, \kappa, \tau, \Omega$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\|u\|_{C^{\alpha}\left(B_{r_{0}}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{1}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Proof. Adapting the scaling arguments as in the Theorem 4.2. together with Theorem 4.4, we get the result.

## 5. Higher Regularity estimates up to the boundary

In this section, we let $\Omega$ be an open set such that $\partial \Omega \cap B_{2}$ is a $C^{1, \gamma}$ hypersurface, with $0 \in \partial \Omega$ and $\gamma>0$. We will assume that $\partial \Omega$ separates $B_{2}$ into two domains. We note that there exists $r_{0} \in(0,1 / 2)$ small only depending on $\Omega$ such that, for all $r \in\left(0,2 r_{0}\right), z \in \partial \Omega \cap B_{3 / 2}$ and $\delta>0$,

$$
\begin{equation*}
C r^{N+\delta} \leq \int_{B_{r}(z)} d^{\delta}(y) d y \tag{5.1}
\end{equation*}
$$

for some constant $C=C(N, s, \delta, \Omega)>0$. On the other hand since $d(y) \leq|y-z|$ for all $z \in \partial \Omega$, for every $r>0$, we have

$$
\begin{equation*}
\int_{B_{r}(z)} d^{\delta}(y) d y \leq C r^{N+\delta} \tag{5.2}
\end{equation*}
$$

with $C=C(N, s, \delta)$.
We consider the cut-off of the distance function $d$ denoted by $d_{2}^{s}:=\varphi_{2} d^{s}$. For $u \in L^{2}\left(\mathbb{R}^{N}\right), z \in \partial \Omega \cap B_{1}$ and $r>0$, we let $P_{r, z}(u)$ be the $L_{l o c}^{2}\left(B_{r}(z)\right)$-projection of $u$ on $\left\langle d_{2}^{s}\right\rangle=\mathbb{R} d_{2}^{s}$, the one-dimensional space spanned by $d_{2}^{s}$. Therefore

$$
\begin{equation*}
\int_{B_{r}(z)}\left(u(y)-P_{r, z}(u)(y)\right) d_{2}^{s}(y) d y=0 \quad \text { and } \quad P_{r, z}(u)(x)=d_{2}^{s}(x) \frac{\int_{B_{r}(z)} u(y) d_{2}^{s}(y) d y}{\int_{B_{r}(z)} d_{2}^{2 s}(y) d y} \tag{5.3}
\end{equation*}
$$

For $z \in B_{1} \cap \partial \Omega$ and $r>0$, we define

$$
\begin{equation*}
Q_{u, z}(r):=\frac{\int_{B_{r}(z)} u(y) d_{2}^{s}(y) d y}{\int_{B_{r}(z)} d_{2}^{2 s}(y) d y} \tag{5.4}
\end{equation*}
$$

Before going on, we explain the arguments in the next two main results of this section. Observe that by Hölder's inequality, for every $r \in\left(0, r_{0}\right]$ and $z \in B_{1} \cap \partial \Omega$,

$$
\left|Q_{u, z}(r)\right| \leq\left\|d_{2}^{s}\right\|_{L^{2}\left(B_{r}(z)\right)}^{-1}\|u\|_{L^{2}\left(B_{r}(z)\right)}=\left\|d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)}^{-1}\|u\|_{L^{2}\left(B_{r}(z)\right)}
$$

Hence by Proposition 3.3 and (5.1), for every $\delta_{0} \in(0, \min (s, 2 s-\beta))$, there exist constants $C, \bar{\varepsilon}_{0}>0$ such that for every $\lambda \in L^{\infty}\left(B_{2} \times B_{2}\right)$ satisfying $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\bar{\varepsilon}_{0}$, for every $f \in \mathcal{M}_{\beta}$ and $u \in \mathcal{S}_{K, 0, f ; \Omega}$ (recall the notation (3.26) ), $r \in\left(0, r_{0}\right]$ and $z \in B_{1} \cap \partial \Omega$, we have

$$
\begin{equation*}
\left|Q_{u, z}(r)\right| \leq C r^{\delta_{0}-s}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right) \tag{5.5}
\end{equation*}
$$

Our objective is to get $u / d^{s} \in C^{s-\beta}\left(B_{r_{0}} \cap \bar{\Omega}\right)$, whenever $\beta \in(0, s)$ and $\Omega$ regular enough. This requires, at least, we already know that $|u| \leq d^{s}$, or equivalently $\left|Q_{u, z}(r)\right| \leq C$. For this purpose, we will use a bootstrap argument in two steps to obtain (5.5) with $\delta_{0}=s$, as long as $\beta<s$ and under more regularity assumption on $K$ and $\partial \Omega$. This will be the content of the next two results.

In order to get the sharp boundary regularity, it will be crucial to quantify the action of the operators $\mathcal{L}_{K}$ on $d^{s}$ for $K \in \mathscr{K}(\lambda, a, \kappa)$. To this end, we first note that by Lemma 9.3, up to decreasing $r_{0}$ if necessary, we may assume that $d_{2}^{s} \in H^{s}\left(B_{2 r_{0}}\right) \cap \mathcal{L}_{s}^{1}$. Next, we introduce $\mathscr{K}(\lambda, a, \kappa, \Omega)$, the class of kernels $K \in \mathscr{K}(\lambda, a, \kappa)$ such that: there exist $\beta^{\prime}=\beta^{\prime}(\Omega, K) \in[0, s)$ and a function $g_{\Omega, K} \in \mathcal{M}_{\beta^{\prime}}$ such that

$$
\begin{equation*}
\mathcal{L}_{K} d_{2}^{s}=\mathcal{L}_{K}\left(\varphi_{2} d^{s}\right)=g_{\Omega, K} \quad \text { in the weak sense, in } B_{2 r_{0}} \cap \Omega \tag{5.6}
\end{equation*}
$$

We note that the class of kernels $\mathscr{K}(\lambda, a, \kappa, \Omega)$ is not empty. This is the case for $K=\mu_{a}$, with $a$ satisfying (1.2), see Section 7 below. We have the following result.

Lemma 5.1. Let $\Omega \subset \mathbb{R}^{N}$ be a $C^{1, \gamma}$ domain as above for some $\gamma>0$. Let $\beta \in(0,2 s), \varrho \in[0, s)$ and $c_{0}, \Lambda, \kappa>0$. Then there exist $C>0$ and $\varepsilon_{1}>0$ with the properties that if

- a satisfies (1.2),
- $\lambda: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow\left[0, k^{-1}\right]$ satisfies $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{1}$,
- $K \in \mathscr{K}(\lambda, a, \kappa, \Omega)$ with $\left\|g_{\Omega, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq 1$, for some $\beta^{\prime} \in[0, s)$,
- $f \in \mathcal{M}_{\beta}$ and $u \in \mathcal{S}_{K, 0, f ; \Omega}$ satisfies

$$
\begin{equation*}
\sup _{z \in B_{1} \cap \partial \Omega}\left|Q_{u, z}(r)\right| \leq c_{0} r^{-\varrho}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right) \quad \text { for every } r \in\left(0, r_{0}\right] \tag{5.7}
\end{equation*}
$$

then, we have

$$
\begin{equation*}
\sup _{z \in B_{1} \cap \partial \Omega}\left\|u-P_{r, z}(u)\right\|_{L^{2}\left(B_{r}(z)\right)}^{2} \leq C r^{N+2\left(2 s-\max \left(\beta, \beta^{\prime}\right)-\varrho\right)}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)^{2} \quad \text { for every } r>0 \tag{5.8}
\end{equation*}
$$

Proof. As in the proof of Proposition (3.2) if (5.8) does not hold, then we can find a sequence $r_{n} \rightarrow 0$, points $z_{n} \in B_{1} \cap \partial \Omega$ and sequences of functions, $a_{n}$ satisfying (1.2), $\lambda_{n}$ with $\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\frac{1}{n}$, $K_{a_{n}} \in \mathscr{K}\left(\lambda_{n}, a_{n}, \kappa, \Omega\right)$ with $\beta_{n}^{\prime} \in[0, s)$ and $\left\|g_{\Omega, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}} \leq 1, f_{n} \in \mathcal{M}_{\beta}$ and $u_{n} \in \mathcal{S}_{K_{a_{n}}, 0, f_{n} ; \Omega}$, with $\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}} \leq 1$ satisfying

$$
\begin{equation*}
\left|Q_{u_{n}, z_{n}}\left(r_{n}\right)\right| \leq c_{0} r_{n}^{-Q}, \tag{5.9}
\end{equation*}
$$

while, letting $\alpha_{n}:=2 s-\max \left(\beta, \beta_{n}^{\prime}\right) \in(s, 2 s)$, we have that

$$
\begin{equation*}
r_{n}^{-N-2\left(\alpha_{n}-\varrho\right)}\left\|u_{n}-P_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r_{n}}\left(z_{n}\right)\right)}^{2} \geq \frac{1}{16} \Theta_{n}\left(r_{n}\right) \geq \frac{n}{32}, \tag{5.10}
\end{equation*}
$$

for all $n \geq 2$. Here also $\Theta_{n}$ is a nonincreasing function on $(0, \infty)$ satisfying

$$
\begin{equation*}
\Theta_{n}(r) \geq r^{-N-2\left(\alpha_{n}-\varrho\right)} \sup _{z \in B_{1} \cap \partial \Omega}\left\|u_{n}-P_{r, z}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r}(z)\right)}^{2} \quad \text { for every } r>0 \text { and } n \geq 2 . \tag{5.11}
\end{equation*}
$$

We define

$$
v_{n}(x)=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\left(\alpha_{n}-\varrho\right)}\left\{u_{n}\left(r_{n} x+z_{n}\right)-P_{r_{n}, z_{n}}\left(u_{n}\right)\left(r_{n} x+z_{n}\right)\right\} .
$$

Since for $r_{n} \leq 1 / 2$, we have $d_{2}^{s}=d^{s}$ on $B_{r_{n}}\left(z_{n}\right)$, making a change of variable in (5.10) and in (5.3), we get

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{16} \quad \text { and } \quad \int_{B_{1}} v_{n}(x) \operatorname{dist}\left(x, \Omega_{n}^{c}\right)^{s} d x=0 \tag{5.12}
\end{equation*}
$$

where

$$
\Omega_{n}:=\frac{1}{r_{n}}\left(\Omega-z_{n}\right) .
$$

We further define $K_{n}(x, y):=r_{n}^{N+2 s} K_{a_{n}}\left(r_{n} x+z_{n}, r_{n} y+z_{n}\right)$ for every $x, y \in \mathbb{R}^{N}$,

$$
\begin{equation*}
\widehat{f}_{n}(x):=r_{n}^{2 s} f_{n}\left(r_{n} x+z_{n}\right) \quad \text { and } \quad g_{n}(x):=r_{n}^{2 s} g_{\Omega, K_{n}}\left(r_{n} x+z_{n}\right) . \tag{5.13}
\end{equation*}
$$

Since $u_{n} \in \mathcal{S}_{K_{a_{n}}, 0, f ; \Omega}$, by (5.6), it is plain that

$$
\begin{cases}\mathcal{L}_{K_{n}} v_{n}(x)=r_{n}^{-\left(\alpha_{n}-\varrho\right)} \Theta_{n}\left(r_{n}\right)^{-1 / 2}\left(\widehat{f}_{n}(x)-Q_{u_{n}, z_{n}}\left(r_{n}\right) g_{n}(x)\right) & \text { in } B_{2 r_{0} / r_{n}}\left(-z_{n}\right) \cap \Omega_{n}  \tag{5.14}\\ v_{n}=0 & \text { in } B_{2 r_{0} / r_{n}}\left(-z_{n}\right) \cap \Omega_{n}^{c}\end{cases}
$$

Claim: There exists $C=C\left(s, N, \beta, \varrho, r_{0}, c_{0}\right)>0$ such that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \alpha_{n}} \quad \text { for every } R \geq 1 \tag{5.15}
\end{equation*}
$$

Let us put $\alpha^{\prime}=\alpha_{n}-\varrho>0$, and we note that $0<s-\varrho<\alpha^{\prime}<2 s-\beta$, for every $n \geq 2$. By a change of variable, we have

$$
\begin{aligned}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} & =\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha^{\prime}}\left\|u-P_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r_{n} R}\left(z_{n}\right)\right)}^{2} \\
& \leq 2 \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha^{\prime}}\left\|u-P_{r_{n} R, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r_{n} R}\left(z_{n}\right)\right)} \\
& +2 \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha^{\prime}}\left\|P_{r_{n} R, z_{n}}\left(u_{n}\right)-P_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r_{n} R}\left(z_{n}\right)\right)}^{2} .
\end{aligned}
$$

Hence by (5.11) and the monotonicity of $\Theta_{n}$, we get

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq 2 R^{2 \alpha^{\prime}}+2 \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha^{\prime}}\left\|P_{r_{n} R, z_{n}}\left(u_{n}\right)-P_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r_{n} R}\left(z_{n}\right)\right)}^{2} . \tag{5.16}
\end{equation*}
$$

Now by (5.1), for all $r \in\left(0, r_{0}\right]$ and $z \in B_{1} \cap \partial \Omega$, we have

$$
\begin{aligned}
C\left|Q_{u_{n}, z}(2 r)-Q_{u_{n}, z}(r) \| B_{r}(z)\right|^{\frac{1}{2}} r^{s} & \leq\left\|P_{2 r, z}\left(u_{n}\right)-P_{r, z}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r}(z)\right)} \\
& \leq\left\|P_{2 r, z}(u)-u\right\|_{L^{2}\left(B_{2 r}(z)\right)}+\left\|P_{r, z}\left(u_{n}\right)-u_{n}\right\|_{L^{2}\left(B_{r}(z)\right)} \\
& \leq(2 r)^{\alpha^{\prime}} \Theta_{n}(2 r)^{1 / 2}(2 r)^{N / 2}+r^{\alpha^{\prime}} \Theta_{n}(r)^{1 / 2} r^{N / 2}
\end{aligned}
$$

Now using the monotonicity of $\Theta_{n}$, we then deduce that there exists a constant $C>0$ such that for every $n \geq 2, r \in\left(0, r_{0}\right]$ and $z \in B_{1} \cap \partial \Omega$,

$$
\left|Q_{u_{n}, z}(2 r)-Q_{u_{n}, z}(r)\right| \leq C r^{\alpha^{\prime}-s} \Theta_{n}(r)^{1 / 2}
$$

Hence, for $m \geq 0$, with $2^{m} \leq \frac{r_{0}}{r}$, using (5.2) and the monotonicity of $\Theta_{n}$, we get

$$
\begin{aligned}
\left\|P_{2^{m} r}\left(u_{n}\right)-P_{r}\left(u_{n}\right)\right\|_{L^{2}\left(B_{2^{m}}(z)\right)} & \leq \sum_{i=0}^{m}\left\|P_{2^{i} r}\left(u_{n}\right)-P_{2^{i-1} r}\left(u_{n}\right)\right\|_{L^{2}\left(B_{2^{i_{r}}}(z)\right)} \\
& \leq C \sum_{i=0}^{m}\left|Q_{u_{n}, z}\left(2^{i} r\right)-Q_{u_{n}, z}\left(2^{i-1} r\right) \| B_{2^{i} r}\right|^{\frac{1}{2}}\left(2^{i} r\right)^{s} \\
& \leq C r^{N / 2+\alpha^{\prime}} \Theta_{n}(r)^{1 / 2} \sum_{i=0}^{m} 2^{i\left(N / 2+\alpha^{\prime}\right)} .
\end{aligned}
$$

As a consequence, we find that

$$
\left\|P_{2^{m} r, z}\left(u_{n}\right)-P_{r, z}\left(u_{n}\right)\right\|_{L^{2}\left(B_{2^{m}}(z)\right)} \leq C r^{N / 2+\alpha^{\prime}} \Theta_{n}(r)^{1 / 2} 2^{m\left(N / 2+\alpha^{\prime}\right)}
$$

