UNIQUE CONTINUATION PROPERTIES FOR RELATIVISTIC SCHRÖDINGER OPERATORS WITH A SINGULAR POTENTIAL

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Abstract. Asymptotics of solutions to relativistic fractional elliptic equations with Hardy type potentials is established in this paper. As a consequence, unique continuation properties are obtained.

1. Introduction

Let $N > 2s$ with $s \in (0,1)$ and Ω be an open subset of $\mathbb{R}^N$. The purpose of the present paper is to establish unique continuation properties for the operator

$$H := (-\Delta + m^2)^s - \frac{a(\frac{x}{|x|})}{|x|^{2s}} - h(x), \tag{1.1}$$

where $m \geq 0$, $a \in C^1(\mathbb{S}^{N-1})$, and

$$h \in C^1(\Omega \setminus \{0\}), \quad |h(x)| + |x \cdot \nabla h(x)| \leq C_h |x|^{-2s+\chi} \text{ as } |x| \to 0, \tag{1.2}$$

for some $C_h > 0$ and $\chi \in (0,1)$. Answers to the problem of unique continuation will be derived from a precise description of the asymptotic behavior of solutions to $Hu = 0$ near 0.

From the mathematical point of view, a reason of interest in potentials of the type $a(x/|x|)|x|^{-2s}$ relies in their criticality with respect to the differential operator $(-\Delta + m^2)^s$; indeed, they have the same homogeneity as the s-laplacian $(-\Delta)^s$, hence they cannot be regarded as a lower order perturbation term. The physical interest in the study of properties of the Hamiltonian in (1.1) is manifest in the case $s = 1/2$; indeed, if $s = 1/2$ and $a \equiv Ze^2$ is constant, then the Hamiltonian (1.1) describes a spin zero relativistic particle of charge e and mass m in the Coulomb field of an infinitely heavy nucleus of charge Z, see e.g. [16,18].

Before going further, let us fix our notion of solutions to $Hu = 0$ in an open set Ω. For every $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $s \in (0,1)$, the relativistic Schrödinger operator with mass $m \geq 0$ is defined as

$$(-\Delta + m^2)^s \varphi(x) = c_{N,s} m^{\frac{N+2s}{2}} P.V. \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\, dy + m^{2s}\varphi(x), \tag{1.3}$$

for every $x \in \mathbb{R}^N$, where $P.V.$ indicates that the integral is meant in the principal value sense and

$$c_{N,s} = 2^{-(N+2s)/2+1} \pi^{-\frac{N}{2}} 2^{2s} \frac{s(1-s)}{\Gamma(2-s)},$$

see Remark 7.3. Here K_ν denotes the modified Bessel function of the second kind with order ν, see appendices B and C in sections 6 and 7. The Dirichlet form associated to $(-\Delta + m^2)^s$ on $C_c^\infty(\mathbb{R}^N)$ is given by

$$\begin{aligned} (u,v)_{H^s_m(\mathbb{R}^N)} :&= \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} d\xi \\ &= \frac{c_{N,s}}{2} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\, dx\, dy \\ &\quad + m^{2s} \int_{\mathbb{R}^N} u(x)v(x) dx, \end{aligned} \tag{1.4}$$

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where $\widehat{u}$ denotes the unitary Fourier transform of u. We define $H^s_m(\mathbb{R}^N)$ as the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm induced by the scalar product (1.4). If $m > 0$, $H^s_m(\mathbb{R}^N)$ is nothing but the standard $H^s(\mathbb{R}^N)$; then, we will write $H^s(\mathbb{R}^N)$ without the subscript "m".

By a weak solution to $Hu = 0$ in Ω, we mean a function $u \in H^s_m(\mathbb{R}^N)$ such that

$$(u, \varphi)_{H^s_m(\mathbb{R}^N)} = \int_\Omega \left(\frac{a(x/|x|)}{|x|^{2s}} u(x) + h(x)u(x) \right) \varphi(x)\, dx, \text{ for all } \varphi \in C^\infty_c(\Omega). \tag{1.5}$$

We notice that the right hand side of (1.5) is well defined in view of the following Hardy type inequality due to Herbst in [16] (see also [28]):

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}}\, dx \le \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2\, d\xi \le \|u\|^2_{H^s_m(\mathbb{R}^N)}, \quad \text{for all } u \in H^s_m(\mathbb{R}^N), \tag{1.6}$$

where

$$\Lambda_{N,s} := 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}.$$

A first aim of this paper is to give a precise description of the behavior near 0 of solutions to the equation $Hu = 0$, from which several unique continuation properties can be derived. The rate and the shape of u can be described in terms of the eigenvalues and the eigenfunctions of the following eigenvalue problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi) = \mu\, \theta_1^{1-2s} \psi, & \text{in } \mathbb{S}^N_+, \\ -\lim_{\theta_1 \to 0^+} \theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi \cdot \mathbf{e}_1 = \kappa_s a(\theta') \psi, & \text{on } \partial \mathbb{S}^N_+, \end{cases} \tag{1.7}$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)} \tag{1.8}$$

and

$$\mathbb{S}^N_+ = \{(\theta_1, \theta_2, \ldots, \theta_{N+1}) \in \mathbb{S}^N : \theta_1 > 0\} = \left\{ \tfrac{z}{|z|} : z \in \mathbb{R}^{N+1},\ z \cdot \mathbf{e}_1 > 0 \right\},$$

with $\mathbf{e}_1 = (1, 0, \ldots, 0)$; we refer to section 2 for a variational formulation of (1.7). From classical spectral theory (see section 2 for the details), if

$$\mu_1(a) := \inf_{\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \setminus \{0\}} \frac{\int_{\mathbb{S}^N_+} \theta_1^{1-2s} |\nabla \psi(\theta)|^2 dS - \kappa_s \int_{\mathbb{S}^{N-1}} a(\theta') \psi^2(0, \theta')\, dS'}{\int_{\mathbb{S}^N_+} \theta_1^{1-2s} \psi^2(\theta)\, dS} > -\infty, \tag{1.9}$$

then problem (1.7) admits a diverging sequence of real eigenvalues with finite multiplicity

$$\mu_1(a) \le \mu_2(a) \le \cdots \le \mu_k(a) \le \cdots,$$

the first one of which coincides with the infimum in (1.9), which is actually attained. Throughout the present paper, we will always assume that

$$\mu_1(a) > -\left(\frac{N-2s}{2} \right)^2. \tag{1.10}$$

Our first result is the following asymptotics of solutions at the singularity, which generalizes to the case $m > 0$ an analogous result obtained by the authors in [9] for $m = 0$.