Now using this in (5.16), we then get, for $2^{m} \leq \frac{r_{0}}{r_{n}}$,

$$
\begin{aligned}
\left\|v_{n}\right\|_{L^{2}\left(B_{2} m\right)}^{2} & \leq 22^{m\left(N+2 \alpha^{\prime}\right)}+2 \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha^{\prime}}\left\|P_{2^{m} r, z_{n}}\left(u_{n}\right)-P_{r, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{2} m_{r}(z)\right)} \\
& \leq 22^{m\left(N+2 \alpha^{\prime}\right)}+C 2^{m N} \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-2 \alpha^{\prime}}\left(r_{n} 2^{m}\right)^{2 \alpha} \Theta_{n}\left(2^{m} r_{n}\right) \\
& \leq 22^{m\left(N+2 \alpha^{\prime}\right)}+C 2^{m\left(N+2 \alpha^{\prime}\right)} \Theta_{n}\left(r_{n}\right)^{-1} \Theta_{n}\left(2^{m} r_{n}\right) \\
& \leq C 2^{m\left(N+2 \alpha^{\prime}\right)}
\end{aligned}
$$

with $C$ is a positive constant depending neither on $n$ nor on $m$. We then conclude that

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \alpha^{\prime}} \quad \text { for every } R \geq 1, \text { with } R r_{n} \leq r_{0} \tag{5.17}
\end{equation*}
$$

We now consider the case $R \geq 1$ and $R r_{n} \geq r_{0}$. Using the fact that $\Theta_{n}\left(r_{n}\right)^{-1} \leq 1$ and $\alpha^{\prime}=\alpha_{n}-\varrho>0$ together with (5.3) and (5.2), we obtain
$\left\|v_{n}\right\|_{L^{2}\left(B_{R}\right)}^{2}=\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N-2 \alpha^{\prime}}\left\|u_{n}-P_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r_{n} R}\left(z_{n}\right)\right)}^{2} \leq r_{n}^{-N-2 \alpha^{\prime}}\left\|u_{n}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}^{2} \leq\left(R r_{0}^{-1}\right)^{N+2 \alpha^{\prime}}$.
This with (5.17) give (5.15), since $\alpha^{\prime}=\alpha_{n}-\varrho$. This finishes the proof of the claim.
Now by (5.15) and Hölder's inequality, we get

$$
\begin{equation*}
\left\|v_{n}\right\|_{L^{1}\left(B_{R}\right)} \leq C R^{N+\alpha_{n}} \quad \text { for all } R \geq 1 \text { and } n \geq 2 \tag{5.18}
\end{equation*}
$$

Since $g_{\Omega, K_{a_{n}}} \in \mathcal{M}_{\beta_{n}^{\prime}}$ (recall (5.13) and (3.1)), we have $\eta_{g_{n}}(1) \leq C r_{n}^{2 s-\beta_{n}^{\prime}}\left\|g_{\Omega, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}}$. Therefore by (5.9), we deduce that

$$
\begin{equation*}
r_{n}^{-\left(\alpha_{n}-\varrho\right)}\left|Q_{u_{n}, z_{n}}\left(r_{n}\right)\right| \eta_{g_{n}}(1) \leq c_{0} C r_{n}^{-\left(\alpha_{n}-\varrho\right)+2 s-\beta_{n}^{\prime}}\left\|g_{\Omega, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}} \leq C\left\|g_{\Omega, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}} \leq C \tag{5.19}
\end{equation*}
$$

with $C>0$ independent on $n$. From now on, we let $n$ large, so that $B_{r_{0} /\left(2 r_{n}\right)} \subset B_{2 r_{0} / r_{n}}\left(-z_{n}\right)$. Since $v_{n}$ satisfy (5.14), then letting $v_{n, M}=\varphi_{M} v_{n}$, we can apply Lemma 9.2 for $1<M<\frac{r_{0}}{2 r_{n}}$, to get

$$
\begin{cases}\mathcal{L}_{K_{n}} v_{n, M}=\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{-\left(\alpha_{n}-\varrho\right)}\left(\widehat{f}_{n}-Q_{u_{n}, z_{n}}\left(r_{n}\right) \widehat{g}_{n}(x)\right)+F_{n} & \text { in } B_{M / 2} \cap \Omega_{n} \\ v_{n, M}=0 & \text { in } B_{M / 2} \cap \Omega_{n}^{c}\end{cases}
$$

where $\left\|F_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C_{0}\left\|v_{n}\right\|_{\mathcal{L}_{s}^{1}}$. Using (5.15) and Hölder's inequality, we get $\left\|F_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C$. It then follows that

$$
\begin{equation*}
\eta_{F_{v_{n}}}(1) \leq C \quad \text { for every } n \geq 2 \tag{5.20}
\end{equation*}
$$

In addition (recalling (5.13)) by (3.1),

$$
\begin{equation*}
\eta_{\widehat{f}_{n}}(1) \leq C r_{n}^{\alpha_{n}} . \tag{5.21}
\end{equation*}
$$

Now by (5.15), Lemma 2.4 (5.19), (5.20) and (5.21), we obtain

$$
\left\{\kappa-\varepsilon \bar{C}\left(1+\Theta_{n}\left(r_{n}\right)^{-1 / 2} r_{n}^{\varrho}+\Theta_{n}\left(r_{n}\right)^{-1 / 2}+\eta_{F_{v_{n}}}(1)\right)\right\}\left[v_{n, M}\right]_{H^{s}\left(B_{M / 4}\right)}^{2} \leq C(M)
$$

Since $\Theta_{n}\left(r_{n}\right)^{-1 / 2} \rightarrow 0$ as $n \rightarrow \infty$, by (5.15), we then deduce that $v_{n}$ is bounded in $H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Hence by Sobolev embedding, $v_{n} \rightarrow v$ in $L_{l o c}^{2}\left(\mathbb{R}^{N}\right)$, for some $v \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. In addition, by (5.18) and since $\alpha_{n}=2 s-\max \left(\beta, \beta_{n}^{\prime}\right)<2 s$, we deduce that $v_{n} \rightarrow v$ in $\mathcal{L}_{s}^{1}$. We also have that $1_{\Omega_{n} \cap B_{1 /\left(2 r_{n}\right)}} \rightarrow 1_{H}$ in $L_{l o c}^{1}$ as $n \rightarrow \infty$, where $H$ is a half-space, with $0 \in \partial H$. Moreover, passing to the limit in (5.12), we get

$$
\begin{equation*}
\|v\|_{L^{2}\left(B_{1}\right)}^{2} \geq \frac{1}{16} \quad \text { and } \quad \int_{B_{1}} v(x) \operatorname{dist}\left(x, \mathbb{R}^{n} \backslash H\right)^{s} d x=0 \tag{5.22}
\end{equation*}
$$

Now, given $\psi \in C_{c}^{\infty}\left(H \cap B_{M}\right)$, since $\Omega$ is of class $C^{1}$, for $n$ large enough, we obtain $\psi \in C_{c}^{\infty}\left(\Omega_{n}\right)$. Since $v_{n}$ satisfy (5.14), then by Lemma 2.5, (5.21) and (5.19), we obtain

$$
\begin{aligned}
\mid \int_{\mathbb{R}^{2 N}}\left(v_{n}(x)\right. & \left.-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \mid \\
& \leq r_{n}^{\varrho} \Theta_{n}\left(r_{n}\right)^{-1 / 2} C(M)\left(1+\left\|g_{\Omega, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}}\right)\left(\|\psi\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}+\left\|\varphi_{M}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)}^{2}\right) \\
& \leq \Theta_{n}\left(r_{n}\right)^{-1 / 2} C(M) .
\end{aligned}
$$

Thanks to Lemma 2.2, letting $n \rightarrow \infty$, we thus get

$$
L_{b} v=0 \quad \text { in } H \quad \text { and } \quad v=0 \quad \text { on } \mathbb{R}^{N} \backslash H
$$

Here $b$ denotes the weak limit of $a_{n}$. Letting $\alpha:=\lim _{n \rightarrow \infty} \alpha_{n} \in[0,2 s)$, by (5.15), we have that $\|v\|_{L^{2}\left(B_{R}\right)}^{2} \leq C R^{N+2 \alpha}$ for every $R \geq 1$. It follows from Lemma 8.3 that $v$ does not change sign on $\mathbb{R}^{N}$, which is in contradiction with (5.22).

The next, result finalizes the two-step bootstrap argument mentioned earlier.
Lemma 5.2. Let $N \geq 1, s \in(0,1), \beta, \delta \in(0, s)$ and $\Omega$ a $C^{1, \gamma}$ domain, with $0 \in \partial \Omega$ as above. Let $K \in \mathscr{K}(\lambda, a, \kappa, \Omega)$ with $\left\|g_{\Omega, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq 1$, for some $\beta^{\prime} \in[0, s-\delta)$ and $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\min \left(\bar{\varepsilon}_{0}, \varepsilon_{1}\right)$, where $\varepsilon_{1}$ and $\bar{\varepsilon}_{0}$ are given by Lemma 5.1 and Proposition 3.3. respectively. Let $f \in \mathcal{M}_{\beta}$, and $u \in$ $H_{l o c}^{s}\left(B_{2}\right) \cap L^{2}\left(\mathbb{R}^{N}\right)$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} u=f \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c}, \tag{5.23}
\end{equation*}
$$

Then there exists $C>0$, only depending on $N, s, \beta, \Lambda, \kappa, \varepsilon_{1}, \bar{\varepsilon}_{0}, \delta$ and $\Omega$, and a function $\psi \in L^{\infty}\left(B_{1} \cap\right.$ $\partial \Omega)$, with $\|\psi\|_{L^{\infty}\left(B_{1} \cap \partial \Omega\right)} \leq C$, such that

$$
\sup _{z \in B_{1} \cap \partial \Omega}\left\|u-\psi(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)}^{2} \leq C r^{N+2\left(2 s-\max \left(\beta, \beta^{\prime}\right)\right)}\left(\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}}\right)^{2} \quad \text { for all } r \in\left(0, r_{0} / 4\right)
$$

Proof. For simplicity, we assume that $\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}} \leq 1$, up to dividing (5.29) with this quantity. Letting $\alpha:=2 s-\max \left(\beta, \beta^{\prime}\right) \in(s, 2 s)$, by Proposition 3.3, for every $\varrho \in(0, \alpha-s)$, there exists $c_{0}>0$, only depending on $N, s, \beta, \varrho, \Omega, \kappa$ and $\Lambda$, such that

$$
\begin{equation*}
\left|Q_{u, z}(r)\right| \leq c_{0} r^{-\varrho} \quad \text { for all } z \in B_{1} \cap \partial \Omega \text { and } r \in\left(0,2 r_{0}\right) \tag{5.24}
\end{equation*}
$$

We can apply Lemma 5.1 to get

$$
\begin{equation*}
\sup _{r>0} \sup _{z \in B_{1} \cap \partial \Omega} r^{-N-2(\alpha-\varrho)}\left\|u-P_{r, z}(u) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)}^{2} \leq C \tag{5.25}
\end{equation*}
$$

with the letter $C$ denoting, here an in the following, a positive constant which may vary from line to line but will depend only on $N, s, \beta, \varrho, \Omega, \kappa$ and $\Lambda$.

Claim: There exits $\psi_{0} \in L^{\infty}\left(\partial \Omega \cap B_{1}\right)$, satisfying $\left\|\psi_{0}\right\|_{L^{\infty}\left(\partial \Omega \cap B_{1}\right)} \leq C$, such that

$$
\begin{equation*}
\left\|u-\psi_{0}(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+\alpha-\varrho} \quad \text { for all } z \in B_{1} \cap \partial \Omega \text { and } r \in\left(0, r_{0} / 2\right) \tag{5.26}
\end{equation*}
$$

Indeed, for $r \in\left(0,2 r_{0}\right)$ and $z \in B_{1} \cap \partial \Omega$, we define

$$
Q_{z}(r):=Q_{u, z}(r)=\frac{\int_{B_{r}(z)} u(y) d^{s}(y) d y}{\int_{B_{r}(z)} d^{2 s}(y) d y}
$$

and recalling (5.3), we have that $P_{r, z}(u)(x)=Q_{z}(r) d^{s}(x)$ because $d_{2}^{s}=d^{s}$ on $B_{r}(z)$.
Let $0<\rho_{2} \leq \rho_{1} / 4 \leq r / 4$. Pick $k \in \mathbb{N}$ and $\sigma \in[1 / 4,1 / 2]$ such that $\rho_{2}=\sigma^{k} \rho_{1}$. Then provided $r \in\left(0,2 r_{0}\right)$, by (5.1) and (5.25), we get

$$
\begin{aligned}
\mid Q_{z}\left(\rho_{1}\right) & -Q_{z}\left(\rho_{2}\right)\left|\leq \sum_{i=0}^{k-1}\right| Q_{z}\left(\sigma^{i+1} \rho_{1}\right)-Q_{z}\left(\sigma^{i} \rho_{1}\right) \mid \\
& \leq C \sum_{i=0}^{k-1}\left(\sigma^{i+1} \rho_{1}\right)^{-\frac{N}{2}-s}\left(\left\|u-P_{\sigma^{i} \rho_{1}, z}(u)\right\|_{L^{2}\left(B_{\sigma^{i} \rho_{1}}(z)\right)}+\left\|u-P_{\sigma^{i+1} \rho_{1}, z}(u)\right\|_{L^{2}\left(B_{\sigma^{i+1} \rho_{1}}(z)\right)}\right) \\
& \leq C \sum_{i=0}^{k-1}\left(\sigma^{i+1} \rho_{1}\right)^{-\frac{N}{2}-s}\left(\left(\sigma^{i} \rho_{1}\right)^{\frac{N}{2}+\alpha-\varrho}+\left(\sigma^{i+1} \rho_{1}\right)^{\frac{N}{2}+\alpha-\varrho}\right) \\
& \leq C \rho_{1}^{\alpha-s-\varrho} \sigma^{-\frac{N}{2}-s} \sum_{i=0}^{k-1} \sigma^{i(\alpha-s-\varrho)} \leq C \rho_{1}^{\alpha-s-\varrho},
\end{aligned}
$$

where we used the fact that $\beta^{\prime} \in(0, s-\delta)$, so that $C$ does not depend on $\delta$ but only on the quantities mentioned above. Therefore, there exists $C>0$ such that for $0<\rho_{2} \leq \rho_{1} / 4 \leq r / 4$, with $r \in\left(0,2 r_{0}\right)$, and $z \in B_{1} \cap \partial \Omega$, we have

$$
\begin{equation*}
\left|Q_{z}\left(\rho_{1}\right)-Q_{z}\left(\rho_{2}\right)\right| \leq C \rho_{1}^{\alpha-s-\varrho} \leq C r^{\alpha-s-\varrho} \tag{5.27}
\end{equation*}
$$

By (5.1) and (5.25), for $r \in\left(0,2 r_{0}\right)$, we get

$$
\left\|P_{r_{0} / 4, z}(u)\right\|_{L^{2}\left(B_{r_{0} / 4}(z)\right)} \leq\left\|u-P_{r_{0} / 4, z}(u)\right\|_{L^{2}\left(B_{r_{0} / 4}(z)\right)}+\|u\|_{L^{2}\left(B_{r_{0} / 4}(z)\right)} \leq C+1
$$

Hence there exists $C>0$, such that for all $z \in B_{1} \cap \partial \Omega$,

$$
\begin{equation*}
\left|Q_{z}\left(r_{0} / 4\right)\right| \leq C \tag{5.28}
\end{equation*}
$$

From (5.27), we deduce that, for every fixed $z \in \partial B_{1} \cap \partial \Omega$ and any sequence $\left(r_{n}\right)_{n \in \mathbb{N}} \subset\left(0, r_{0} / 4\right]$ tending to zero, $\left(Q_{z}\left(r_{n}\right)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence, which is bounded by (5.28). We can thus define $\psi_{0}(z):=\lim _{r \rightarrow 0} Q_{z}(r)$. Now by (5.1) and (5.27) (letting $\left.\rho_{2} \rightarrow 0\right)$, for $r \in\left(0, r_{0} / 2\right)$, we get

$$
C r^{-N / 2-s}\left\|P_{r, z}(u)-\psi_{0}(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)} \leq\left|Q_{z}(r)-\psi_{0}(z)\right| \leq C r^{\alpha-s-\varrho}
$$

This in particular yields $\left|\psi_{0}(z)\right| r_{0}^{s} \leq C r_{0}^{\alpha-s-\varrho}+Q_{z}\left(r_{0} / 4\right) \leq C$, by (5.28). Consequently $\left\|\psi_{0}\right\|_{L^{\infty}\left(\partial B_{1} \cap \partial \Omega\right)} \leq$ $C$. Finally using (5.25) and the above inequality, we can estimate

$$
\begin{aligned}
\left\|u-\psi_{0}(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)} & \leq\left\|u-P_{r, z}(u)\right\|_{L^{2}\left(B_{r}(z)\right)}+\left\|P_{r, z}(u)-\psi_{0}(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)} \\
& \leq C r^{N / 2+\alpha-\varrho}
\end{aligned}
$$

This proves (5.26), as claimed.
Now (5.26), implies in particular that $\|u\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+s}$. Hence by Hölder's inequality and since $\alpha-\varrho>s$, there exists $c_{1}=c_{1}(N, s, \beta, \Omega, \Lambda, \varrho, \kappa, \delta)>0$ such that

$$
\left|Q_{u, z}(r)\right| \leq c_{1} \quad \text { for every } r \in\left(0, r_{0} / 2\right) \text { and } z \in B_{1} \cap \partial \Omega
$$

We can therefore apply Lemma 5.1 with $\varrho=0$ and thus use the same argument above starting from (5.25). We then conclude that there exists $\psi \in L^{\infty}\left(B_{1} \cap \partial \Omega\right)$, with $\|\psi\|_{L^{\infty}\left(B_{1} \cap \partial \Omega\right)} \leq C$ and such that

$$
\left\|u-\psi(z) d^{s}\right\|_{L^{2}\left(B_{r}(z)\right)} \leq C r^{N / 2+\alpha} \quad \text { for all } z \in B_{1} \cap \partial \Omega \text { and } r \in\left(0, r_{0} / 4\right)
$$

Combining Lemma 5.2 and the interior estimates in Theorem 4.2, we get the following result.
Corollary 5.3. Let $N \geq 1, s \in(0,1), \beta, \delta \in(0, s)$ and $\Omega$ a $C^{1, \gamma}$ domain, with $0 \in \partial \Omega$ as above. Let a satisfy 1.2 and $K \in \mathscr{K}(\lambda, a, \kappa)$. Suppose that $\left\|g_{\Omega, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}$, as defined in (5.6), with $\beta^{\prime} \in[0, s-\delta)$. Let $f \in \mathcal{M}_{\beta}$, and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} u=f \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c} \tag{5.29}
\end{equation*}
$$

Then there exist $C, \varepsilon_{2}>0$ and $r_{1}>0$, only depending on $N, s, \beta, \Lambda, \kappa, c_{0}, \delta$ and $\Omega$, such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{2}$, we have

$$
\left\|u / d^{s}\right\|_{C^{s-\max \left(\beta, \beta^{\prime}\right)\left(B_{r_{1}} \cap \bar{\Omega}\right)}} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) .
$$

Proof. We assume that $\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}} \leq 1$. Consider $\psi \in L^{\infty}\left(B_{1} \cap \partial \Omega\right)$ given by Lemma 5.2 Let $x_{0} \in \Omega \cap B_{r_{0} / 4}$ and $z_{0} \in \partial \Omega \cap B_{1}$ be such that $\left|x_{0}-z_{0}\right|=d\left(x_{0}\right) \leq r_{0} / 4$. Put $\rho=d\left(x_{0}\right) / 2$, so that $B_{\rho}\left(x_{0}\right) \subset B_{3 \rho}\left(z_{0}\right)$ and $B_{\rho}\left(x_{0}\right) \subset \subset \Omega$. We define $v(x)=u(x)-\psi\left(z_{0}\right) d_{2}^{s}(x)$ and $w_{\rho}(x):=\rho^{-s} v\left(\rho x+x_{0}\right)$. Then, letting $f_{\rho}(x):=\rho^{s} f\left(\rho x+x_{0}\right)$ and $g_{\rho}(x)=\rho^{s} g_{\Omega, K}\left(\rho x+x_{0}\right)$, we then have

$$
\begin{equation*}
\mathcal{L}_{K_{\rho, x_{0}}} w_{\rho}=f_{\rho}-\psi\left(z_{0}\right) g_{\rho} \quad \text { in } B_{1} \tag{5.30}
\end{equation*}
$$

with $K_{\rho, x_{0}}(x, y)=\rho^{N+2 s} K\left(\rho x+x_{0}, \rho y+x_{0}\right)$. We note that $K_{\rho, x_{0}} \in \mathscr{K}\left(\lambda_{\rho, x_{0}}, a, \kappa\right)$, where $\lambda_{\rho, x_{0}}(x, y)=$ $\lambda\left(\rho x+x_{0}, \rho y+y_{0}\right)$. In particular, decreasing $r_{0}$ if necessary,

$$
\left\|\lambda_{\rho, x_{0}}\right\|_{L^{\infty}\left(B_{1} \times B_{1}\right)} \leq\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}
$$

Therefore, by Lemma 5.2, we can find $\varepsilon_{2}>0$ small, independent on $\rho$ such that, if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{2}$, we have

$$
\begin{equation*}
\left\|w_{\rho}\right\|_{L^{2}\left(B_{1}\right)}=\rho^{-N / 2}\|v\|_{L^{2}\left(B_{\rho}\left(x_{0}\right)\right)} \leq \rho^{-N / 2} \rho^{-s}\left\|u-\psi\left(z_{0}\right) d^{s}\right\|_{L^{2}\left(B_{3 \rho}\left(z_{0}\right)\right)} \leq C \rho^{s-\max \left(\beta, \beta^{\prime}\right)} \tag{5.31}
\end{equation*}
$$

By Hölder's inequality, we also have $\|v\|_{L^{1}\left(B_{r}\left(z_{0}\right)\right)} \leq C r^{N+2 s-\max \left(\beta, \beta^{\prime}\right)}$, for every $r>0$. We can thus proceed as in the proof of Theorem 4.4 to get $\left\|w_{\rho}\right\|_{\mathcal{L}_{s}^{1}} \leq C \rho^{s-\max \left(\beta, \beta^{\prime}\right)}$, with $C$ independent on $\beta^{\prime}$. We note that by (3.1), $\left\|f_{\rho}\right\|_{\mathcal{M}_{\beta}} \leq \rho^{s-\beta}$ and $\left\|g_{\rho}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq \rho^{s-\beta^{\prime}}\left\|g_{\Omega, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}$. It then follows from (5.30), Corollary 4.1 and (5.31), that

$$
\begin{aligned}
\left\|w_{\rho}\right\|_{C^{s}\left(B_{1 / 2}\right)} & \leq C\left(\left\|w_{\rho}\right\|_{L^{2}\left(B_{1}\right)}+\left\|w_{\rho}\right\|_{\mathcal{L}_{s}^{1}}+\left\|f_{\rho}-\psi\left(z_{0}\right) g_{\rho}\right\|_{\mathcal{M}_{s}}\right) \\
& \leq C\left(\left\|w_{\rho}\right\|_{L^{2}\left(B_{1}\right)}+\left\|w_{\rho}\right\|_{\mathcal{L}_{s}^{1}}+\left\|f_{\rho}-\psi\left(z_{0}\right) g_{\rho}\right\|_{\mathcal{M}_{\max \left(\beta, \beta^{\prime}\right)}}\right) \\
& \leq C \rho^{s-\max \left(\beta, \beta^{\prime}\right)}
\end{aligned}
$$

Hence,

$$
\left\|w_{\rho}\right\|_{C^{s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{1 / 2}\right)} \leq C \rho^{s-\max \left(\beta, \beta^{\prime}\right)}
$$

Scaling back, and since $d_{2}^{s}=d^{s}$ on $B_{\rho / 2}\left(x_{0}\right)$, we get

$$
\left\|u-\psi\left(z_{0}\right) d^{s}\right\|_{L^{\infty}\left(B_{\rho / 2}\left(x_{0}\right)\right)} \leq C \rho^{2 s-\max \left(\beta, \beta^{\prime}\right)} \quad \text { and } \quad\left[u-\psi\left(z_{0}\right) d^{s}\right]_{C^{s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{\rho / 2}\left(x_{0}\right)\right)} \leq C \rho^{s}
$$

Since $\|\psi\|_{L^{\infty}\left(B_{1} \cap \partial \Omega\right)} \leq C$, the two inequalities above imply that

$$
\left[u / d^{s}\right]_{C^{s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{\rho / 2}\left(x_{0}\right)\right)} \leq C,
$$

which yields (see the proof of Proposition 1.1 in [49])

$$
\left[u / d^{s}\right]_{C^{s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{r_{1}} \cap \bar{\Omega}\right)} \leq C
$$

for some $r_{1} \leq r_{0} / 4$ and $C>0$, only depending on $\Omega, N, s, \beta, \Lambda, \kappa$ and $\gamma$. By Lemma 5.2. $\|u\|_{L^{2}\left(B_{r}\right)} \leq$ $C r^{N / 2+s}$. Then using similar arguments as in the proof of Theorem 4.4, we find that

$$
\left\|u / d^{s}\right\|_{L^{\infty}\left(B_{r_{1}} \cap \Omega\right)} \leq C
$$

Finally in the general case $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$, we can use similar cut-off arguments as in the proof of Corollary 4.1 to get the estimate involving only $\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}$ in the place of $\|u\|_{L^{2}\left(\mathbb{R}^{N}\right)}$. The proof of the corollary is thus finished.

## 6. Higher order interior Regularity

For $K$ a kernel satisfying (2.2), we define the functions
$J_{e, K}(x ; y)=\frac{1}{2}(K(x, x+y)+K(x, x-y)) \quad$ and $\quad J_{o, K}(x ; y)=\frac{1}{2}(K(x, x+y)-K(x, x-y))$.
We suppose in the following in this section that, for $2 s>1$, the function $x \mapsto P V \int_{\mathbb{R}^{N}} y J_{o, K}(x ; y) d y$ belongs to $L_{l o c}^{1}\left(B_{2} ; \mathbb{R}^{N}\right)$. We then consider the map $j_{o, K}: B_{2} \rightarrow \mathbb{R}^{N}$ defined as

$$
\begin{equation*}
j_{o, K}(x):=(2 s-1)_{+} P V \int_{\mathbb{R}^{N}} y J_{o, K}(x ; y) d y=(2 s-1)_{+} \sum_{i=1}^{N} e_{i} P V \int_{\mathbb{R}^{N}} y_{i} J_{o, K}(x ; y) d y \tag{6.1}
\end{equation*}
$$

where $\ell_{+}:=\max (\ell, 0)$ for all $\ell \in \mathbb{R}$.
We note that if $u \in C^{2 s+\varepsilon}(\Omega) \cap \mathcal{L}_{s}^{1}$, for some $\varepsilon>0$ and an open set $\Omega$, then for every $\psi \in C_{c}^{\infty}(\Omega)$, we have

$$
\begin{align*}
& \frac{1}{2} \int_{\mathbb{R}^{2 N}}(u(x)-u(y))(\psi(x)-\psi(y)) K(x, y) d x d y \\
&=\int_{\mathbb{R}^{N}} \psi(x)\left[P V \int_{\mathbb{R}^{N}}(u(x)-u(x+y)) J_{e, K}(x ; y) d y\right] d x \\
&+\int_{\mathbb{R}^{N}} \psi(x)\left[P V \int_{\mathbb{R}^{N}}(u(x)-u(x+y)) J_{o, K}(x ; y) d y\right] d x \tag{6.2}
\end{align*}
$$

Moreover for every $x \in \Omega$, we have

$$
\begin{equation*}
P V \int_{\mathbb{R}^{N}}(u(x)-u(x+y)) J_{e, K}(x ; y) d y=\frac{1}{2} \int_{\mathbb{R}^{N}}(2 u(x)-u(x+y)-u(x-y)) J_{e, K}(x ; y) d y \tag{6.3}
\end{equation*}
$$

We consider the family of affine functions

$$
q_{t, T}(x)=t+(2 s-1)_{+} T \cdot x \quad t \in \mathbb{R} \text { and } T \in \mathbb{R}^{N}
$$

For $z \in \mathbb{R}^{N}$, we define the following finite dimensional subspace of $L^{2}\left(B_{r}(z)\right)$, given by

$$
\mathcal{H}_{z}:=\left\{q_{t, T}(\cdot-z): t \in \mathbb{R}, T \in \mathbb{R}^{N}\right\}
$$

For $u \in L_{l o c}^{2}\left(\mathbb{R}^{N}\right), r>0$ and $z \in \mathbb{R}^{N}$, we let $\mathbf{P}_{r, z}(u) \in \mathcal{H}_{z}$ its $L^{2}\left(B_{r}(z)\right)$-projection on $\mathcal{H}_{z}$. Then

$$
\begin{equation*}
\int_{B_{r}(z)}\left(u(x)-\mathbf{P}_{r, z}(u)(x)\right) p(x) d x=0 \quad \text { for every } p \in \mathcal{H}_{z} \tag{6.4}
\end{equation*}
$$

Lemma 6.1. Let $s \in(1 / 2,1), \beta \in(0,2 s-1), \Lambda, \kappa>0$ and $\delta \in(0,2 s-1)$. Then there exist $C>0$ and $\varepsilon_{0}>0$ such that for every

- a satisfying (1.2),
- $\lambda: \mathbb{R}^{N} \times \mathbb{R}^{N} \rightarrow\left[0, k^{-1}\right]$ satisfying $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{0}$,
- $K \in \mathscr{K}(\lambda, a, \kappa)$ satisfying $\left\|\varphi_{2} j_{o, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq 1$ (see (6.1)), for some $\beta^{\prime} \in[0,2 s-1-\delta)$,
- $f \in \mathcal{M}_{\beta}$ and $u \in \mathcal{S}_{K, 0, f}$ satisfying $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}} \leq 1$,
we have

$$
\begin{equation*}
\sup _{r>0} r^{-\left(2 s-\max \left(\beta, \beta^{\prime}\right)\right)} \sup _{z \in B_{1}}\left\|u-\boldsymbol{P}_{r, z}(u)\right\|_{L^{\infty}\left(B_{r}(z)\right)} \leq C \tag{6.5}
\end{equation*}
$$

provided $2 s-\beta>1$ if $2 s>1$.