Theorem 1.1. *Let $u \in H^s_m(\mathbb{R}^N)$ be a nontrivial weak solution to*

$$(-\Delta + m^2)^s u(x) - \frac{a(\frac{x}{|x|})}{|x|^{2s}} u(x) - h(x)u(x) = 0$$

in an open set $\Omega \subset \mathbb{R}^N$ containing the origin, with $s \in (0,1)$, $N > 2s$, $m \ge 0$, h satisfying assumption (1.2), and $a \in C^1(\mathbb{S}^{N-1})$. Then there exists an eigenvalue $\mu_{k_0}(a)$ of (1.7) and an eigenfunction ψ associated to $\mu_{k_0}(a)$ such that

$$\tau^{-\frac{2s-N}{2} - \sqrt{\left(\frac{2s-N}{2}\right)^2 + \mu_{k_0}(a)}} u(\tau x) \to |x|^{-\frac{N-2s}{2} + \sqrt{\left(\frac{2s-N}{2}\right)^2 + \mu_{k_0}(a)}} \psi\left(0, \tfrac{x}{|x|}\right) \quad \text{as } \tau \to 0^+,$$

in $C^{1,\alpha}_{\rm loc}(B_1'\setminus\{0\})$ *for some* $\alpha\in(0,1)$, *where* $B_1':=\{x\in\mathbb{R}^N: |x|<1\}$, *and, in particular,*

$$\tau^{-\frac{2s-N}{2}-\sqrt{\left(\frac{2s-N}{2}\right)^2+\mu_{k_0}(a)}}u(\tau\theta')\to\psi(0,\theta')\quad \text{in } C^{1,\alpha}(\mathbb{S}^{N-1})\quad \text{as } \tau\to 0^+,$$

where $\mathbb{S}^{N-1}=\partial\mathbb{S}^N_+$.

The proof of Theorem 1.1 is based on an Almgren type monotonicity formula (see [1, 14]) for a Caffarelli-Silvestre type extended problem. Indeed, for every $u\in H^s(\mathbb{R}^N)$ there exists a unique $w=\mathcal{H}(u)\in H^1(\mathbb{R}^{N+1}_+;t^{1-2s})$ weakly solving

$$\begin{cases}-\operatorname{div}(t^{1-2s}\nabla w)+m^2t^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+,\\ w=u, & \text{on } \partial\mathbb{R}^{N+1}_+=\{0\}\times\mathbb{R}^N,\end{cases}$$

where $\mathbb{R}^{N+1}_+=\{z=(t,x): t\in(0,+\infty),\ x\in\mathbb{R}^N\}$ and $H^1(\mathbb{R}^{N+1}_+;t^{1-2s})$ is defined as the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|w\|_{H^1(\mathbb{R}^{N+1}_+;t^{1-2s})}=\left(\int_{\mathbb{R}^{N+1}_+}t^{1-2s}\Big(|\nabla w(t,x)|^2+w^2(t,x)\Big)dt\,dx\right)^{1/2}.$$

Furthermore,

$$-\lim_{t\to0^+}t^{1-2s}\frac{\partial w}{\partial t}(x)=\kappa_s(-\Delta+m^2)^su(x),\quad \text{in } H^{-s}(\mathbb{R}^N),$$

in a weak sense, see Theorem 7.1 in Appendix B. Therefore $u\in H^s(\mathbb{R}^N)$ weakly solves $H(u)=0$ in Ω in the sense of (1.5) if and only if its extension $w=\mathcal{H}(u)$ satisfies

$$\begin{cases}-\operatorname{div}(t^{1-2s}\nabla w(t,x))+m^2t^{1-2s}w=0, & \text{in } \mathbb{R}^{N+1}_+,\\ w(0,x)=u(x), & \text{in } \mathbb{R}^N,\\ -\lim_{t\to0^+}t^{1-2s}\frac{\partial w}{\partial t}(t,x)=\kappa_s\Big(\frac{a(x/|x|)}{|x|^{2s}}w+hw\Big), & \text{in } \Omega,\end{cases}\tag{1.11}$$

in a weak sense. The asymptotics provided in Theorem 1.1 follows from combining an Almgren type monotonicity formula for problem (1.11) with a blow-up analysis; see [10–12] for the combination of such methods to prove not only unique continuation but also the precise asymptotics of solutions. We also refer to [4, 9] for monotonicity formulas in fractional problems.

As a particular case of Theorem 1.1, if $a\equiv0$ we obtain the following result.

Corollary 1.2. *Let* Ω *be an open bounded subset of* $\mathbb{R}^N$ *and* $u\in H^s_m(\mathbb{R}^N)$ *be a nontrivial weak solution to*

$$(-\Delta+m^2)^su(x)=h(x)u(x),\quad \text{in } \Omega,\tag{1.12}$$

with $s\in(0,1)$ *and* $h\in C^1(\Omega)$. *Then, for every* $x_0\in\Omega$, *there exists an eigenvalue* $\mu_{k_0}=\mu_{k_0}(0)$ *of problem* (1.7) *with* $a\equiv0$ *and an eigenfunction* ψ *associated to* μ_{k_0} *such that*

$$\tau^{-\frac{2s-N}{2}-\sqrt{\left(\frac{2s-N}{2}\right)^2+\mu_{k_0}}}u(x_0+\tau(x-x_0)) \to |x-x_0|^{-\frac{N-2s}{2}+\sqrt{\left(\frac{2s-N}{2}\right)^2+\mu_{k_0}}}\psi\Big(0,\tfrac{x-x_0}{|x-x_0|}\Big)\quad \text{as } \tau\to0^+,\tag{1.13}$$

in $C^{1,\alpha}(\{x\in\mathbb{R}^N: x-x_0\in B_1'\})$.

A relevant application of the asymptotic analysis contained in Theorem 1.1 and Corollary 1.2 is the validity of some unique continuation principles. A direct consequence of Theorem 1.1 is the following *strong unique continuation property*, which extends to the case $m>0$ an analogous result obtained for $m=0$ in [9].

Theorem 1.3. *Suppose that all the assumptions of Theorem 1.1 hold true. Let* $u\in H^s_m(\mathbb{R}^N)$ *be a weak solution to*

$$(-\Delta+m^2)^su(x)-\frac{a(\frac{x}{|x|})}{|x|^{2s}}u(x)-h(x)u(x)=0$$

in an open set $\Omega\subset\mathbb{R}^N$ *containing the origin. If* $u(x)=o(|x|^n)=o(1)|x|^n$ *as* $|x|\to0$ *for all* $n\in\mathbb{N}$, *then* $u\equiv0$ *in* Ω.

We mention that recently some strong unique continuation properties for fractional laplacian have been proved by several authors, see [9, 13, 21, 24, 29]. Corollary 1.2 allows also to prove the following unique continuation principle from sets of positive measures, which implies, as an interesting application, that the nodal sets of eigenfunctions for $(-\Delta+m^2)^s$ have zero Lebesgue measure.

Theorem 1.4. *Suppose that u is as in Corollary 1.2. If $u \equiv 0$ on a set $E \subset \Omega$ of positive measure, then $u \equiv 0$ in Ω.*

A direct application of Theorem 1.4 can be found in [13], where the authors proved the case $N=1$ and $m=0$.

Remark 1.5. *We point out that the results presented above still hold for the more general nonlinear problem*

$$(-\Delta+m^2)^s u = \frac{a(\frac{x}{|x|})}{|x|^{2s}}u(x) + h(x)u(x) + f(x,u),$$

which was considered in [9] for $m=0$. Assuming that

$$\begin{aligned} &f \in C^1(\Omega\times\mathbb{R}), \quad t \mapsto F(x,t) \in C^1(\Omega\times\mathbb{R}), \\ &|f(x,t)t| + |f'_t(x,t)t^2| + |\nabla_x F(x,t)\cdot x| \leq C_f\,|t|^p \text{ for a.e. } x\in\Omega \text{ and all } t\in\mathbb{R}, \end{aligned} \tag{1.14}$$

where $2 < p \leq 2^*(s) = \frac{2N}{N-2s}$, $F(x,t) = \int_0^t f(x,r)\,dr$, the asymptotics of Theorem 1.1 and the unique continuation principles of Theorems 1.3 and 1.4 still hold. Since the presence of the nonlinear term introduces essentially the same difficulties already treated in [9], we present here the details of proofs only for the linear problem focusing on the differences from [9] due to the introduction of the relativistic correction.