Proof. Then as in the proof of Proposition 3.2 if (6.5) does not hold, then we can find a sequence $r_{n} \rightarrow 0$, points $z_{n} \in B_{1}$ and sequences of functions, $a_{n}$ satisfying (1.2), $\lambda_{n}$ with $\left\|\lambda_{n}\right\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\frac{1}{n}$, $K_{a_{n}} \in \mathscr{K}\left(\lambda_{n}, a_{n}, \kappa\right)$ with $\left\|\varphi_{2} j_{o, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}} \leq 1$ and $\beta_{n}^{\prime} \in(0,2 s-1-\delta), f_{n} \in \mathcal{M}_{\beta}$ and $u_{n} \in \mathcal{S}_{K_{a_{n}}, 0, f_{n}}$, with $\left\|u_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left\|f_{n}\right\|_{\mathcal{M}_{\beta}} \leq 1$, such that

$$
\begin{equation*}
r_{n}^{-\left(2 s-\max \left(\beta, \beta_{n}^{\prime}\right)\right)}\left\|u_{n}-\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n}}\left(z_{n}\right)\right)} \geq \frac{1}{16} \Theta_{n}\left(r_{n}\right) \geq \frac{n}{32} \tag{6.6}
\end{equation*}
$$

Here also $\Theta_{n}$ is a nonincreasing function on $(0, \infty)$ satisfying

$$
\begin{equation*}
\Theta_{n}(r) \geq r^{-\left(2 s-\max \left(\beta, \beta_{n}^{\prime}\right)\right)} \sup _{z \in B_{1}}\left\|u_{n}-\mathbf{P}_{r, z}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r}(z)\right)} \quad \text { for every } r>0 \text { and } n \geq 2 \tag{6.7}
\end{equation*}
$$

To alleviate the notations, we put $\beta_{n}:=\max \left(\beta, \beta_{n}^{\prime}\right) \leq \max (\beta, 2 s-\delta)<2 s$, for every $n \geq 2$. We define

$$
w_{n}(x)=\Theta_{n}\left(r_{n}\right) r_{n}^{-\left(2 s-\beta_{n}\right)}\left[u_{n}\left(r_{n} x+z_{n}\right)-\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\left(r_{n} x+z_{n}\right)\right]
$$

so that

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{\infty}\left(B_{1}\right)}^{2} \geq \frac{1}{16} \tag{6.8}
\end{equation*}
$$

and, thanks to (6.4), by a change of variable,

$$
\begin{equation*}
\int_{B_{1}} w_{n}(x) p(x) d x=0 \quad \text { for every } p \in \mathcal{H}_{0} \tag{6.9}
\end{equation*}
$$

Claim: There exists $C=C(s, \beta, N, \delta)>0$ such that

$$
\begin{equation*}
\left\|w_{n}\right\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{2 s-\beta_{n}} \quad \text { for every } R \geq 1 \tag{6.10}
\end{equation*}
$$

To prove this claim, we note that by a change of variable, we have

$$
\begin{aligned}
\left\|w_{n}\right\|_{L^{\infty}\left(B_{R}\right)} & =\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-N} r_{n}^{-\left(2 s-\beta_{n}\right)}\left\|u-\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n} R}\left(z_{n}\right)\right)} \\
& \leq \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left\|u-\mathbf{P}_{r_{n} R, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n} R}\left(z_{n}\right)\right)} \\
& +\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left\|\mathbf{P}_{r_{n} R, z_{n}}\left(u_{n}\right)-\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n} R}\left(z_{n}\right)\right)} .
\end{aligned}
$$

We write $\mathbf{P}_{r, z}\left(u_{n}\right)(x)=t(r)+T(r) \cdot(x-z)$, for $r>0$ and $z \in B_{1}$. Then, we have

$$
\begin{aligned}
\left(\left|t(2 r)-t_{z}(r)\right|^{2}+|T(2 r)-T(r)|^{2} r^{2}\right)\left|B_{r}\right|^{2} & =\left\|\mathbf{P}_{2 r, z}\left(u_{n}\right)-\mathbf{P}_{r, z}\left(u_{n}\right)\right\|_{L^{2}\left(B_{r}(z)\right)}^{2} \\
& \left.\leq 2\left\|\mathbf{P}_{2 r, z}\left(u_{n}\right)-u_{n}\right\|_{L^{2}\left(B_{2 r}(z)\right)}^{2}+2\left\|\mathbf{P}_{r, z}\left(u_{n}\right)-u_{n}\right\|_{L^{2}\left(B_{r}(z)\right)}^{2}\right) \\
& \leq 2(2 r)^{2\left(2 s-\beta_{n}\right)} \Theta_{n}(2 r)^{2}(2 r)^{N}+2 r^{2\left(2 s-\beta_{n}\right)} \Theta_{n}(r)^{2} r^{N} \\
& \leq C r^{N} \Theta_{n}(r)^{2} r^{2\left(2 s-\beta_{n}\right)}
\end{aligned}
$$

where we have used the monotonicity of $\Theta_{n}$. We then have, for every $r>0$,

$$
|t(2 r)-t(r)|+|T(2 r)-T(r)| r \leq C \Theta_{n}(r) r^{2 s-\beta_{n}}
$$

Hence, since $2 s-\beta_{n} \geq 1$ if $2 s>1$, for every integer $m \geq 1$, we get

$$
\begin{aligned}
\left|T\left(2^{m} r\right)-T(r)\right| & =\sum_{i=1}^{m}\left|T\left(2^{i} r\right)-T\left(2^{i-1} r\right)\right| \\
& \leq C \Theta_{n}\left(2^{i-1} r\right) r^{2 s-\beta_{n}-1} \sum_{i=1}^{m} 2^{(i-1)\left(2 s-\beta_{n}-1\right)} \leq C \Theta_{n}(r)\left(2^{m} r\right)^{2 s-\beta_{n}-1}
\end{aligned}
$$

with $C>0$ a constant independent on $m$ and on $n \geq 2$, since $2 s-\beta_{n}-1 \geq \min (2 s-1-\beta, d)>0$. Similarly, we also have that $\left|t\left(2^{m} r\right)-t(r)\right| \leq C \Theta_{n}(r)\left(2^{m} r\right)^{2 s-\beta_{n}}$.

Now for $R \geq 1$, letting $m$ be the smallest integer such that $2^{m-1} \leq R \leq 2^{m}$, we then get

$$
\begin{aligned}
\left\|w_{n}\right\|_{L^{\infty}\left(B_{R}\right)}^{2} & =\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left\|u-\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n} R}\left(z_{n}\right)\right)} \\
& \leq \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left\|u-\mathbf{P}_{r_{n} R, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n} R}\left(z_{n}\right)\right)} \\
& +\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left\|\mathbf{P}_{r_{n} R, z_{n}}\left(u_{n}\right)-\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\right\|_{L^{\infty}\left(B_{r_{n} R}\left(z_{n}\right)\right)} \\
& \leq C \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left(r_{n} R\right)^{\left(2 s-\beta_{n}\right)} \Theta_{n}\left(R r_{n}\right) \\
& +\Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left(\left|t\left(R r_{n}\right)-t\left(r_{n}\right)\right|+\left|T\left(R r_{n}\right)-T\left(r_{n}\right)\right| r_{n} R\right) \\
& \leq C \Theta_{n}\left(r_{n}\right)^{-1} r_{n}^{-\left(2 s-\beta_{n}\right)}\left(r_{n} R\right)^{2 s-\beta_{n}} \Theta_{n}\left(R r_{n}\right) .
\end{aligned}
$$

By the monotonicity of $\Theta_{n}$, we get the claim.
It follows from (6.10) that

$$
\begin{equation*}
\left\|w_{n}\right\|_{\mathcal{L}_{s}^{1}} \leq C \quad \text { for every } n \geq 2 \tag{6.11}
\end{equation*}
$$

We define $K_{n}(x, y):=r_{n}^{N+2 s} K_{a_{n}}\left(r_{n} x+z_{n}, r_{n} y+z_{n}\right)$, and we note that

$$
J_{o, K_{n}}(x ; y)=r_{n}^{N+2 s} J_{o, K_{a_{n}}}\left(r_{n} x+z_{n} ; r_{n} y\right) .
$$

We put $\mathbf{P}_{n}(x):=\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)\left(r_{n} x+z_{n}\right)$ and let $\psi \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$. We use (6.3), to get

$$
\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(\mathbf{P}_{n}(x)-\mathbf{P}_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y \\
& \quad=0+\int_{\mathbb{R}^{N}} \psi(x)\left[P V \int_{\mathbb{R}^{N}}\left(\mathbf{P}_{n}(x)-\mathbf{P}_{n}(x+y)\right) J_{o, K_{n}}(x ; y) d y\right] d x
\end{aligned}
$$

Therefore writing $\mathbf{P}_{r_{n}, z_{n}}\left(u_{n}\right)(x)=t_{n}+(2 s-1)_{+} T_{n} \cdot\left(x-z_{n}\right)$, we see that

$$
P V \int_{\mathbb{R}^{N}}\left(\mathbf{P}_{n}(x)-\mathbf{P}_{n}(x+y)\right) J_{o, K_{n}}(x ; y) d y=(2 s-1)_{+}\left(r_{n} T_{n}\right) \cdot\left(r_{n}^{2 s} j_{o, K_{a_{n}}}\left(r_{n} x+z_{n}\right)\right) .
$$

We then conclude that

$$
\begin{equation*}
\mathcal{L}_{K_{n}} w_{n}=r_{n}^{-\left(2 s-\beta_{n}\right)} \Theta_{n}\left(r_{n}\right)^{-1}\left(\bar{f}_{n}+(2 s-1)_{+} h_{n}\right) \quad \text { in } B_{1 / 2 r_{n}}, \tag{6.12}
\end{equation*}
$$

where, noting that $\varphi_{2} \equiv 1$ on $B_{2}$ and recalling (6.1),

$$
\bar{f}_{n}(x):=r_{n}^{2 s} f_{n}\left(r_{n} x+z_{n}\right) \quad \text { and } \quad h_{n}(x)=\left(r_{n} T_{n}\right) \cdot\left(r_{n}^{2 s} j_{o, K_{a_{n}}}\left(r_{n} x+z_{n}\right)\right) \varphi_{2}\left(r_{n} x+z_{n}\right) .
$$

Since $\left\|u_{n}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq 1$, then $\left|T_{n}\right| \leq r_{n}^{-1}$. Therefore, since by assumption, $\left\|\varphi_{2} j_{o, K_{a_{n}}}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}} \leq 1$, we deduce that

$$
\begin{equation*}
\left\|\bar{f}_{n}\right\|_{\mathcal{M}_{\beta}}+\left\|h_{n}\right\|_{\mathcal{M}_{\beta_{n}^{\prime}}} \leq 2 r_{n}^{2 s-\beta_{n}} \leq 2 \tag{6.13}
\end{equation*}
$$

Next, we note that $K_{n} \in \mathscr{K}\left(\widetilde{\lambda}_{n}, a_{n}, \kappa\right)$, with $\widetilde{\lambda}_{n}(x, y)=\lambda_{n}\left(r_{n} x+z_{n}, r_{n} y+z_{n}\right)$. By assumption, $\left\|\widetilde{\lambda}_{n}\right\|_{L^{\infty}\left(B_{1 /\left(2 r_{n}\right)} \times B_{1 /\left(2 r_{n}\right)}\right)} \leq \frac{1}{n}$. Now by Corollary 4.1 (6.11) and (6.13), we deduce that $w_{n}$ is bounded in $C_{\text {loc }}^{\delta}\left(\mathbb{R}^{N}\right)$, for some $\delta>0$. In addition thanks to (6.10), up to a subsequence, it converges in $\mathcal{L}_{s}^{1} \cap C_{\text {loc }}^{\delta / 2}\left(\mathbb{R}^{N}\right)$ to some $w \in C_{\text {loc }}^{\delta}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$. Moreover, by (6.8) and (6.9), we deduce that

$$
\begin{equation*}
\|w\|_{L^{\infty}\left(B_{1}\right)} \geq \frac{1}{16} \tag{6.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{B_{1}} w(x) p(x) d x=0 \quad \text { for every } p \in \mathcal{H}_{0} \tag{6.15}
\end{equation*}
$$

We apply Lemma 2.4 (after a cut-off argument as in the proof of Proposition (3.2), use (6.10) and (6.13) to get

$$
\{\kappa-\varepsilon \bar{C}\}\left[\varphi_{M} w_{n}\right]_{H^{s}\left(B_{M / 2}\right)}^{2} \leq C(M) \quad \text { whenever } 1<M<\frac{1}{2 r_{n}},
$$

where we used the fact that $\Theta_{n}\left(r_{n}\right)^{-1} \leq 1$, for every $n \geq 2$. Therefore, provided $\varepsilon$ is small enough, we find that $w_{n}$ is bounded in $H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$ and thus $w \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. Now by Lemma 2.5 and (6.13), we have

$$
\left|\int_{\mathbb{R}^{2 N}}\left(v_{n}(x)-v_{n}(y)\right)(\psi(x)-\psi(y)) K_{n}(x, y) d x d y\right| \leq \Theta_{n}\left(r_{n}\right)^{-1} C(M)
$$

Letting $n \rightarrow \infty$ in the above inequality and using Lemma 2.2 we find that $L_{b} w=0$ in $\mathbb{R}^{N}$, with $b$ the limit of $a_{n}$ in the weak-star topology of $L^{\infty}\left(S^{N-1}\right)$. By (6.10) and Lemma 8.3, $w \in \mathcal{H}_{0}$, which contradicts (6.15) and (6.14).

We now have the following $C^{2 s-\beta}$ regularity estimates for $2 s-\beta>1$ - understanding that $C^{2 s-\beta}=C^{1,2 s-\beta-1}$ if $2 s-\beta>1$ by an abuse of notation.

Corollary 6.2. Let $s \in(1 / 2,1), \beta, \delta \in(0,2 s-1)$ and $\kappa, \Lambda>0$. Let a satisfy (1.2) and $K \in \mathscr{K}(\lambda, a, \kappa)$ satisfy $\left\|\varphi_{2} j_{o, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq c_{0}$, for some $c_{0} \geq 0$ and $\beta^{\prime} \in[0,2 s-1-\delta)$. Let $f \in \mathcal{M}_{\beta}$ and $u \in H^{s}\left(B_{2}\right) \cap \mathcal{L}_{s}^{1}$ satisfy

$$
\mathcal{L}_{K} u=f \quad \text { in } B_{2}
$$

Then there exists $\varepsilon_{0}>0$, only depending on $N, s, \beta, \kappa, c_{0}, \delta$ and $\Lambda$, such that if $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{0}$, then $u \in C^{2 s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{1 / 2}\right)$. Moreover, there exists $C=C\left(N, s, \beta, \Lambda, \kappa, c_{0}, \delta\right)$ such that

$$
\|u\|_{C^{2 s-\max \left(\beta, \beta^{\prime}\right)\left(B_{1 / 2}\right)}} \leq C\left(\|u\|_{L^{2}\left(B_{2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Proof. We first assume that $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\|f\|_{\mathcal{M}_{\beta}} \leq 1$ and $\left\|j_{o, K}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq 1$. By a well known iteration argument (see e.g [52] or the proof of Lemma (5.2), we find that

$$
|u(x)-u(z)-(2 s-1) T(z) \cdot(x-z)| \leq C|x-z|^{2 s-\max \left(\beta, \beta^{\prime}\right)} \quad \text { for every } x, z \in B_{1}
$$

with $\|T\|_{L^{\infty}\left(B_{1}\right)} \leq C$, provided $\|\lambda\|_{L^{\infty}\left(B_{2} \times B_{2}\right)}<\varepsilon_{0}$, with $\varepsilon_{0}$ given by Lemma 6.1 In particular, since $2 s-\max \left(\beta, \beta^{\prime}\right)>1$ then $\nabla u(z)=(2 s-1) T(z)$. Note that since $\beta^{\prime} \in[0,2 s-1-\delta)$, the constant $C$ does not depend on $\beta^{\prime}$ but on $\delta$ (see the proof of Lemma 5.2). By a classical extension theorem (see e.g. 54] [Page 177], we deduce that $u \in C^{2 s-\max \left(\beta, \beta^{\prime}\right)}\left(\overline{B_{1 / 2}}\right)$. Moreover

$$
\|u\|_{C^{2 s-\max \left(\beta, \beta^{\prime}\right)}\left(\overline{B_{1 / 2}}\right)} \leq C .
$$

Now for the general case $u \in H^{s}\left(B_{2}\right), f \in \mathcal{M}_{\beta}$ and $\left\|\varphi_{2} j_{o, K}\right\|_{\mathcal{M}_{\beta}} \leq c_{0}$, we use cut-off and scaling arguments as in the proof of Corollary 4.1 to get

$$
\|u\|_{C^{2 s-\max \left(\beta, \beta^{\prime}\right)}\left(B_{1 / 4}\right)} \leq C\left(\|u\|_{L^{\infty}\left(B_{1 / 2}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Now, decreasing $\varepsilon_{0}$ if necessary, by Corollary 4.1 we have

$$
\|u\|_{L^{\infty}\left(B_{1 / 2}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{1}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

The proof of the corollary is thus finished.

## 7. Proof of the main results

Proof of Theorem 1.1. Suppose that $\Omega$ is domain of class $C^{1, \gamma}$, with $0 \in \partial \Omega$. We consider $\Omega^{\prime}$ a bounded domain of class $C^{1, \gamma}$ which coincides with $\partial \Omega$ in a neighborhood of 0 . We let $\bar{r}>0$ small so that the distance function $d=\operatorname{dist}\left(\cdot, \mathbb{R}^{N} \backslash \Omega\right)$ is of class $C^{1, \gamma}$ in $\Omega \cap B_{4 \bar{r}}$ and $d_{\Omega^{\prime}}(x):=\operatorname{dist}\left(x, \mathbb{R}^{N} \backslash \Omega^{\prime}\right)=d(x)$ for every $x \in \bar{\Omega} \cap B_{4 \bar{r}}$. Now, for $x \in \Omega \cap B_{\bar{r}}$, we have

$$
(-\Delta)_{a}^{s}\left(\varphi_{2 \bar{r}} d^{s}\right)=(-\Delta)_{a}^{s}\left(\varphi_{2 \bar{r}}\left(d^{s}-d_{\Omega^{\prime}}^{s}\right)\right)+(-\Delta)_{a}^{s}\left(\varphi_{2 \bar{r}} d_{\Omega^{\prime}}^{s}\right)
$$

By [45, Proposition 2.3 and 2.6] and Lemma 9.2] for $\gamma \neq s$, there exists a constant $C=C(\Omega, N, s, \Lambda)>$ 0 , such that

$$
\left|(-\Delta)_{a}^{s}\left(\varphi_{2 \bar{r}} d_{\Omega^{\prime}}^{s}\right)(x)\right| \leq C d_{\Omega^{\prime}}^{(s-\gamma)_{+}}(x) \quad \text { for every } x \in \Omega^{\prime}
$$

Since $d^{s}-d_{\Omega^{\prime}}^{s}=0$ on $\Omega \cap B_{4 \bar{r}}$, we get $\left|(-\Delta)_{a}^{s}\left(\varphi_{2 \bar{r}}\left(d^{s}-d_{\Omega^{\prime}}^{s}\right)\right)\right| \leq C$ on $\Omega \cap B_{\bar{r}}$. We define $g_{\Omega, \mu_{a}}(x)=$ $(-\Delta)_{a}^{s}\left(\varphi_{2 \bar{r}} d^{s}\right)(x)$ for $x \in \Omega \cap B_{\bar{r}}$ and $g_{\Omega, \mu_{a}}(x)=0$ for $x \in \mathbb{R}^{N} \backslash\left(\Omega \cap B_{\bar{r}}\right)$. Then, there exists a constant $C=C(\Omega, N, s, \Lambda)>0$ such that

$$
\left|g_{\Omega, \mu_{a}}(x)\right| \leq C \max \left(d^{\gamma-s}(x), 1\right) \quad \text { for every } x \in \mathbb{R}^{N}
$$

Letting $\beta^{\prime}=(s-\gamma)_{+}$, for $\gamma \neq s$, we then deduce that $\left\|g_{\Omega, \mu_{a}}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq C_{1}(s, N, \Omega, \gamma, \Lambda)$. When $\gamma=s$, we can let $\beta^{\prime}=\varepsilon$ with $\varepsilon<\beta$. Moreover, up to scaling $\Omega$ to $\frac{1}{\delta} \Omega$, for some small $\delta>0$ depending only on $C_{1}$, we may assume that $\left\|g_{\Omega, \mu_{a}}\right\|_{\mathcal{M}_{\beta^{\prime}}} \leq 1$ and that $\partial \Omega$ seperates $B_{2}$ into two domains.
By Theorem4.4 there exists $C>0$, only depending on $N, s, \Omega, \beta, \Lambda, \gamma$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\|u\|_{L^{\infty}\left(B_{\bar{r} / 2}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{\bar{r}}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right)
$$

Then applying Corollary 5.3, we get the Theorem 1.1(ii). Now Theorem 1.1(i) follows immediately from Corollary 4.1 and Corollary 6.2 .

Proof of Theorem 1.3. It suffices to apply Corollary 4.1 to get $L^{\infty}$-bound and then apply Theorem 4.2 to get the result for $2 s-\max \left(\beta, \beta^{\prime}\right)>1$. If $2 s \leq 1$, then the result follows from Corollary 4.1,

Proof of Corollary 1.4. Using a scaling and a covering argument as in the proof of Theorem 4.2 together with Theorem 1.3, we get the result.

Proof of Theorem [1.6. It suffices to apply Theorem 4.4 to get $L^{\infty}$-bound and then apply Corollary 5.3 .

Proof of Corollary 1.7. Using a scaling and a covering argument as in the proof of Theorem 4.2 and applying Theorem 1.6, we get the result.

Proof of Theorem 1.5. By assumption, for every $x_{0} \in \overline{B_{1}}$ and $\varepsilon>0$, there exists $r_{\varepsilon}=r\left(\varepsilon, x_{0}\right) \in$ $(0,1 / 100)$ such that

$$
\left|\lambda_{K}(x, r, \theta)-\lambda_{K}\left(x_{0}, 0, \theta\right)\right|<\varepsilon \quad \text { for all } x \in B_{16 r_{\varepsilon}}\left(x_{0}\right) \text { and all } r \in\left(0,16 r_{\varepsilon}\right)
$$

This implies that

$$
\begin{equation*}
\left|K(x, y)-\mu_{a}(x, y)\right|<\varepsilon \mu_{1}(x, y) \quad \text { for } x \neq y \in B_{8 r_{\varepsilon}}\left(x_{0}\right) \tag{7.1}
\end{equation*}
$$

where $a(\theta)=\lambda_{K}\left(x_{0}, 0, \theta\right)$, which is even, since $\lambda_{o, K}\left(x_{0}, 0, \theta\right)=0$. Letting $K_{\varepsilon}(x, y)=r_{\varepsilon}^{N+2 s} K\left(r_{\varepsilon} x+\right.$ $\left.x_{0}, r_{\varepsilon} y+x_{0}\right), v_{\varepsilon}(x)=v\left(r_{\varepsilon} x+x_{0}\right)$ and $f_{\varepsilon}(x)=r_{\varepsilon}^{2 s} f\left(r_{\varepsilon} x+x_{0}\right)$, we then have

$$
\begin{equation*}
\mathcal{L}_{K_{\varepsilon}} v_{\varepsilon}=f_{\varepsilon} \quad B_{8} . \tag{7.2}
\end{equation*}
$$

We note that

$$
\widetilde{\lambda}_{K_{\varepsilon}}(x, r, \theta)=\widetilde{\lambda}_{K}\left(r_{\varepsilon} x+x_{0}, r_{\varepsilon} r, \theta\right)
$$

and thus, by assumption,

$$
\widetilde{\lambda}_{o, K_{\varepsilon}}(x, r, \theta)=\widetilde{\lambda}_{o, K}\left(r_{\varepsilon} x+x_{0}, r_{\varepsilon} r, \theta\right) \leq C r^{\alpha+(2 s-1)_{+}} .
$$

Clearly, by (7.1),

$$
\begin{equation*}
\left|K_{\varepsilon}(x, y)-\mu_{a}(x, y)\right|<\varepsilon \mu_{1}(x, y) \quad \text { for } x \neq y \in B_{8} \tag{7.3}
\end{equation*}
$$

By Corollary 4.1 provided $\varepsilon$ is small, for every $\varrho \in(0, s / 2)$, we have

$$
\begin{equation*}
\left\|v_{\varepsilon}\right\|_{C^{2 s-\varrho}\left(B_{7}\right)} \leq C\left(\left\|v_{\varepsilon}\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}+\left\|f_{\varepsilon}\right\|_{L^{\infty}\left(B_{8}\right)}\right) \tag{7.4}
\end{equation*}
$$

provided $2 s-\varrho \neq 1$. We let $v_{1, \varepsilon}:=\varphi_{1} v_{\varepsilon} \in H^{s}\left(\mathbb{R}^{N}\right) \cap C_{c}^{2 s-\varrho}\left(B_{2}\right)$. Then, we have

$$
\begin{equation*}
\mathcal{L}_{K_{\varepsilon}} v_{1, \varepsilon}=f_{1, \varepsilon} \quad \text { in } B_{2} \tag{7.5}
\end{equation*}
$$

with

$$
f_{1, \varepsilon}(x)=f_{\varepsilon}(x)+G_{v_{\varepsilon}}(x)
$$

where

$$
\begin{aligned}
G_{v_{\varepsilon}}(x) & =\int_{\mathbb{R}^{N}} v_{\varepsilon}(y)\left(\varphi_{1}(x)-\varphi_{1}(y)\right) K_{\varepsilon}(x, y) d y \\
& =\int_{\mathbb{R}^{N}}\left(v_{\varepsilon}(y)-v_{\varepsilon}(x)\right)\left(\varphi_{1}(x)-\varphi_{1}(y)\right) K_{\varepsilon}(x, y) d y+v_{\varepsilon}(x) \int_{\mathbb{R}^{N}}\left(\varphi_{1}(x)-\varphi_{1}(y)\right) K_{\varepsilon}(x, y) d y \\
& =\widetilde{G}_{\varepsilon}(x)+\frac{v_{\varepsilon}(x)}{2} \int_{S^{N-1}} \int_{0}^{\infty}\left(2 \varphi_{1}(x)-\varphi_{1}(x+r \theta)-\varphi_{1}(x-r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{e, K_{\varepsilon}}(x, r, \theta) d r d \theta \\
& +\frac{v_{\varepsilon}(x)}{2} \int_{S^{N-1}} \int_{0}^{\infty}\left(\varphi_{1}(x+r \theta)-\varphi_{1}(x-r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{o, K_{\varepsilon}}(x, r, \theta) d r d \theta
\end{aligned}
$$

where

$$
\widetilde{G}_{\varepsilon}(x):=\int_{S^{N-1}} \int_{0}^{\infty}\left(v_{\varepsilon}(x+r \theta)-v_{\varepsilon}(x)\right)\left(\varphi_{1}(x)-\varphi_{1}(x+r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{K_{\varepsilon}}(x, r, \theta) d r d \theta
$$

Since $v_{\varepsilon} \in C^{2 s-\varrho}\left(B_{7}\right) \cap C^{\alpha}\left(\mathbb{R}^{N}\right)$, and $\varphi_{1} \in C^{1}\left(\mathbb{R}^{N}\right)$, while the map $x \mapsto \widetilde{\lambda}_{K_{\varepsilon}}(x, r, \theta) \in C^{\alpha}\left(\mathbb{R}^{N}\right)$, direct computations show that

$$
\left\|\widetilde{G}_{v_{\varepsilon}}\right\|_{C^{\beta}\left(B_{3}\right)} \leq C\left(\left\|v_{\varepsilon}\right\|_{C^{2 s-\varrho\left(B_{4}\right)}}+\left\|v_{\varepsilon}\right\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right)
$$

for every $\beta \in(0, \min (\alpha, 2 s-\varrho))$. Now since $\varphi_{1} \in C^{2}\left(\mathbb{R}^{N}\right)$ and

$$
\begin{equation*}
\left|\widetilde{\lambda}_{o, K_{\varepsilon}}\left(x_{1}, r, \theta\right)-\widetilde{\lambda}_{o, K_{\varepsilon}}\left(x_{2}, r, \theta\right)\right| \leq C \min \left(r,\left|x_{1}-x_{2}\right|\right)^{\alpha+(2 s-1)_{+}} \tag{7.6}
\end{equation*}
$$

then we can find an $\alpha_{0}>0$, only depending on $\alpha, s$ and $\varrho$, such that for all $\beta \in\left(0, \alpha_{0}\right)$,

$$
\left\|G_{v_{\varepsilon}}\right\|_{C^{\beta}\left(B_{3}\right)} \leq C\left(\left\|v_{\varepsilon}\right\|_{C^{2 s-\varrho\left(B_{4}\right)}}+\left\|v_{\varepsilon}\right\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right)
$$

By this, we deduce that

$$
\begin{align*}
\left\|f_{1, \varepsilon}\right\|_{C^{\beta}\left(B_{3}\right)} & \leq C\left(\left\|f_{\varepsilon}\right\|_{C^{\beta}\left(B_{3}\right)}+\left\|v_{\varepsilon}\right\|_{C^{2 s-\varrho}\left(B_{4}\right)}+\left\|v_{\varepsilon}\right\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right) \\
& \leq C\left(\left\|f_{\varepsilon}\right\|_{C^{\beta}\left(B_{8}\right)}+\left\|v_{\varepsilon}\right\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right) \tag{7.7}
\end{align*}
$$

where, we used (7.4) for the last inequality, provided $\alpha_{0}<2 s-\varrho$.
We consider a nonegative function $\eta \in C_{c}^{\infty}(\mathbb{R})$ satisfying $\eta(t)=1$ for $|t| \leq 1$ and $\eta(t)=0$ for $|t| \geq 2$. We put $\eta_{\delta}(t)=\eta(t / \delta)$. We now define

$$
K_{\varepsilon, \delta}(x, y)=\eta_{\delta}(|x-y|) \mu_{a}(x, y)+\left(1-\eta_{\delta}(|x-y|)\right) K_{\varepsilon}(x, y)
$$

and we note that, by (7.3),

$$
\begin{equation*}
\left|K_{\varepsilon, \delta}(x, y)-\mu_{a}(x, y)\right|<\varepsilon\left(1+\|\eta\|_{L^{\infty}(\mathbb{R})}\right) \quad \text { for } x \neq y \in B_{8} \tag{7.8}
\end{equation*}
$$

In addition,

$$
\begin{equation*}
\tilde{\lambda}_{K_{\varepsilon, \delta}}(x, r, \theta)=\eta_{\delta}(r) a(\theta)+\left(1-\eta_{\delta}(r)\right) \tilde{\lambda}_{K_{\varepsilon}}(x, r, \theta) \tag{7.9}
\end{equation*}
$$

Now for $\varepsilon, \delta>0$, we consider $w_{\varepsilon, \delta} \in H^{s}\left(\mathbb{R}^{N}\right)$, the (unique) weak solution to

$$
\begin{cases}\mathcal{L}_{K_{\varepsilon, \delta}} w_{\varepsilon, \delta}=f_{1, \varepsilon} & \text { in } B_{2}  \tag{7.10}\\ w_{\varepsilon, \delta}=v_{1, \varepsilon}=0 & \text { in } \mathbb{R}^{N} \backslash B_{2}\end{cases}
$$

Multiplying (7.10) by $w_{\varepsilon, \delta}$, integrating, using the symmetry of $K_{\varepsilon, \delta}$ and Hölder's inequality, we deduce that

$$
\left[w_{\varepsilon, \delta}\right]_{H^{s}\left(\mathbb{R}^{N}\right)}^{2} \leq\left\|f_{1, \varepsilon}\right\|_{L^{2}\left(B_{2}\right)}\left\|w_{\varepsilon, \delta}\right\|_{L^{2}\left(B_{2}\right)}
$$

Hence by the Poincaré inequality, we find that

$$
\begin{equation*}
\left\|w_{\varepsilon, \delta}\right\|_{H^{s}\left(\mathbb{R}^{N}\right)} \leq C\left\|f_{1, \varepsilon}\right\|_{L^{2}\left(B_{2}\right)} \tag{7.11}
\end{equation*}
$$

where here and in the following, the letter $C$ denotes a constant, which may vary from line to line but independent on $\delta, f$ and $v$. Thanks to (7.8), we can apply Theorem 4.4 together with (7.11), to get

$$
\begin{equation*}
\left\|w_{\varepsilon, \delta}\right\|_{C^{s-\varrho\left(\mathbb{R}^{N}\right)}} \leq C\left(\left\|w_{\varepsilon, \delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{1, \varepsilon}\right\|_{L^{\infty}\left(B_{2}\right)}\right) \leq C\left\|f_{1, \varepsilon}\right\|_{L^{\infty}\left(B_{2}\right)} \tag{7.12}
\end{equation*}
$$

provided $\varepsilon$ small, independent on $\delta$. Furthermore by Corollary 4.1 provided $\varepsilon$ small and independent on $\delta$, using (7.11), we have that

$$
\begin{equation*}
\left\|w_{\varepsilon, \delta}\right\|_{C^{2 s-\varrho}\left(B_{1}\right)} \leq C\left(\left\|w_{\varepsilon, \delta}\right\|_{L^{2}\left(\mathbb{R}^{N}\right)}+\left\|f_{1, \varepsilon}\right\|_{L^{\infty}\left(B_{2}\right)}\right) \leq C\left\|f_{1, \varepsilon}\right\|_{L^{\infty}\left(B_{2}\right)} \tag{7.13}
\end{equation*}
$$