Beside the above unique continuation properties (UCPs), several results of independent interest will be proved in this paper. Indeed, to prove the UCPs, we transform, in the spirit of [9], problems of the type

$$(-\Delta+m^2)^s u(x) = G(x,u), \quad \text{in } \Omega, \tag{1.15}$$

into the problem

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w) + m^2 t^{1-2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ w = u, & \text{on } \mathbb{R}^N, \\ -\lim_{t\to 0} t^{1-2s}\frac{\partial w}{\partial t} = \kappa_s(-\Delta+m^2)^s u = \kappa_s G(x,w), & \text{in } \Omega. \end{cases} \tag{1.16}$$

Such extension is a generalization of the Caffarelli-Silvestre extension [4] and it is a particular case of more general extension theorems proved in Section 6. We actually derive asymptotics of solutions and unique continuation for problems of type (1.15) as a consequence of asymptotics and unique continuation for the corresponding extended problem (1.16).

In sections 2, 3 and 4 we present some preliminary results including some Hardy type inequalities, Schauder estimates for boundary value problems related to (1.16) and a Pohozaev type identity. These latter preparatory results will be used in the study of the monotonicity properties of the Almgren type frequency function associated to the extended problem (1.11); in section 5 a blow-up analysis of the extended problem will be also performed thus leading to the proof of Theorem 1.1 and, as consequences of Theorem 1.1, of Corollary 1.2 and Theorems 1.3 and 1.4. Finally, in Section 7 we describe some properties of the relativistic Schrödinger operator $(-\Delta+m^2)^s$.

2. Hardy type inequalities

Let us denote, for every $R>0$,

$$\begin{aligned} B_R^+ &= \{z=(t,x)\in\mathbb{R}^{N+1}_+ \,:\, |z|<R\}, \quad B'_R := \{x\in\mathbb{R}^N : |x|<R\}, \\ S_R^+ &= \{z=(t,x)\in\mathbb{R}^{N+1}_+ \,:\, |z|=R\}. \end{aligned}$$

For every $R>0$, we define the space $H^1(B_R^+;t^{1-2s})$ as the completion of $C^\infty(\overline{B_R^+})$ with respect to the norm

$$\|w\|_{H^1(B_R^+;t^{1-2s})} = \bigg(\int_{B_R^+} t^{1-2s}\Big(|\nabla w(t,x)|^2 + w^2(t,x)\Big)dt\,dx\bigg)^{1/2}.$$

We also define $H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$ as the completion of $C^\infty(\overline{\mathbb{S}^N_+})$ with respect to the norm

$$\|\psi\|_{H^1(\mathbb{S}^N_+;\theta_1^{1-2s})} = \left(\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\big(|\nabla_{\mathbb{S}^N}\psi(\theta)|^2 + \psi^2(\theta)\big)dS\right)^{1/2}$$

and

$$L^2(\mathbb{S}^N_+;\theta_1^{1-2s}) := \Big\{\psi : \mathbb{S}^N_+ \to \mathbb{R} \text{ measurable such that } \textstyle\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi^2(\theta)\,dS < +\infty\Big\}.$$

We recall the Sobolev trace inequality: there exists $S_{N,s} > 0$ such that, for all $w \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$

$$\left(\int_{\mathbb{R}^N} |w(0,x)|^{2N/(N-2s)}\,dx\right)^{(N-2s)/N} dx \le S_{N,s}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w(t,x)|^2 dt\,dx,$$

where $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ is defined as the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm $\big(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w(t,x)|^2 dt\,dx\big)^{1/2}$ (see e.g. [9] for details). Using a change of variables and writing $w(z) = f(|z|)\psi(z/|z|)$, with $f \in C^\infty_c(0,\infty)$, we can easily prove that there exists a well defined continuous trace operator

$$H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \to L^{2N/(N-2s)}(\partial\mathbb{S}^N_+) = L^{2N/(N-2s)}(\mathbb{S}^{N-1}),$$

so that, for some $C_{N,s} > 0$,

$$\|\psi(0,\cdot)\|^2_{L^{2N/(N-2s)}(\mathbb{S}^{N-1})} \le C_{N,s}\left(\int_{\mathbb{S}^N_+}\theta_1^{1-2s}|\nabla\psi(\theta)|^2\,dS + \int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi^2(\theta)\,dS\right) \tag{2.1}$$

for all $\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$.

In order to construct an orthonormal basis of $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$ for expanding solutions to $Hu = 0$ in Fourier series, we are naturally lead to consider the eigenvalue problem (1.7), which admits the following variational formulation: we say that $\mu \in \mathbb{R}$ is an eigenvalue of problem (1.7) if there exists $\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \setminus \{0\}$ (called eigenfunction) such that

$$Q(\psi,\vartheta) = \mu\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi(\theta)\vartheta(\theta)\,dS, \quad \text{for all } \vartheta \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}),$$

where

$$Q : H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \times H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \to \mathbb{R},$$
$$Q(\psi,\vartheta) = \int_{\mathbb{S}^N_+}\theta_1^{1-2s}\nabla\psi(\theta)\cdot\nabla\vartheta(\theta)\,dS - \kappa_s\int_{\mathbb{S}^{N-1}} a(\theta')\psi(0,\theta')\vartheta(0,\theta')\,dS'.$$

If $a \in L^{N/(2s)}(\mathbb{S}^{N-1})$ and (1.9) holds, then we can prove that the bilinear form Q is continuous and weakly coercive on $H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$. Moreover, since the weight θ_1^{1-2s} belongs to the second Muckenhoupt class, the embedding

$$H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \hookrightarrow\hookrightarrow L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$$

is compact. From classical spectral theory, problem (1.7) admits a diverging sequence of real eigenvalues with finite multiplicity $\mu_1(a) \le \mu_2(a) \le \cdots \le \mu_k(a) \le \cdots$ the first of which coincides with the infimum in (1.9) and then admits the variational characterization

$$\mu_1(a) = \min_{\psi\in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})\setminus\{0\}} \frac{Q(\psi,\psi)}{\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi^2(\theta)\,dS}. \tag{2.2}$$

We assume that (1.10) holds. To each $k \ge 1$, we associate an $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$-normalized eigenfunction $\psi_k \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$, $\psi_k \not\equiv 0$ corresponding to the k-th eigenvalue $\mu_k(a)$, i.e. satisfying

$$Q(\psi_k,\vartheta) = \mu_k(a)\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi_k(\theta)\vartheta(\theta)\,dS, \quad \text{for all } \vartheta \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}). \tag{2.3}$$

In the enumeration $\mu_1(a) \le \mu_2(a) \le \cdots \le \mu_k(a) \le \cdots$, we repeat each eigenvalue as many times as its multiplicity; thus exactly one eigenfunction ψ_k corresponds to each index $k \in \mathbb{N}$, $k \ge 1$. We can choose the functions ψ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$.