On the other hand, multiplying (7.10) by $w_{\varepsilon, \delta}-v_{1, \varepsilon}$, we see that

$$
\begin{equation*}
\left[w_{\varepsilon, \delta}-v_{1, \varepsilon}\right]_{H^{s}\left(\mathbb{R}^{N}\right)} \leq \int_{\mathbb{R}^{N} \times \mathbb{R}^{N}}\left(v_{1, \varepsilon}(x)-v_{1, \varepsilon}(y)\right)^{2}\left|K_{\varepsilon, \delta}(x, y)-K_{\varepsilon}(x, y)\right| d x d y \tag{7.14}
\end{equation*}
$$

Hence, by the Poincaré inequality, the dominated convergence and (7.12), as $\delta \rightarrow 0$, we may assume that $w_{\varepsilon, \delta} \rightarrow v_{1, \varepsilon}$ in $C^{s-2 \varrho}\left(\mathbb{R}^{N}\right) \cap C^{2 s-2 \varrho}\left(\overline{B_{1 / 2}}\right)$. Now for $\delta>0$, we have

$$
\mathcal{L}_{\mu_{a}} w_{\varepsilon, \delta}=f_{1, \varepsilon}+H_{\varepsilon, \delta} \quad \text { in } B_{2},
$$

where $H_{\varepsilon} \in C^{\min (\alpha, s-\varrho)}\left(\mathbb{R}^{N}\right)$ is given by

$$
H_{\varepsilon, \delta}(x):=\int_{S^{N-1}} \int_{0}^{\infty}\left(w_{\varepsilon, \delta}(x)-w_{\varepsilon, \delta}(x+r \theta)\right)\left(1-\eta_{\delta}(r)\right) r^{-1-2 s}\left(\widetilde{\lambda}_{\mu_{a}}(x, r, \theta)-\widetilde{\lambda}_{K_{\varepsilon}}(x, r, \theta)\right) d r d \theta
$$

It follows from [47, Theorem 1.1] and (7.7) that $w_{\varepsilon, \delta} \in C_{l o c}^{2 s+\alpha_{1}}\left(B_{2}\right)$, for some $\alpha_{1}>0$, only depending on $\alpha, s$ and $\varrho$. Now by (6.2),

$$
\begin{equation*}
L_{K_{\varepsilon, \delta}}\left(x, w_{\varepsilon, \delta}\right)=F_{\varepsilon, \delta} \quad \text { in } B_{2} \tag{7.15}
\end{equation*}
$$

where the $x$-dependent operator $L_{K_{\varepsilon, \delta}}(x, \cdot)$ is given by

$$
\begin{equation*}
L_{K_{\varepsilon, \delta}}(x, u)=\frac{1}{2} \int_{\mathbb{R}^{N}}(2 u(x)-u(x+y)-u(x-y)) J_{e, K_{\varepsilon, \delta}}(x ; y) d y \tag{7.16}
\end{equation*}
$$

and

$$
F_{\varepsilon, \delta}(x):=f_{1, \varepsilon}(x)+\frac{1}{2} \int_{S^{N-1}} \int_{0}^{\infty}\left(w_{\varepsilon, \delta}(x+r \theta)-w_{\varepsilon, \delta}(x-r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{o, K_{\varepsilon, \delta}}(x, r, \theta) d r d \theta
$$

We observe that $\widetilde{\lambda}_{o, K_{\varepsilon, \delta}}(x, r, \theta)=\left(1-\eta_{\delta}(r)\right) \widetilde{\lambda}_{o, K_{\varepsilon}}(x, r, \theta)$. Hence by (7.6) and the fact that $w_{\varepsilon, \delta} \in$ $C^{2 s-\varrho}\left(B_{1}\right) \cap C^{s-\varrho}\left(\mathbb{R}^{N}\right)$, then provided $\varrho<\alpha$, there exists $\alpha_{2}>0$, depending only on $s, \alpha$ and $\varrho$ such that

$$
\left\|F_{\varepsilon, \delta}\right\|_{C^{\beta}\left(B_{1 / 2}\right)} \leq C\left(\left\|f_{1, \varepsilon}\right\|_{C^{\beta}\left(B_{2}\right)}+\left\|w_{\varepsilon, \delta}\right\|_{C^{2 s-\varrho}\left(B_{1}\right)}+\left\|w_{\varepsilon, \delta}\right\|_{C^{s-\varrho\left(\mathbb{R}^{N}\right)}}\right)
$$

for all $\beta \in\left(0, \alpha_{2}\right)$. Hence by (7.12) and (7.13),

$$
\begin{equation*}
\left\|F_{\varepsilon, \delta}\right\|_{C^{\beta}\left(B_{1 / 2}\right)} \leq C\left\|f_{1, \varepsilon}\right\|_{C^{\beta}\left(B_{2}\right)} . \tag{7.17}
\end{equation*}
$$

It is easy to see, from (17.9) and our assumption that, for very $x_{1}, x_{2} \in \mathbb{R}^{N}$,

$$
\begin{aligned}
\int_{B_{2 \rho} \backslash B_{\rho}} & \left|J_{e, K_{\varepsilon}, \delta}\left(x_{1} ; y\right)-J_{e, K_{\varepsilon, \delta}}\left(x_{2} ; y\right)\right| d y= \\
& \int_{S^{N-1}} \int_{\rho}^{2 \rho}\left|\widetilde{\lambda}_{e, K_{\varepsilon, \delta}}\left(x_{1}, r, \theta\right)-\widetilde{\lambda}_{e, K_{\varepsilon, \delta}}\left(x_{2}, r, \theta\right)\right| r^{-1-2 s} d r d \theta \leq C\left|x_{1}-x_{2}\right|^{\alpha} \rho^{-2 s}
\end{aligned}
$$

We now apply [51, Theorem 1.1] to the equation (7.15) and use (7.17) together with (7.12), to deduce that, there exists $\bar{\alpha} \in\left(0, \min \left(\alpha_{0}, \alpha_{1}, \alpha_{2}\right)\right)$, independent on $\delta, v$ and $f$, such that

$$
\left\|w_{\varepsilon, \delta}\right\|_{C^{2 s+\beta}\left(B_{1 / 4}\right)} \leq C\left(\left\|F_{\varepsilon, \delta}\right\|_{C^{\beta}\left(B_{1}\right)}+\left\|w_{\varepsilon, \delta}\right\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right) \leq C\left\|f_{1, \varepsilon}\right\|_{C^{\beta}\left(B_{2}\right)}
$$

whenever $\beta \in(0, \bar{\alpha})$, with $2 s+\beta \notin \mathbb{N}$. In view of (7.7) and recalling that $w_{\varepsilon, \delta} \rightarrow v_{\varepsilon}$ in $C^{s-2 \varrho}\left(\mathbb{R}^{N}\right) \cap$ $C^{2 s-2 \varrho}\left(\overline{B_{1 / 2}}\right)$, then decreasing $\bar{\alpha}$ if necessary, we can send $\delta \rightarrow 0$ and get

$$
\left\|v_{\varepsilon}\right\|_{C^{2 s+\beta}\left(B_{1 / 4}\right)} \leq C\left(\left\|f_{\varepsilon}\right\|_{C^{\beta}\left(B_{8}\right)}+\left\|v_{\varepsilon}\right\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right)
$$

provided $2 s+\beta<2$ —noting that if $2 s>1$ then choosing $\varrho$ small, we have $\nabla w_{\varepsilon, \delta} \rightarrow \nabla v_{\varepsilon}$ pointwise on $B_{1 / 4}$. Scaling and translating back, we have thus proved that for every $x_{0} \in \overline{B_{1}}$, there exists $\delta_{x_{0}} \in(0,1)$ such that

$$
\|v\|_{C^{2 s+\beta}\left(B_{\delta_{x_{0}}}\left(x_{0}\right)\right)} \leq C\left(x_{0}\right)\left(\|f\|_{C^{\beta}\left(B_{2}\right)}+\|v\|_{C^{\beta}\left(\mathbb{R}^{N}\right)}\right)
$$

with $C\left(x_{0}\right)$ is a constant depending only on $x_{0}, N, s, \alpha, \beta$ and $\kappa$. Now by a covering argument as in the proof of Theorem 4.2, we get the desired estimate.
7.1. Proof of Theorem 1.8, We start with the following result which provides a global diffeomorphism that locally flattens the boundary of $\partial \Omega$ near the origin.

Lemma 7.1. Let $\Omega$ be an open set with boundary of class $C^{k, \gamma}$, for some $k \geq 1$ and $\gamma \in[0,1]$. Suppose that $0 \in \partial \Omega$ and that the interior unit normal of $\partial \Omega$ at 0 coincides with $e_{N}$. Then there exists $\rho_{0}>0$ such that for every $\rho \in\left(0, \rho_{0}\right)$, there exists a (global) diffeomorphism $\Phi_{\rho}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ with the following properties

- $D \Phi_{\rho}-i d \in C_{c}^{k, \gamma}\left(B_{2 \rho} ; \mathbb{R}^{N}\right)$,
- $\Phi_{\rho}(0)=0$ and $D \Upsilon_{\rho}(0)=i d$,
- $\left\|D \Upsilon_{\rho}-i d\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \rightarrow 0$, as $\rho \rightarrow 0$,
- $\Phi_{\rho}\left(B_{r}^{\prime}, x_{N}\right) \subset \Omega$ if and only if $x_{N} \in(0, \rho)$,
- $\Phi_{\rho}\left(B_{\rho}^{\prime}, 0\right) \subset \partial \Omega$,
- the distance function to $\Omega^{c}$, satisfies $d\left(\Phi_{\rho}(x)\right)=x_{N}$, for all $x \in B_{\rho}^{+}$,
- $\Phi_{\rho}$ is volume preserving in $\mathbb{R}^{N}$, i.e. $\operatorname{Det} D \Phi_{\rho}(x)=1$ for every $x \in \mathbb{R}^{N}$.

Here $B_{\rho}^{\prime}$ denotes the ball in $\mathbb{R}^{N-1}$ centered at 0 , with radius $\rho>0$.
 $\left(x^{\prime}, \phi(x)\right) \in \partial \Omega$ is a parameterization of a neighborhood of 0 in $\partial \Omega$. Since the normal of $\partial \Omega$ at 0 coincides with $e_{N}$ and $\Omega$ is of class $C^{1}$, we get

$$
\begin{equation*}
\|\nabla \phi\|_{L^{\infty}\left(B_{\rho}\right)} \rightarrow 0 \quad \text { as } \rho \rightarrow 0 \tag{7.18}
\end{equation*}
$$

Moreover decreasing $\rho_{1}$, if necessary, we have

$$
\left(x^{\prime}, x_{N}+\phi\left(x^{\prime}\right)\right) \in \Omega \quad \text { for all } x_{N} \in\left(0, \rho_{1}\right)
$$

Let $\eta \in C_{c}^{\infty}\left(B_{1}^{\prime}\right)$, with $\eta \equiv 1$ on $B_{1 / 2}^{\prime}$. For $\rho \in\left(0, \rho_{1}\right)$, we consider $\eta_{\rho} \in C_{c}^{\infty}\left(B_{\rho}^{\prime}\right)$ given by $\eta_{\rho}(x)=$ $\eta(x / \rho)$. We now define $\Phi_{\rho}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$, by

$$
\Phi_{\rho}\left(x^{\prime}, x_{N}\right):=\left(x^{\prime}, x_{N}+\eta_{\rho}\left(x^{\prime}\right) \phi\left(x^{\prime}\right)\right)
$$

By (7.18), we have

$$
\left\|D \Phi_{\rho}-i d\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)} \leq C \rho^{-1}\|\phi\|_{L^{\infty}\left(B_{\rho}\right)}+C\|\nabla \phi\|_{L^{\infty}\left(B_{\rho}\right)} \rightarrow 0 \quad \text { as } \rho \rightarrow 0
$$

This implies that there exists $\rho_{0}>0$ such that for every $\rho \in\left(0, \rho_{0}\right)$ and for every $x \in \mathbb{R}^{N}$, the Jacboian of $\Phi_{\rho}$ at $x, \operatorname{Det} D \Phi_{\rho}(x)=1$. Since, $\lim _{|x| \rightarrow \infty}\left|\Phi_{\rho}(x)\right| \rightarrow+\infty$, it follows from Hadamard's Global Inversion Theorem (see e.g. [26]) that $\Phi_{\rho}$ is a global diffeomorphism, for every $\rho \in\left(0, \rho_{0}\right)$. Clearly $\Phi_{\rho}$ satisfies all properties stated in the lemma.

Proof of Theorem 1.8 (completed). We assume that the interior unit normal of $\partial \Omega$ at 0 coincides with $e_{N}$. Consider $\Phi_{\rho} \in C^{1,1}\left(\mathbb{R}^{N} ; \mathbb{R}^{N}\right)$, given by Lemma 7.1. In the following, we fix $\rho>0$ small, so that

$$
\begin{equation*}
\sup _{x \in \mathbb{R}^{N}}\left|D \Phi_{\rho}(x)-i d\right|<\frac{1}{4} \tag{7.19}
\end{equation*}
$$

By Theorem 4.4 there exists $C>0$, only depending on $N, s, \Omega, \beta, \Lambda, \gamma, \delta$ and $\|V\|_{\mathcal{M}_{\beta}}$, such that

$$
\begin{equation*}
\|u\|_{L^{\infty}\left(B_{\rho}\right)} \leq C\left(\|u\|_{L^{2}\left(B_{2 \rho}\right)}+\|u\|_{\mathcal{L}_{s}^{1}}+\|f\|_{\mathcal{M}_{\beta}}\right) . \tag{7.20}
\end{equation*}
$$

We then have that $\mathcal{L}_{K} u=f-u V$ on $\Omega$. Letting $U(x)=u\left(\Phi_{\rho}(x)\right)$ and $F(x)=f\left(\Phi_{\rho}(x)\right)-$ $U(x) V\left(\Phi_{\rho}(x)\right)$, then by a change of variable, we have

$$
\frac{1}{2} \int_{\mathbb{R}^{2 N}}(U(x)-U(y))(\psi(x)-\psi(y)) K\left(\Phi_{\rho}(x), \Phi_{\rho}(y)\right) d x d y=\int_{\mathbb{R}^{N}} F(x) \psi(x) d x
$$

for every $\psi \in C_{c}^{\infty}\left(B_{\rho / 2}^{+}\right)$. Therefore, $U \in H^{s}\left(B_{\rho}\right) \cap \mathcal{L}_{s}^{1}$,

$$
\mathcal{L}_{K_{\rho}} U=F \quad \text { on } B_{\rho / 2}^{+}, \quad \text { and } \quad U=0 \quad \text { on } B_{\rho / 2}^{-}
$$

where $K_{\rho}(x, y):=K\left(\Phi_{\rho}(x), \Phi_{\rho}(y)\right), B_{\rho}^{-}:=B_{\rho} \cap\left\{x_{N}<0\right\}$ and $B_{\rho}^{+}:=B_{\rho} \cap\left\{x_{N}>0\right\}$. In particular, we have

$$
\begin{equation*}
\mathcal{L}_{K_{\rho}} U=\varphi_{\rho / 4} F \quad \text { on } B_{\rho / 4}^{+}, \quad \text { and } \quad U=0 \quad \text { on } B_{\rho / 4}^{-}, \tag{7.21}
\end{equation*}
$$

and we note that by (7.20), $\varphi_{\rho / 4} F \in \mathcal{M}_{\beta}$. We observe that

$$
\begin{aligned}
K_{\rho}(x, x+y) & =K\left(\Phi_{\rho}(x), \Phi_{\rho}(x+y)\right) \\
& =K\left(\Phi_{\rho}(x), \Phi_{\rho}(x)+\left[\Phi_{\rho}(x+y)-\Phi_{\rho}(x)\right]\right) .
\end{aligned}
$$