The following results will be useful to prove Hard-type inequalities for the potential $a(x/|x|)|x|^{-2s}$ with a belonging to some L^p space; indeed, the Hardy inequality for this potential involves only $\mu_1(a)$ whose corresponding eigenfunction is simple.

Lemma 2.1. *If $a \in L^{N/2s}(\mathbb{S}^{N-1})$ and a satisfies (1.9), then $\mu_1(a)$ is attained by a positive minimizer. Moreover, the mapping $a \mapsto \mu_1(a)$ is continuous in $L^q(\mathbb{S}^{N-1})$ for every $q > N/(2s)$.*

Proof. The first assertion is classical thanks to the Sobolev-trace inequality on $\mathbb{S}^N_+$ (2.1), so we skip the details. Now let $q > N/(2s)$ and $a_n \in L^q(\mathbb{S}^{N-1})$ such that $a_n \to a$ in $L^q(\mathbb{S}^{N-1})$ (and a_n, a satisfy (1.9)). For every $\psi \in C^\infty(\overline{\mathbb{S}^N_+})$, $\psi \not\equiv 0$, using Hölder inequality, we can see that

$$\begin{aligned}\mu_1(a_n) &\le \frac{\int_{\mathbb{S}^N_+} \theta_1^{1-2s} |\nabla\psi(\theta)|^2\, dS - \kappa_s \int_{\mathbb{S}^{N-1}} a_n(\theta')\psi^2(0,\theta')\, dS'}{\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\psi^2(\theta)\, dS} \\ &\le \frac{\int_{\mathbb{S}^N_+} \theta_1^{1-2s} |\nabla\psi(\theta)|^2\, dS - \kappa_s \int_{\mathbb{S}^{N-1}} a(\theta')\psi^2(0,\theta')\, dS'}{\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\psi^2(\theta)\, dS} \\ &\quad + \kappa_s \frac{\|a_n - a\|_{L^{N/(2s)}(\mathbb{S}^{N-1})} \|\psi(0,\cdot)\|^2_{L^{2N/(N-2s)}(\mathbb{S}^{N-1})}}{\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\psi^2(\theta)\, dS}.\end{aligned}$$

So, choosing ψ to be a minimizer for $\mu_1(a)$, we get

$$\mu_1(a_n) \le \mu_1(a) + o(1), \quad \text{as } n \to \infty.$$

Define $\mathcal{C}_\delta = \{\sigma \in \mathbb{S}^N_+ : \operatorname{dist}(\sigma, \partial\mathbb{S}^N_+) < \delta\}$ for all $\delta > 0$. Let $\chi_\delta \in C^\infty(\mathbb{S}^N)$ be such that $\chi_\delta = 1$ on $\mathcal{C}_\delta$ and $\chi_\delta = 0$ on $\mathbb{S}^N \setminus \mathcal{C}_{2\delta}$. Next, let ψ_n be a positive minimizer for $\mu_1(a_n)$ normalized so that $\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\psi_n^2(\theta)\, dS = 1$. Then

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_n) = \mu_1(a_n)\, \theta_1^{1-2s}\psi_n, & \text{in } \mathbb{S}^N_+, \\ -\lim_{\theta_1\to 0^+} \theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi_n \cdot \mathbf{e}_1 = \kappa_s a_n(\theta')\psi_n, & \text{on } \partial\mathbb{S}^N_+. \end{cases}$$

Multiply the above equation by $\psi_n\chi_\delta^2$ and integrate by parts to get

$$\begin{aligned}&\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\chi_\delta^2|\nabla\psi_n|^2(\theta)\, dS + 2\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\nabla\psi_n \cdot \chi_\delta\psi_n\nabla\chi_\delta(\theta)\, dS \\ &\qquad \le \mu_1(a) + o(1) + \kappa_s \int_{\mathbb{S}^{N-1}} a_n\chi_\delta^2\psi_n^2(0,\theta')\, dS'.\end{aligned}$$

Hence by Hölder's inequality

$$\begin{aligned}&\int_{\mathbb{S}^N_+} \theta_1^{1-2s}|\nabla(\chi_\delta\psi_n)|^2(\theta)\, dS - \int_{\mathbb{S}^N_+} \theta_1^{1-2s}|\nabla\chi_\delta|^2\psi_n^2(\theta)\, dS \\ &\qquad \le \mu_1(a) + o(1) + \kappa_s\|a_n\|_{L^{N/2s}(\mathcal{C}_{2\delta})}\|\chi_d\psi_n(0,\cdot)\|^2_{L^{2N/(N-2s)}(\mathbb{S}^{N-1})}\end{aligned}$$

and thus

$$\begin{aligned}&\int_{\mathbb{S}^N_+} \theta_1^{1-2s}|\nabla(\chi_\delta\psi_n)|^2(\theta)\, dS \\ &\le C(a,N,s,\delta)\Big(1 + \|a_n\|_{L^q(\mathbb{S}^{N-1})}|\mathcal{C}_{2\delta}|^{(2sq-N)/2sq}\|\chi_\delta\psi_n(0,\cdot)\|^2_{L^{2N/(N-2s)}(\mathbb{S}^{N-1})}\Big)\end{aligned}$$

for some positive constant $C(a,N,s,\delta)$ depending only on a, N, s, δ. Therefore, provided δ is small, by the Sobolev inequality we infer

$$\int_{\mathcal{C}_\delta} \theta_1^{1-2s}|\nabla\psi_n|^2(\theta)\, dS = \int_{\mathcal{C}_\delta} \theta_1^{1-2s}|\nabla(\chi_\delta\psi_n)|^2(\theta)\, dS \le 2C(a,N,s,\delta) \quad \text{for all } n \in \mathbb{N}.$$

Similar arguments can be performed on geodesic balls of $\mathbb{S}^N_+$ with radius δ. By covering $\overline{\mathbb{S}^N_+ \setminus \mathcal{C}_{\delta/2}}$ with such finite small balls and with a classical argument of partition of unity, we conclude that

$$\int_{\mathbb{S}^N_+} \theta_1^{1-2s}|\nabla\psi_n|^2(\theta)\, dS \le \text{const}, \qquad |\mu_1(a_n)| \le \text{const}.$$

It turns out that, up to subsequences, ψ_n converges weakly in $H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$ and strongly in $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$ to some nontrivial function ψ, which can be easily proved to be the positive (or negative) normalized eigenfunction associated to $\mu_1(a)$; it then follows easily that the convergence holds for all the sequence (not only up to subsequences) and that $\mu_1(a_n)\to\mu_1(a)$ as $n\to\infty$. □

Lemma 2.2. *If $a\in L^q(\mathbb{S}^{N-1})$, with $q>N/(2s)$, then*

$$\int_{B_r^+} t^{1-2s}|\nabla w|^2\,dt\,dx-\kappa_s\int_{B_r'}\frac{a(x/|x|)}{|x|^{2s}}w^2\,dx+\frac{N-2s}{2r}\int_{S_r^+}t^{1-2s}w^2\,dS \geq\left(\mu_1(a)+\left(\frac{N-2s}{2}\right)^2\right)\int_{B_r^+}t^{1-2s}\frac{w^2}{|z|^2}\,dz \tag{2.4}$$

for all $r>0$ and $w\in H^1(B_r^+;t^{1-2s})$.

Proof. By scaling, it is enough to prove the inequality for $r=1$. Let $w\in C^\infty(\overline{B_1^+})$. We have that

$$\begin{aligned}&\int_{B_1^+} t^{1-2s}|\nabla w|^2\,dt\,dx-\kappa_s\int_{B_1'}\frac{a(x/|x|)}{|x|^{2s}}w^2\,dx+\frac{N-2s}{2}\int_{S_1^+}t^{1-2s}w^2\,dS\\ &=\int_{B_1^+}t^{1-2s}\left(\nabla w(z)\cdot\frac{z}{|z|}\right)^2dz+\left(\frac{N-2s}{2}\right)\int_{S_1^+}t^{1-2s}w^2dS\\ &\quad+\int_0^1\frac{\rho^{N+1-2s}}{\rho^2}\left(\int_{\mathbb{S}^N_+}\theta_1^{1-2s}|\nabla_{\mathbb{S}^N}w(\rho\theta)|^2\,dS-\kappa_s\int_{\mathbb{S}^{N-1}}a(\theta')w^2(\rho\theta')\right)d\rho.\end{aligned} \tag{2.5}$$

From [9, Lemma 2.4] we have that

$$\int_{B_1^+}t^{1-2s}\left(\nabla w(z)\cdot\frac{z}{|z|}\right)^2dz+\left(\frac{N-2s}{2}\right)\int_{S_1^+}t^{1-2s}w^2dS\geq\left(\frac{N-2s}{2}\right)^2\int_{B_1^+}t^{1-2s}\frac{w^2}{|z|^2}\,dz, \tag{2.6}$$

whereas, from (2.2) it follows that

$$\begin{aligned}&\int_0^1\frac{\rho^{N+1-2s}}{\rho^2}\left(\int_{\mathbb{S}^N_+}\theta_1^{1-2s}|\nabla_{\mathbb{S}^N}w(\rho\theta)|^2\,dS-\kappa_s\int_{\mathbb{S}^{N-1}}a(\theta')w^2(\rho\theta')\right)d\rho\\ &\geq\mu_1(a)\int_0^1\frac{\rho^{N+1-2s}}{\rho^2}\left(\int_{\mathbb{S}^N_+}\theta_1^{1-2s}w^2(\rho\theta)dS\right)d\rho=\mu_1(a)\int_{B_1^+}t^{1-2s}\frac{w^2}{|z|^2}\,dz.\end{aligned} \tag{2.7}$$

The conclusion follows from (2.5), (2.6), and (2.7) and density of $C^\infty(\overline{B_1^+})$ in $H^1(B_1^+;t^{1-2s})$. □

Corollary 2.3. *If $a\in L^q(\mathbb{S}^{N-1})$, with $q>N/(2s)$, satisfies (1.10), then there exists $C_{a,N,s}>0$ such that*

$$\begin{aligned}&\int_{B_r^+} t^{1-2s}|\nabla w|^2\,dt\,dx-\kappa_s\int_{B_r'}\frac{a(x/|x|)}{|x|^{2s}}w^2\,dx+\frac{N-2s}{2r}\int_{S_r^+}t^{1-2s}w^2\,dS\\ &\qquad\geq C_{a,N,s}\left(\int_{B_r^+}t^{1-2s}|\nabla w|^2\,dt\,dx+\frac{N-2s}{2r}\int_{S_r^+}t^{1-2s}w^2\,dS\right)\end{aligned}$$

for all $r>0$ and $w\in H^1(B_r^+;t^{1-2s})$.

Proof. By scaling, it is enough to prove the inequality for $r=1$. We argue by contradiction and assume that, for every $\varepsilon>0$ there exists $w_\varepsilon\in H^1(B_1^+;t^{1-2s})$ such that

$$\begin{aligned}&\int_{B_1^+} t^{1-2s}|\nabla w_\varepsilon|^2\,dt\,dx-\kappa_s\int_{B_1'}\frac{a(x/|x|)}{|x|^{2s}}w_\varepsilon^2\,dx+\frac{N-2s}{2}\int_{S_1^+}t^{1-2s}w_\varepsilon^2\,dS\\ &\qquad<\varepsilon\left(\int_{B_1^+}t^{1-2s}|\nabla w_\varepsilon|^2\,dt\,dx+\frac{N-2s}{2}\int_{S_1^+}t^{1-2s}w_\varepsilon^2\,dS\right)\end{aligned}$$

i.e.

$$\int_{B_1^+}t^{1-2s}|\nabla w_\varepsilon|^2\,dt\,dx+\frac{N-2s}{2}\int_{S_1^+}t^{1-2s}w_\varepsilon^2\,dS-\kappa_s\int_{B_1'}\frac{(1-\varepsilon)^{-1}a(x/|x|)}{|x|^{2s}}w_\varepsilon^2\,dx<0.$$

From Lemma 2.2 it follows that

$$\left(\mu_1\Big(\frac{a}{1-\varepsilon}\Big)+\Big(\frac{N-2s}{2}\Big)^2\right)\int_{B_1^+} t^{1-2s}\frac{w_\varepsilon^2}{|z|^2}\,dz<0$$

and hence

$$\mu_1\Big(\frac{a}{1-\varepsilon}\Big)+\Big(\frac{N-2s}{2}\Big)^2<0$$

On the other hand, from Lemma 2.1, letting $\varepsilon\to 0$, we obtain $\mu_1(a)\le -\big(\frac{N-2s}{2}\big)^2$, thus contradicting assumption (1.10). □

The following corollary follows from Proposition 6.2 and Corollary 2.3.

Corollary 2.4. *If $a\in L^q(\mathbb{S}^{N-1})$, with $q>N/(2s)$, satisfies* (1.10) *and $C_{a,N,s}>0$ is as in Corollary 2.3, then*

$$\int_{\mathbb{R}^N}|\xi|^{2s}\widehat{u}^2\,d\xi-\int_{\mathbb{R}^N}\frac{a(x/|x|)}{|x|^{2s}}u^2\,dx\ge C_{a,N,s}\int_{\mathbb{R}^N}|\xi|^{2s}\widehat{u}^2\,d\xi \tag{2.8}$$

for all $u\in H_0^s(\mathbb{R}^N)$.

Remark 2.5. *We notice that, if $q>N/(2s)$, then the best constant in inequality* (2.8) *depends continuously on $a\in L^q(\mathbb{S}^{N-1})$. Indeed, if $C_{a,N,s}$ is the best constant in* (2.8)*, arguing as in [26, Lemma 1.1] and exploiting the compactness of the map $H^1(\mathbb{S}^N_+;\theta_1^{1-2s})\to\mathbb{R}$, $\psi\mapsto\int_{\mathbb{S}^{N-1}}a\psi^2$ (which easily follows from* (2.1)*), we obtain that*

$$\begin{aligned}
C_{a,N,s}&=\inf_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})\setminus\{0\}}\frac{\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla w|^2\,dt\,dx-\kappa_s\int_{\mathbb{R}^N}|x|^{-2s}a(x/|x|)w^2\,dx}{\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla w|^2\,dt\,dx}\\
&=1-\sup_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})\setminus\{0\}}\frac{\kappa_s\int_{\mathbb{R}^N}|x|^{-2s}a(x/|x|)w^2\,dx}{\int_{\mathbb{R}^{N+1}_+}t^{1-2s}|\nabla w|^2\,dt\,dx}\\
&=1-\max_{\psi\in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})\setminus\{0\}}\frac{\kappa_s\int_{\mathbb{S}^{N-1}}a(\theta')\psi^2(0,\theta')\,dS'}{\int_{\mathbb{S}^N_+}\theta_1^{1-2s}|\nabla\psi(\theta)|^2dS+\big(\frac{N-2s}{2}\big)^2\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi^2(\theta)\,dS}.
\end{aligned}$$

From the above characterization of $C_{a,N,s}$ it is then easy to prove that, if $a_n\to a$ in $L^q(\mathbb{S}^{N-1})$, then $C_{a_n,N,s}\to C_{a,N,s}$ as $n\to+\infty$.