We define $A \in C^{0,1}\left(\mathbb{R}^{N} \times[0, \infty) \times S^{N-1} ; \mathbb{R}^{N}\right)$, by

$$
A(x, r, \theta):=\int_{0}^{1} D \Phi_{\rho}(x+r \tau \theta) \theta d \tau
$$

so that, for $r>0$,

$$
K_{\rho}(x, x+r \theta)=K\left(\Phi_{\rho}(x), \Phi_{\rho}(x)+r A(x, r, \theta)\right) .
$$

Now by (7.19), we can find constants $C, C^{\prime}>0$ such that, for every $x_{1}, x_{2} \in B_{2}, r_{1}, r_{2} \in[0,2)$ and $\theta \in S^{N-1}$,

$$
\begin{equation*}
\left|A\left(x_{1}, r_{1}, \theta\right)-A\left(x_{2}, r_{2}, \theta\right)\right| \leq C\left(\left|x_{1}-x_{2}\right|+\left|r_{1}-r_{2}\right|\right) \tag{7.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{\left|A\left(x_{1}, r_{1}, \theta\right)\right|} \geq C^{\prime} \tag{7.23}
\end{equation*}
$$

Since $\widetilde{\lambda}_{K} \in C^{s+\delta}\left(B_{2} \times[0,2) \times S^{N-1}\right)$, we then have that $K_{\rho} \in \widetilde{\mathscr{K}}\left(\kappa^{\prime}\right)$ and satisfies (2.2), for some $\kappa^{\prime}>0$, only depending on $\Omega, N, s$ and $\kappa$, with

$$
\widetilde{\lambda}_{K_{\rho}}(x, r, \theta)=|A(x, r, \theta)|^{-N-2 s} \widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|, \frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right) .
$$

By (7.22) and (7.23), it is clear that

$$
\begin{equation*}
\left|\widetilde{\lambda}_{K_{\rho}}\left(x_{1}, r_{1}, \theta\right)-\widetilde{\lambda}_{K_{\rho}}\left(x_{2}, r_{2}, \theta\right)\right| \leq C\left(\left|x_{1}-x_{2}\right|^{s+\delta}+\left|r_{1}-r_{2}\right|^{s+\delta}\right) \tag{7.24}
\end{equation*}
$$

We note that by assumption,

$$
\begin{aligned}
\left\lvert\, \widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|, \frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right)\right. & \left.-\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|,-\frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right) \right\rvert\, \\
& \leq c_{0} r^{s+\delta}|A(x, r, \theta)| \leq C r^{s+\delta}
\end{aligned}
$$

Next, we put

$$
G(x, r, \theta):=\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|, \frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right) .
$$

Then, using the above estimate, (7.22) and (7.23), we get

$$
\begin{aligned}
& |G(x, r, \theta)-G(x, r,-\theta)| \\
& \leq\left|\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|, \frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right)-\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|,-\frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right)\right| \\
& +\left|\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r, \theta)|,-\frac{A(x, r, \theta)}{|A(x, r, \theta)|}\right)-\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r,-\theta)|,-\frac{A(x, r, \theta)}{|A(x, r,-\theta)|}\right)\right| \\
& +\left|\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r,-\theta)|,-\frac{A(x, r, \theta)}{|A(x, r,-\theta)|}\right)-\widetilde{\lambda}_{K}\left(\Phi_{\rho}(x), r|A(x, r,-\theta)|, \frac{A(x, r,-\theta)}{|A(x, r,-\theta)|}\right)\right| \\
& \leq C r^{s+\delta}+C\left|\frac{1}{|A(x, r, \theta)|}-\frac{1}{|A(x, r,-\theta)|}\right|^{s+\delta}+C|A(x, r, \theta)+A(x, r,-\theta)|^{s+\delta} \\
& \leq C r^{s+\delta}
\end{aligned}
$$

with $C$ depends only on $\Omega, N, s$, and $\delta$. From this, (7.22) and (7.23), we easily deduce that

$$
\begin{equation*}
\left|\widetilde{\lambda}_{K_{\rho}}(x, r, \theta)-\widetilde{\lambda}_{K_{\rho}}(x, r,-\theta)\right| \leq C\left(r^{s+\delta}+r\right) \tag{7.25}
\end{equation*}
$$

We put $d_{+}(x)=\max \left(x_{N}, 0\right)$. Using (6.2), for $\psi \in C_{c}^{\infty}\left(B_{2}^{+}\right)$, we then have

$$
\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(d_{+}^{s}(x)-d_{+}^{s}(y)\right)(\psi(x)-\psi(y)) K_{\rho}(x, y) d x d y \\
& \quad=\int_{\mathbb{R}^{N}} \psi(x) \int_{S^{N-1}} \int_{0}^{\infty}\left(d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{e, K_{\rho}}(x, 0, \theta) d r d \theta d x \\
& \quad+\int_{\mathbb{R}^{N}} \psi(x) \int_{S^{N-1}} \int_{0}^{\infty}\left(d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right) r^{-1-2 s}\left(\widetilde{\lambda}_{e, K_{\rho}}(x, r, \theta)-\widetilde{\lambda}_{e, K_{\rho}}(x, 0, \theta)\right) d r d \theta d x \\
& \quad+\int_{\mathbb{R}^{N}} \psi(x) \int_{S^{N-1}} \int_{0}^{\infty}\left(d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{o, K_{\rho}}(x, r, \theta) d r d \theta d x
\end{aligned}
$$

Using the fact that $(-\Delta)_{1}^{s} d_{+}^{s}=0$ on $\left\{x_{N}>0\right\}$, we see that

$$
\int_{S^{N-1}} \int_{0}^{\infty}\left(d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{e, K_{\rho}}(x, 0, \theta) d r d \theta=0
$$

It follows that

$$
\begin{equation*}
\mathcal{L}_{K_{\rho}} d_{+}^{s}=F_{e}+F_{o} \quad \text { in } B_{2}^{+} \tag{7.26}
\end{equation*}
$$

where

$$
F_{e}(x):=\int_{S^{N-1}} \int_{0}^{\infty}\left(d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right) r^{-1-2 s}\left(\widetilde{\lambda}_{e, K_{\rho}}(x, r, \theta)-\widetilde{\lambda}_{e, K_{\rho}}(x, 0, \theta)\right) d r d \theta
$$

and

$$
F_{o}(x):=\int_{S^{N-1}} \int_{0}^{\infty}\left(d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right) r^{-1-2 s} \widetilde{\lambda}_{o, K_{\rho}}(x, r, \theta) d r d \theta
$$

Now from (7.24) and (7.25), we obtain

$$
\left|\widetilde{\lambda}_{e, K_{\rho}}(x, r, \theta)-\widetilde{\lambda}_{e, K_{\rho}}(x, 0, \theta)\right|+\left|\widetilde{\lambda}_{o, K_{\rho}}(x, r, \theta)\right| \leq C \min \left(r^{s+\delta}+r, 1\right)
$$

This with the fact that $\left|d_{+}^{s}(x)-d_{+}^{s}(x+r \theta)\right| \leq C r^{s}$ imply that

$$
\left\|F_{e}\right\|_{L^{\infty}\left(B_{1}\right)}+\left\|F_{o}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C
$$

In view of Lemma 9.2 and (7.26), we find that

$$
\mathcal{L}_{K_{\rho}}\left(\varphi_{1} d_{+}^{s}\right)=g_{\rho} \quad \text { in } B_{1 / 2}^{+}
$$

with $\left\|g_{\rho}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C$. By this, (7.21), (7.24) and (7.25), we can thus apply Corollary 1.7 to get

$$
\left\|U / d_{+}^{s}\right\|_{C^{s-\beta}}\left(\overline{B_{\varrho}^{+}}\right) \leq C\left(\|U\|_{L^{2}\left(B_{\rho}\right)}+\|U\|_{\mathcal{L}_{s}^{1}}+\|F\|_{\mathcal{M}_{\beta}}\right)
$$

for some $C>0$ and $\varrho \in(0, \rho / 4)$, only depending on $N, s, \Omega, \delta, \beta, \kappa, \rho$. Since $d\left(\Phi_{\rho}(x)\right)=x_{N}$ on $B_{\rho}^{+}$, then by a change of variable and using (7.20), we get the desired result.

## 8. Appendix 1: Liouville theorems

In this section we consider $\mathcal{H}$ being either $\mathbb{R}^{N}$ or the half-space $\mathbb{R}_{+}^{N}=\left\{x \in \mathbb{R}^{N}: x_{N}>0\right\}$. We prove a classification result for all functions $u \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$ satisfying $L_{b} u=0$ in $\mathcal{H}$ and $u=0$ in $\mathcal{H}^{c}$, provided $b$ is a weak limit of $a_{n}$ satisying (1.2) and $u$ satisfying some growth conditions. We note that in the case $u \in L_{l o c}^{\infty}\left(\mathbb{R}^{N}\right)$ such classification results (for more general nonolcal operators $L_{b}$ ) are proved in 47. We will need the following result for the proof of the Liouville theorems.

Lemma 8.1. Let $\Omega$ be an open set with $0 \in \partial \Omega$ and let a satisfy (1.2). We consider $u \in H_{\text {loc }}^{s}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$ satisfying

$$
(-\Delta)_{a}^{s} u=0 \quad \text { in } B_{2} \cap \Omega, \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c}
$$

Then there exists $C=C(N, s, \Lambda, \Omega)>0$ such that

$$
\|u\|_{L^{\infty}\left(B_{1} \cap \Omega\right)} \leq C\left(\|u\|_{\mathcal{L}_{s}^{1}}+\|u\|_{L^{2}\left(B_{2}\right)}\right) .
$$

Proof. The interior $L_{l o c}^{\infty}\left(B_{2} \cap \Omega\right)$ estimate follows from [17], where the authors used the De Giorgi iteration argument. We note that in [17], it is assumed that $u \in H^{s}\left(\mathbb{R}^{N}\right)$ but by carefully looking at their arguments, we see that this can be weakened to $u \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$. For the $L^{\infty}\left(B_{1} \cap \Omega\right)$ estimate, the proof is precisely the same.

In the following, for $b$ satisfying (2.5) and $f \in H^{s}\left(\mathbb{R}^{N}\right)$, we put

$$
[f]_{H_{b}^{s}\left(\mathbb{R}^{N}\right)}:=\left(\int_{\mathbb{R}^{2 N}}(f(x)-f(y))^{2} \mu_{b}(x, y) d x d y\right)^{1 / 2}
$$

Recall the Poincaré-type inequality related to this seminorm, see 24, 27,

$$
\begin{equation*}
C\|f\|_{L^{2}(A)} \leq[f]_{H_{b}^{s}\left(\mathbb{R}^{N}\right)} \quad \text { for every } f \in H^{s}\left(\mathbb{R}^{N}\right), \text { with } f=0 \text { on } A^{c} \tag{8.1}
\end{equation*}
$$

whenever $|A|<\infty$. Here $C$ is positive constant, only depending on $N, s, b$ and $A$. We note that if $b$ satisfies (1.2), then the constant $C$ in (8.1) can be chosen to depend only on $N, s, \Lambda$ and $A$.

Lemma 8.2. Let $\Omega$ be an open set with $0 \in \partial \Omega$. Suppose that there exists a sequence of functions $a_{n}$ satisfying (1.2) and $a_{n} \stackrel{*}{\rightharpoonup} b$ in $L^{\infty}\left(S^{N-1}\right)$. We consider $u \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right) \cap \mathcal{L}_{s}^{1}$ satisfying

$$
L_{b} u=0 \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad u=0 \quad \text { in } B_{2} \cap \Omega^{c} .
$$

Then there exists $C=C(N, s, \Lambda, \Omega)>0$ such that

$$
\|u\|_{L^{\infty}\left(B_{1} \cap \Omega\right)} \leq C\left(\|u\|_{\mathcal{L}_{s}^{1}}+\|u\|_{L^{2}\left(B_{2}\right)}\right) .
$$

Proof. Let $M \geq 1989$, so that $\varphi_{M} u \in H^{s}\left(\mathbb{R}^{N}\right)$. We let $v_{n, M} \in H^{s}\left(\mathbb{R}^{N}\right)$ be the (unique) solution to

$$
\begin{equation*}
(-\Delta)_{a_{n}}^{s} v_{n, M}=0 \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad v_{n, M}=\varphi_{M} u \quad \text { in }\left(B_{2} \cap \Omega\right)^{c} . \tag{8.2}
\end{equation*}
$$

By Lemma 8.2,

$$
\begin{equation*}
\left\|v_{n, M}\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)} \leq C\left(\left\|v_{n, M}\right\|_{\mathcal{L}_{s}^{1}}+\left\|v_{n, M}\right\|_{L^{2}\left(B_{2}\right)}\right) \tag{8.3}
\end{equation*}
$$

Moreover,
$(-\Delta)_{a_{n}}^{s}\left(v_{n, M}-\varphi_{M} u\right)=-(-\Delta)_{a_{n}}^{s}\left(\varphi_{M} u\right) \quad$ in $B_{2} \cap \Omega \quad$ and $\quad v_{n, M}-\varphi_{M} u=0 \quad$ in $\left(B_{2} \cap \Omega\right)^{c}$.
Multiplying this with $v_{n, M}-\varphi_{M} u$, integrating and using Hölder's inequality, we deduce that

$$
\left[v_{n, M}-\varphi_{M} u\right]_{H_{a_{n}}^{s}\left(\mathbb{R}^{N}\right)}^{2} \leq\left[v_{n, M}-\varphi_{M} u\right]_{H_{a_{n}}^{s}\left(\mathbb{R}^{N}\right)}\left[\varphi_{M} u\right]_{H_{a_{n}}^{s}\left(\mathbb{R}^{N}\right)}
$$

It follows that $\left[v_{n, M}-\varphi_{M} u\right]_{H_{a_{n}}^{s}\left(\mathbb{R}^{N}\right)} \leq\left[\varphi_{M} u\right]_{H_{a_{n}}^{s}\left(\mathbb{R}^{N}\right)}$ and thus by (8.1) and (1.2), we deduce that the sequence $\left(v_{n, M}\right)_{n}$ is bounded in $H^{s}\left(\mathbb{R}^{N}\right)$. Hence, up to a subsequence, $\left(v_{n, M}\right)_{n}$ converges in $L^{2}\left(B_{2} \cap \Omega\right)$ and in $\mathcal{L}_{s}^{1}$ to some function $v_{M}$. Passing to the limit in (8.2) as $n \rightarrow \infty$ and using Lemma 2.2, we find that

$$
\begin{equation*}
L_{b} v_{M}=0 \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad v_{M}=\varphi_{M} u \quad \text { in }\left(B_{2} \cap \Omega\right)^{c} \tag{8.4}
\end{equation*}
$$

Moreover passing to the limit in (8.3), we get

$$
\begin{equation*}
\left\|v_{M}\right\|_{L^{\infty}\left(B_{1} \cap \Omega\right)} \leq C\left(\left\|v_{M}\right\|_{\mathcal{L}_{s}^{1}}+\left\|v_{M}\right\|_{L^{2}\left(B_{2}\right)}\right) \tag{8.5}
\end{equation*}
$$