Combining Corollary 2.3 with [9, Lemma 2.5] we obtain the following estimate.

Corollary 2.6. *If $a\in L^q(\mathbb{S}^{N-1})$, with $q>N/(2s)$, satisfies* (1.10)*, there exists $C'_{a,N,s}>0$ such that*

$$\int_{B_r^+}t^{1-2s}|\nabla w|^2\,dt\,dx-\kappa_s\int_{B'_r}\frac{a(x/|x|)}{|x|^{2s}}w^2\,dx+\frac{N-2s}{2r}\int_{S_r^+}t^{1-2s}w^2\,dS\ge C'_{a,N,s}\int_{B'_r}\frac{w^2}{|x|^{2s}}\,dx$$

for all $r>0$ and $w\in H^1(B_r^+;t^{1-2s})$.

3. Schauder estimates for degenerate elliptic equations

As stated in Section 1, for $u\in H^s(\mathbb{R}^N)$, the nonlocal equation

$$(-\Delta+m^2)^s u=G(x,u),\quad \text{in }\Omega,$$

can be reformulated as a local problem by considering its extension in $\mathbb{R}^{N+1}_+$. Indeed, letting $w\in H^1(\mathbb{R}^{N+1}_+;t^{1-2s})$ be the unique weak solution to the problem

$$\begin{cases}-\mathrm{div}(t^{1-2s}\nabla w)+m^2t^{1-2s}w=0, & \text{in }\mathbb{R}^{N+1}_+,\\ w=u, & \text{on }\mathbb{R}^N,\end{cases}$$

we have that

$$-\lim_{t\to 0}t^{1-2s}\frac{\partial w}{\partial t}=\kappa_s(-\Delta+m^2)^s u,\quad \text{in }\Omega,$$

in a weak sense. This will be proved in the appendix A. This naturally leads to the study of regularity properties of solutions to

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla w) + m^2 t^{1-2s} w = F(x,w), & \text{in } \Omega \times (0,T), \\ -\lim_{t\to 0} t^{1-2s} \frac{\partial w}{\partial t} = G(x,w), & \text{in } \Omega, \end{cases}$$

which is the content of this section.

Before going on, let us state the following weighted Sobolev inequality whose proof is essentially contained in the book of Opic and Kufner, [20].

Lemma 3.1. *Let $N > 2s$. Then there exists a constant $S_{N,s} > 0$ such that for every $v \in C^1_c(\mathbb{R}^{N+1})$ we have*

$$\left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|v|^{\frac{2N_s}{N_s-2}}\,dt\,dx\right)^{\frac{N_s-2}{N_s}} \le S_{N,s}\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla v|^2\,dt\,dx, \tag{3.1}$$

where $N_s = N + 2 - 2s$.

Proof. We have, see [20, Section 19], that

$$\left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|v|^{\frac{2N_s}{N_s-2}}\,dt\,dx\right)^{\frac{N_s-2}{N_s}} \le C_{N,s}\left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla v|^2\,dt\,dx + \int_{\mathbb{R}^{N+1}_+} t^{1-2s}v^2\,dt\,dx\right).$$

Using simple scaling argument, we obtain (3.1). □

We will also need the following result.

Lemma 3.2. *Let $a, b \in L^p(B'_1)$, for some $p > \frac{N}{2s}$ and $c, d \in L^q(B^+_1; t^{1-2s})$, for some $q > \frac{N+2-2s}{2}$. Let $w \in H^1(B^+_1; t^{1-2s})$ be such that*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla w) + t^{1-2s}c(z)w = t^{1-2s}d(z), & \text{in } B^+_1, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t w = a(x)w + b(x), & \text{on } B'_1. \end{cases}$$

Then there exits a constant $C > 0$ depending only on $N, s, \|a\|_{L^p(B'_1)}, \|c\|_{L^q(B^+_1;t^{1-2s})}$ such that

$$\|w\|_{H^1(B^+_1;t^{1-2s})} \le C\left(\|w\|_{L^2(B^+_1;t^{1-2s})} + \|b\|_{L^p(B'_1)} + \|d\|_{L^q(B^+_1;t^{1-2s})}\right).$$

Proof. The proof is not difficult taking into account the weighted Sobolev inequality (3.1) together with the Sobolev-trace inequality: for every $v \in C^1_c(\mathbb{R}^{N+1})$

$$C_{N,s}\int_{\mathbb{R}^N} |v(0,x)|^{\frac{2N}{N-2s}}\,dx \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla v(t,x)|^2\,dt\,dx.$$

We skip the details. □

Proposition 3.3. *Let $a, b \in L^p(B'_1)$, for some $p > N/(2s)$ and $c, d \in L^q(B^+_1; t^{1-2s})$, for some $q > \frac{N+2-2s}{2}$. Let $w \in H^1(B^+_1; t^{1-2s})$ be a weak solution of*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla w) + t^{1-2s}c(z)w \le t^{1-2s}d(z), & \text{in } B^+_1, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t w \le a(x)w + b(x), & \text{on } B'_1. \end{cases} \tag{3.2}$$

Then

$$\sup_{B^+_{1/2}} w^+ \le C\Big(\|w^+\|_{L^2(B^+_1;t^{1-2s})} + \|b^+\|_{L^p(B'_1)} + \|d^+\|_{L^q(B^+_1;t^{1-2s})}\Big),$$

where $w^+ = \max\{0, w\}$, and $C > 0$ depends only on $N, s, \|a^+\|_{L^p(B'_1)}, \|c^-\|_{L^q(B^+_1;t^{1-2s})}$.

Proof. Let $k = \max(\|d^+\|_{L^q(B^+_1;t^{1-2s})}, \|b^+\|_{L^p(B'_1)})$ or an arbitrary positive small number if $\max(\|d^+\|_{L^q(B^+_1;t^{1-2s})}, \|b^+\|_{L^p(B'_1)}) = 0$. For every $L > 0$, set $\overline{w} = w^+ + k$ and

$$\overline{w}_L = \begin{cases} \overline{w}, & \text{if } w < L, \\ k + L, & \text{if } w \ge L. \end{cases}$$

Put

$$W = \overline{w}_L^{\frac{\beta}{2}}\overline{w}, \qquad \varphi = \eta^2(\overline{w}_L^{\beta}\overline{w} - k^{\beta+1}) \in H^1(B_1^+; t^{1-2s}),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_c^1(B_1^+ \cup B_1')$. Following [17,25], testing (3.2) with φ, integration by parts, we have

$$\int_{B_1^+} t^{1-2s}|\nabla(\eta W)|^2 \leq (1+\beta)^{\frac{\delta}{\theta}} C \int_{B_1^+} t^{1-2s} W^2(|\nabla\eta|^2 + \eta^2) + 2(1+\beta)\int_{B_1^+} t^{1-2s}\Big(c^- + \frac{d^+}{k}\Big)(\eta W)^2, \tag{3.3}$$

for some positive constants δ, θ depending only on N, s and C depending only on $N, s, \|a^+\|_{L^p(B_1)}$. By using Hölder inequality, we get

$$\int_{B_1^+} t^{1-2s}\Big(c^- + \frac{d^+}{k}\Big)(\eta W)^2 \leq (\|c^-\|_{L^q(B_1^+;t^{1-2s})} + 1)\|(\eta W)^2\|_{L^{\frac{q}{q-1}}(B_1^+;t^{1-2s})} =: C_1\|(\eta W)^2\|_{L^{\frac{q}{q-1}}(B_1^+)}. \tag{3.4}$$

Since $1 < \frac{q}{q-1} < \frac{N+2-2s}{N-2s}$, by interpolation and Young's inequalities, we have

$$2C_1(1+\beta)\|(\eta W)^2\|_{L^{\frac{q}{q-1}}(B_1^+)} \leq \frac{1}{2S_{N,s}}\|(\eta W)^2\|_{L^{\frac{N+2-2s}{N-2s}}(B_1^+)} + (1+\beta)^{\frac{\delta}{\theta}} C\|(\eta W)\|_{L^1(B_1^+)}.$$

By the weighted Sobolev inequality (3.1), we have

$$\|(\eta W)^2\|_{L^{\frac{N+2-2s}{N-2s}}(B_1^+)} \leq S_{N,s}\int_{B_1^+} t^{1-2s}|\nabla(\eta W)|^2.$$

Using the two inequalities above in (3.4), we get

$$2(1+\beta)\int_{B_1^+} t^{1-2s}\Big(c^- + \frac{d^+}{k}\Big)(\eta W)^2 \leq \frac{1}{2}\int_{B_1^+} t^{1-2s}|\nabla(\eta W)|^2 + (1+\beta)^{\frac{\delta}{\theta}} C\|\eta W\|^2_{L^2(B_1^+)}$$

Putting this in (3.3), we obtain

$$\int_{B_1^+} t^{1-2\sigma}|\nabla(\eta W)|^2 \leq C(1+\beta)^{\frac{\delta}{\theta}}\int_{B_1^+} t^{1-2\sigma}(\eta^2 + |\nabla\eta|^2)W^2.$$

At this point, the argument in [25, Proposition 3.1] yields the result. □

The next result is a weak Harnack inequality.

Proposition 3.4. *Let $a, b \in L^p(B_1')$ for some $p > N/(2s)$ and $c, d \in L^q(B_1^+; t^{1-2s})$ for some $q > \frac{N+2-2s}{2}$. Let $w \in H^1(B_1^+; t^{1-2s})$ be a nonnegative weak solution of*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w) + c(z)t^{1-2s}w \geq t^{1-2s}d(z), & \text{in } B_1^+,\\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t w \geq a(x)w + b(x), & \text{on } B_1'. \end{cases} \tag{3.5}$$

Then for some $p_0 > 0$ and any $0 < r < r' < 1$ we have that

$$\inf_{\overline{B}_r^+} w + \|b^-\|_{L^p(B_1')} + \|d^-\|_{L^q(B_1^+;t^{1-2s})} \geq C\|w\|_{L^{p_0}(t^{1-2s}, B_{r'}^+)},$$

where $C > 0$ depends only on $N, s, r, r', \|a^-\|_{L^p(B_1')}, \|c^+\|_{L^q(B_1^+;t^{1-2s})}$.

Proof. Set $\overline{w} = w + k > 0$, for some positive k to be determined and $v = \overline{w}^{-1}$. Let Φ be any nonnegative function in $H^1(B_1^+; t^{1-2s})$ with compact support in $B_1^+ \cup B_1'$. Multiplying both sides of the first inequality in (3.5) by $\overline{w}^{-2}\Phi$ and integrating by parts, we obtain

$$\int_{B_1^+} t^{1-2s}\nabla v\nabla\Phi + \int_{B_1^+} t^{1-2s}\tilde{c}(z)v\Phi - \int_{B_1'} \tilde{a}v\Phi \leq 0,$$

where

$$\tilde{a} = \frac{a^-w + b^-}{\overline{w}}, \qquad \tilde{c} = \frac{c^+w + d^-}{\overline{w}}.$$

If $\max(\|b^-\|_{L^p(B_1')}, \|d^-\|_{L^q(B_1^+;t^{1-2s})}) \neq 0$ then we choose $k = \max(\|b^-\|_{L^p(B_1')}, \|d^-\|_{L^q(B_1^+;t^{1-2s})})$. Otherwise, choose an arbitrary $k > 0$ which will be sent to zero. Therefore Proposition 3.3 (see also [17]) implies that for any $r' \in (r,1)$ and any $p > 0$

$$\sup_{B_r^+} v \leq C\|v\|_{L^p(B_{r'}^+;t^{1-2s})}.$$

Following exactly the same arguments as in [25], we get the result. □

We now prove local Schauder estimates.

Proposition 3.5. *Let $a, b \in L^p(B_1')$, for some $p > \frac{N}{2s}$ and $c, d \in L^q(B_1^+; t^{1-2s})$, for some $q > \frac{N+2-2s}{2}$. Let $w \in H^1(B_1^+; t^{1-2s})$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w) + t^{1-2s}c(z)w = t^{1-2s}d(z), & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t w = a(x)w + b(x), & \text{on } B_1'. \end{cases} \tag{3.6}$$

Then $w \in C^{0,\alpha}(\overline{B_{1/2}^+})$ and in addition

$$\|w\|_{C^{0,\alpha}(\overline{B_{1/2}^+})} \leq C\left(\|w\|_{L^2(B_1^+)} + \|b\|_{L^p(B_1')} + \|d\|_{L^q(B_1^+;t^{1-2s})}\right),$$

with $C, \alpha > 0$ depending only on $N, s, \|a\|_{L^p(B_1')}, \|c\|_{L^q(B_1^+;t^{1-2s})}$.

Proof. The proof is a consequence of Propositions 3.3 and 3.4 with a standard scaling argument for which we refer to [15]. □

Remark 3.6. *Let $w \in H^1(B_1^+; t^{1-2s})$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(t^{1-2s}A(z)\nabla w) + t^{1-2s}c(z)w = t^{1-2s}d(z), & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t w = a(x)w + b(x), & \text{on } B_1', \end{cases} \tag{3.7}$$

with a, b, c, d as in Proposition 3.5 and the matrix A satisfying

$$C_1|\xi|^2 \leq A(z)\xi\cdot\xi \leq C_2|\xi|^2 \quad \text{for all } z \in B_1^+, \quad \xi \in \mathbb{R}^N,$$

with $C_1, C_2 > 0$. Then the same conclusion as in Proposition 3.5 holds taking into account the constants C_1, C_2.