Next, letting $w_{M}:=v_{M}-\varphi_{M} u$, using Lemma 9.2, we find that

$$
\begin{equation*}
L_{b} w_{M}=-L_{b}\left(\varphi_{M} u\right)=-G_{M} \quad \text { in } B_{2} \cap \Omega \quad \text { and } \quad w_{M}=0 \quad \text { in }\left(B_{2} \cap \Omega\right)^{c} \tag{8.6}
\end{equation*}
$$

with $\left\|G_{M}\right\|_{L^{\infty}\left(B_{2}\right)} \leq C \int_{|y| \geq M}|u(y) \| y|^{-N-2 s} d y$. Here, $C$ is a positive constant only depending on $N, s$ and $\Lambda$. Multiplying the first equation in (8.6) by $w_{M}$, integrating and using (8.1), we obtain

$$
\begin{aligned}
{\left[w_{M}\right]_{H_{b}^{s}\left(\mathbb{R}^{N}\right)}^{2} } & \leq\left\|G_{M}\right\|_{L^{\infty}\left(B_{2}\right)} \int_{B_{2} \cap \Omega}\left|w_{M}(x)\right| d x \\
& \leq C \int_{|y| \geq M}|u(y) \| y|^{-N-2 s} d y\left[w_{M}\right]_{H_{b}^{s}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

where $C=C(s, N, \Lambda, \Omega, b)$. This then implies that

$$
\left[w_{M}\right]_{H_{b}^{s}\left(\mathbb{R}^{N}\right)} \leq C \int_{|y| \geq M}|u(y)||y|^{-N-2 s} d y
$$

We then deduce that $w_{M} \rightarrow 0$ in $L^{2}\left(\mathbb{R}^{N}\right)$ as $M \rightarrow \infty$, by (8.1). In addition, we have

$$
\left\|v_{M}-u\right\|_{\mathcal{L}_{s}^{1}} \leq\left\|v_{M}-\varphi_{M} u\right\|_{\mathcal{L}_{s}^{1}}+\left\|\left(1-\varphi_{M}\right) u\right\|_{\mathcal{L}_{s}^{1}}=\int_{B_{2} \cap \Omega} \frac{\left|w_{M}(y)\right|}{1+|y|^{N+2 s}} d y+\left\|\left(1-\varphi_{M}\right) u\right\|_{\mathcal{L}_{s}^{1}}
$$

We conclude that $v_{M} \rightarrow u$ in $\mathcal{L}_{s}^{1}$ as $M \rightarrow \infty$. Letting $M \rightarrow \infty$ in (8.5), we get

$$
\|u\|_{L^{\infty}\left(B_{1} \cap \Omega\right)} \leq C\left(\|u\|_{\mathcal{L}_{s}^{1}}+\|u\|_{L^{2}\left(B_{2}\right)}\right)
$$

Lemma 8.3. Suppose that there exists a sequence of functions $a_{n}$ satisfying (1.2) and $a_{n} \xrightarrow{*} b$ in $L^{\infty}\left(S^{N-1}\right)$. We consider $u \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$ satisfying

$$
\left\{\begin{array}{l}
L_{b} u=0 \quad \text { in } \mathcal{H}, \tag{8.7}
\end{array} \quad u=0 \quad \text { in } \mathbb{R}^{N} \backslash \mathcal{H}, ~ 子 u \|_{L^{2}\left(B_{R}\right)}^{2} \leq R^{N+2 \gamma} \quad \text { for some } \gamma<2 s \text { and for every } R \geq 1 .\right.
$$

Then $u$ is an affine function if $\mathcal{H}=\mathbb{R}^{N}$, while $u$ is proportional to $\max \left(x_{N}, 0\right)^{s}$ if $\mathcal{H}=\mathbb{R}_{+}^{N}$.
Proof. We put $v_{R}(z)=u(R z)$, for $R \geq 1$ and $z \in \mathbb{R}^{N}$. Since $L_{b} v_{R}=0$ in $\mathcal{H}$ and $v_{R}=0$ on $\mathcal{H}^{c}$, by Lemma 8.2,

$$
\left\|v_{R}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C\left(\left\|v_{R}\right\|_{\mathcal{L}_{s}^{1}}+\left\|v_{R}\right\|_{L^{2}\left(B_{2}\right)}\right)
$$

Scaling back, we get

$$
\|u\|_{L^{\infty}\left(B_{R}\right)} \leq C\left(R^{2 s} \int_{\mathbb{R}^{N}} \frac{|u(x)|}{R^{N+2 s}+|x|^{N+2 s}} d x+2^{\gamma} R^{\gamma}\right)
$$

Now, using Hölder's inequality and (8.7), we get $\|u\|_{L^{1}\left(B_{R}\right)} \leq C R^{N+\gamma}$. We then have

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} \frac{|u(x)|}{R^{N+2 s}+|x|^{N+2 s}} d x & \leq R^{-N-2 s} \int_{B_{R}}|u(x)| d x+\int_{|x| \geq R}|u(x)||x|^{-N-2 s} d x \\
& \leq C R^{-N-2 s} \int_{B_{R}}|u(x)| d x+\sum_{i=0}^{\infty} \int_{2^{i} R \leq|x| \leq 2^{i+1} R}|u(x)||x|^{-N-2 s} d x \\
& \leq C R^{-2 s+\gamma}+C R^{-2 s+\gamma} \sum_{i=0}^{\infty} 2^{-(N+2 s) i} 2^{(i+1)(N+\gamma)} \\
& \leq C\left(1+\sum_{i=0}^{\infty} 2^{-i(2 s-\gamma)}\right) R^{-2 s+\gamma} \leq C R^{-2 s+\gamma}
\end{aligned}
$$

We deduce that

$$
\|u\|_{L^{\infty}\left(B_{R}\right)} \leq C R^{\gamma} \quad \text { for all } R \geq 1
$$

It follows from the Liouville theorems in [45], that $u$ is an affine function if $\mathcal{H}=\mathbb{R}^{N}$, while $u$ is proportional to $\max \left(x_{N}, 0\right)^{s}$ when $\mathcal{H}=\mathbb{R}_{+}^{N}$.

## 9. Appendix 2: Some technical results

The following result is a Caccioppoli type inequality, see e.g. [17, 25, 41, for other versions. Note in the following lemma that $K$ could be any nonegative and nontrivial symmetric function on $\mathbb{R}^{N} \times \mathbb{R}^{N}$.

Lemma 9.1. Let $R>0$ and $K$ satisfy (2.2). Let $v \in H_{l o c}^{s}\left(B_{2 R}\right) \cap \mathcal{L}_{s}^{1}$ and $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{N}\right)$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} v=f \quad \text { in } B_{2 R} \cap \Omega, \quad \text { and } \quad v=0 \quad \text { in } B_{2 R} \cap \Omega^{c} . \tag{9.1}
\end{equation*}
$$

Then for every $\varepsilon>0$ and for every $\varphi \in C_{c}^{\infty}\left(B_{2 R}\right)$, we have

$$
\begin{aligned}
(1-\varepsilon) \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi^{2}(y) K(x, y) d y d x \leq & \int_{\mathbb{R}^{N}}|f(x) \| v(x)| \varphi^{2}(x) d x \\
& +\varepsilon^{-1} \int_{\mathbb{R}^{2 N}} v^{2}(x)(\varphi(x)-\varphi(y))^{2} K(x, y) d y d x
\end{aligned}
$$

Proof. Direct computations give

$$
\begin{aligned}
(v(x)-v(y))\left[v(x) \varphi^{2}(x)-v(y) \varphi^{2}(y)\right] & =(v(x)-v(y))^{2} \varphi^{2}(y)+v(x)(v(x)-v(y))\left[\varphi^{2}(x)-\varphi^{2}(y)\right] \\
& =(v(x)-v(y))^{2} \varphi^{2}(y)+\varphi(x) v(x)(v(x)-v(y))(\varphi(x)-\varphi(y)) \\
& +\varphi(y) v(x)(v(x)-v(y))(\varphi(x)-\varphi(y))
\end{aligned}
$$

Testing the equation (9.1) with $v \varphi^{2}$, and use the identity above together with the symmetry of $K$, to get

$$
\begin{align*}
\frac{1}{2} \int_{\mathbb{R}^{2 N}}(v(x)-v(y))^{2} \varphi^{2}(y) K(x, y) d y d x= & -\frac{1}{2} \int_{\mathbb{R}^{2 N}} \varphi(x) v(x)(v(x)-v(y))(\varphi(x)-\varphi(y)) K(x, y) d y d x \\
& -\frac{1}{2} \int_{\mathbb{R}^{2 N}} \varphi(y) v(x)(v(x)-v(y))(\varphi(x)-\varphi(y)) K(x, y) d x d y \\
& +\int_{\mathbb{R}^{N}} f(x) v(x) \varphi^{2}(x) d x \tag{9.2}
\end{align*}
$$

By Hölder and Young's inequalities, we get

$$
\begin{aligned}
& \left|\int_{\mathbb{R}^{2 N}} \varphi(y) v(x)(v(x)-v(y))(\varphi(x)-\varphi(y)) K(x, y) d y d x\right| \\
& \leq \int_{\mathbb{R}^{N}}|v(x)|\left(\int_{\mathbb{R}^{N}} \varphi^{2}(y)(v(x)-v(y))^{2} K(x, y) d y\right)^{1 / 2}\left(\int_{\mathbb{R}^{N}}(\varphi(x)-\varphi(y))^{2} K(x, y) d y\right)^{1 / 2} d x \\
& \leq\left(\int_{\mathbb{R}^{2 N}} \varphi^{2}(y)(v(x)-v(y))^{2} K(x, y) d y d x\right)^{1 / 2}\left(\int_{\mathbb{R}^{2 N}} v^{2}(x)(\varphi(x)-\varphi(y))^{2} K(x, y) d y d x\right)^{1 / 2} \\
& \leq \varepsilon \int_{\mathbb{R}^{2 N}} \varphi^{2}(y)(v(x)-v(y))^{2} K(x, y) d y d x+\varepsilon^{-1} \int_{\mathbb{R}^{2 N}} v^{2}(x)(\varphi(x)-\varphi(y))^{2} K(x, y) d y d x
\end{aligned}
$$

By similar arguments, we also have

$$
\begin{aligned}
& \left|\int_{\mathbb{R}^{2 N}} \varphi(x) v(x)(v(x)-v(y))(\varphi(x)-\varphi(y)) K(x, y) d y d x\right| \\
& \leq \varepsilon \int_{\mathbb{R}^{2 N}} \varphi^{2}(x)(v(x)-v(y))^{2} K(x, y) d y d x+\varepsilon^{-1} \int_{\mathbb{R}^{2 N}} v^{2}(x)(\varphi(x)-\varphi(y))^{2} K(x, y) d y d x
\end{aligned}
$$

Using the above two estimates above in (9.2), we get the result.
The following result provides a localization of solutions for nonlocal equations.
Lemma 9.2. Let $K$ satisfy (2.2) or $K=\mu_{b}$, for some $b \in L^{\infty}\left(S^{N-1}\right)$ with $b>0$. Let $v \in H_{\text {loc }}^{s}\left(B_{2 R}\right) \cap$ $\mathcal{L}_{s}^{1}$ and $f \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$ satisfy

$$
\begin{equation*}
\mathcal{L}_{K} v=f \quad \text { in } B_{2 R} \cap \Omega \quad \text { and } \quad v=0 \quad \text { in } B_{2 R} \cap \Omega^{c} \tag{9.3}
\end{equation*}
$$

for $R>0$. We let $v_{R}:=\varphi_{R} v$. Then

$$
\mathcal{L}_{K} v_{R}=f+G_{v, R} \quad \text { in } B_{R / 2} \cap \Omega
$$

where $G_{v, R}(x)=\varphi_{R / 2}(x) \int_{\mathbb{R}^{N}} v(y)\left(\varphi_{R}(x)-\varphi_{R}(y)\right) K(x, y) d y$.
Proof. For simplicity, we assume that $K=\mu_{b}$ for some even function $b \in L^{\infty}\left(S^{N-1}\right)$. Recall the identity, which follows from the symmetry of $K$,

$$
\mathcal{L}_{K}\left(v \varphi_{R}\right)=v \mathcal{L}_{K} \varphi_{R}+\varphi_{R} \mathcal{L}_{K} v-I\left(v, \varphi_{R}\right)
$$

with $I(v, \varphi)(x)=\int_{\mathbb{R}^{N}}(v(x)-v(y))\left(\varphi_{R}(x)-\varphi_{R}(y)\right) K(x, y) d y$. Since $\varphi_{R} \equiv 1$ on $B_{R}$, we have

$$
\begin{aligned}
I(v, \varphi)(x) & =\int_{|y| \geq R}(v(x)-v(y))\left(1-\varphi_{R}(y)\right) K(x, y) d y \\
& =v(x) \int_{|y| \geq R}\left(1-\varphi_{R}(y)\right) K(x, y) d y-\int_{|y| \geq R} v(y)\left(1-\varphi_{R}(y)\right) K(x, y) d y
\end{aligned}
$$

We note that

$$
-\varphi_{R / 2}(x) \mathcal{L}_{K} \varphi_{R}(x)+\varphi_{R / 2}(x) \int_{|y| \geq R}\left(1-\varphi_{R}(y)\right) K(x, y) d y=0
$$

Therefore letting

$$
G_{v, R}(x):=\varphi_{R / 2}(x) \int_{|y| \geq R} v(y)\left(1-\varphi_{R}(y)\right) K(x, y) d y
$$

it follows that

$$
\mathcal{L}_{K}\left(\varphi_{R} v\right)=f+G_{v, R} \quad \text { in } B_{R / 2} \cap \Omega
$$

The proof is thus finished.
We close this section with the following result.
Lemma 9.3. Let $\Omega$ be $C^{1}$ open set with $0 \in \partial \Omega$ and such that the interior normal of $\partial \Omega$ at 0 coincides with $e_{N}$. Then $d^{s} \in H^{s}\left(B_{r}\right)$, for some $r>0$.

Proof. Let $d_{+}(x):=\max \left(x_{N}, 0\right)$. Then since $(-\Delta)_{1}^{s} d_{+}^{s}(x)=0$ for every $x \in \mathbb{R}_{+}^{N}$, by Lemma 9.2, we have

$$
(-\Delta)_{1}^{s} v_{R}^{+}(x)=G_{R}(x) \quad \text { for every } x \in B_{R}^{+}
$$

with $v_{R}^{+}=\varphi_{2 R} d_{+}^{s}$ and $G_{R} \in L^{\infty}\left(\mathbb{R}^{N}\right)$. Now multiplying the above equation by $v_{R}^{+} \varphi_{R / 2}^{2}$ (supported in $B_{R}^{+}$, integrating on $B_{R}^{+}$and using the symmetry of $\mu_{1}$, we see that

$$
\frac{1}{2} \int_{\mathbb{R}^{2 N}}\left(v_{R}^{+}(x)-v_{R}^{+}(x)\right)\left(v_{R}^{+} \varphi_{R / 2}^{2}(x)-v_{R}^{+} \varphi_{R / 2}^{2}(y)\right) \mu_{1}(x, y) d x d y=\int_{\mathbb{R}^{N}} v_{R}^{+}(x) \varphi_{R / 2}^{2}(x) G_{R}(x) d x
$$

We may now apply Lemma 9.1 (or eventually following its proof), to deduce that

$$
\int_{\mathbb{R}^{2 N}}\left(v_{R}^{+}(x)-v_{R}^{+}(y)\right)^{2} \varphi_{R / 2}^{2}(y) \mu_{1}(x, y) d y d x<\infty
$$

This implies that $d_{+}^{s} \in H_{l o c}^{s}\left(\mathbb{R}^{N}\right)$. To conclude, we use the parameterization $\Phi_{\rho}$ given by Lemma 7.1 and make changes of variables, to get

$$
\begin{aligned}
\int_{\Phi_{\rho}\left(B_{\rho}\right)} \int_{\Phi_{\rho}\left(B_{\rho}\right)} \frac{\left(d^{s}(x)-d^{s}(y)\right)^{2}}{|x-y|^{N+2 s}} d x d y & =\int_{B_{\rho}} \int_{B_{\rho}} \frac{\left(d_{+}^{s}(x)-d_{+}^{s}(y)\right)^{2}}{\left|\Phi_{\rho}(x)-\Phi_{\rho}(y)\right|^{N+2 s}} d x d y \\
& \leq C \int_{B_{\rho}} \int_{B_{\rho}} \frac{\left(d_{+}^{s}(x)-d_{+}^{s}(y)\right)^{2}}{|x-y|^{N+2 s}} d x d y<\infty
\end{aligned}
$$

provided $\rho$ is small enough. The proof is thus finished.

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