Proposition 3.7. *Let $a, b \in C^k(B_1')$ and $\nabla_x^i c, \nabla_x^i d \in L^\infty(B_1^+)$, for some $k \geq 1$ and $i = 0, \dots, k$. Let $w \in H^1(B_1^+; t^{1-2s})$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w) + t^{1-2s}c(z)w = t^{1-2s}d(z), & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t w = a(x)w + b(x), & \text{on } B_1'. \end{cases} \tag{3.8}$$

Then for $i = 1, \dots, k$ we have that $w \in C^{i,\alpha}(B_r^+)$, for some $r \in (0,1)$ depending only on k, and in addition

$$\sum_{i=1}^k \|\nabla_x^i w\|_{C^{0,\alpha}(\overline{B_r^+})} \leq C\Big(\|w\|_{L^2(B_1^+;t^{1-2s})} + \|a\|_{C^k(\overline{B_r'})} + \|b\|_{C^k(\overline{B_r'})} + \sum_{i=1}^k \|\nabla_x^i c, \nabla_x^i d\|_{L^\infty(B_1^+)}\Big),$$

with $C, \alpha > 0$ depending only on $N, s, k, r, \|a\|_{L^\infty(B_{1/2}')}, \|c\|_{L^\infty(B_{1/2}^+)}$.

Proof. Let $h \in \mathbb{R}^N$ such that $|h| < \frac{1}{2}$. Then we have

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w^h) + t^{1-2s}c(z)w^h = -t^{1-2s}c^h(z)w + t^{1-2s}d^h(z), & \text{in } B_{1/2}^+ \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t w^h = a(x)w^h + a^h(x)w + b^h(x), & \text{on } B_{1/2}', \end{cases}$$

where we denote $f^h(t,x) = \frac{f(t,x+h)-f(t,x)}{h}$, for $t \geq 0$. Applying Lemma 3.2, Proposition 3.3 and Proposition 3.5 we get

$$\begin{aligned}
&\|w^h\|_{H^1(B^+_{1/2};t^{1-2s})} + \|w^h\|_{C^{0,\alpha}(\overline{B^+_{1/4}})}\\
&\leq C\left(\|w^h\|_{L^2(B^+_{1/2};t^{1-2s})} + \|a^h w + b^h\|_{L^\infty(B'_{1/2})} + \|c^h w + d^h\|_{L^\infty(B^+_{1/2};t^{1-2s})}\right)\\
&\leq C\left(\|\nabla w\|_{L^2(B^+_{1/2};t^{1-2s})} + \|w\|_{C^0(\overline{B^+_{1/2}})} + \|\nabla_x a, \nabla_x b\|_{L^\infty(B'_{1/2})} + \|\nabla_x c, \nabla_x d\|_{L^\infty(B^+_{1/2};t^{1-2s})}\right)\\
&\leq C\left(\|w\|_{L^2(B^+_{1/2};t^{1-2s})} + \sum_{i=0}^{1}\|\nabla^i_x a, \nabla^i_x b\|_{L^\infty(B'_{1/2})} + \sum_{i=0}^{1}\|\nabla^i_x c, \nabla^i_x d\|_{L^\infty(B^+_{1/2})}\right)
\end{aligned}$$

for some positive constant C depending only on $N, s, \|a\|_{L^\infty(B'_{1/2})}, \|c\|_{L^\infty(B^+_{1/2})}$. Therefore using Fatou's Lemma, we obtain $W_1 := \nabla_x w \in H^1(B^+_{1/2}; t^{1-2s}) \cap C^0(B^+_{1/2})$,

$$\|\nabla_x w\|_{L^\infty(B^+_{1/4})} \leq C\left(\|w\|_{L^2(B^+_{1/2};t^{1-2s})} + \sum_{i=0}^{1}\|\nabla^i_x a, \nabla^i_x b\|_{L^\infty(B'_{1/2})} + \sum_{i=0}^{1}\|\nabla^i_x c, \nabla^i_x d\|_{L^\infty(B^+_{1/2})}\right), \tag{3.9}$$

and

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla W_1) + t^{1-2s}c(z)W_1 = t^{1-2s}d_1(z), & \text{in } B^+_{1/2}\\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t W_1 = a(x)W_1 + b_1(x), & \text{on } B'_{1/2}, \end{cases}$$

where $d_1(z) = -\nabla_x c(z) w + \nabla_x d(z)$ and $b_1(x) = \nabla_x a(x) w + \nabla_x b(x)$. Hence by Proposition 3.5 and (3.9), we have

$$\begin{aligned}
\|W_1\|_{C^{0,\alpha}(\overline{B^+_{1/4}})} &\leq C\left(\|W_1\|_{L^\infty(B^+_{1/2})} + \|b_1(x)\|_{L^\infty(B'_{1/2})} + \|d_1(z)\|_{L^\infty(B^+_{1/2})}\right)\\
&\leq C\left(\|w\|_{L^2(B^+_1;t^{1-2s})} + \sum_{i=0}^{1}\|\nabla^i_x a, \nabla^i_x b\|_{L^\infty(B'_{1/2})} + \sum_{i=0}^{1}\|\nabla^i_x c, \nabla^i_x d\|_{L^\infty(B^+_{1/2})}\right),
\end{aligned}$$

with $C > 0$ depending only on $N, s, \|a\|_{L^\infty(B'_{1/2})}, \|c\|_{L^\infty(\overline{B^+_{1/2}})}$. Iterating this procedure we get the the desired estimate for $W_i = \nabla^i_x w$. □

4. A Pohozaev type identity

In order to differentiate the Almgren frequency function associated to the extended problem (see section 5), we need to derive a Pohozaev type identity, which first requires the following regularity result.

Lemma 4.1. *Let $v \in H^1(B^+_1; t^{1-2s})$ satisfy*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla v) + m^2 t^{1-2s} v = 0, & \text{in } B^+_1,\\ -\lim_{t\to 0^+} t^{1-2s} v_t = g, & \text{on } B'_1, \end{cases} \tag{4.1}$$

where $g \in C^{0,\gamma}(B'_r)$, $\gamma \in [0, 2-2s)$ (meaning that $C^{0,\gamma} = L^\infty$ if $\gamma = 0$). Then for every $t_0 > 0$ sufficiently small there exist positive constants C and $\alpha \geq 0$ (with $\alpha > 0$ if $\gamma > 0$), depending only on N, s, t_0, m, γ such that

$$\|t^{1-2s} v_t\|_{C^{0,\alpha}([0,t_0)\times B'_{1/8})} \leq C\left(\|v\|_{L^2(B^+_1;t^{1-2s})} + \|g\|_{C^\gamma(B'_{1/2})}\right). \tag{4.2}$$

Proof. If $m = 0$, this was proved in [3]. We will assume in the following that $m > 0$. Next pick $\eta \in C^\infty_c(\overline{B'_1})$ with $\eta = 1$ on $B'_{1/2}$ and $\eta = 0$ on $\mathbb{R}^N \setminus \overline{B'_{3/2}}$. Then we have that $\eta g \in L^2(\mathbb{R}^N)$. By minimization arguments, there exists $W \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ satisfying

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla W) + m^2 t^{1-2s} W = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ -\lim_{t\to 0^+} t^{1-2s} W_t = \eta g, & \text{on } \mathbb{R}^N. \end{cases}$$

We define $w = -t^{1-2s} W_t$ and we observe that $w \in L^2(\mathbb{R}^{N+1}_+; t^{-1+2s})$ and

$$\begin{cases} -\operatorname{div}(t^{-1+2s}\nabla w) + m^2 t^{-1+2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+ \\ w = \eta g, & \text{on } \mathbb{R}^N. \end{cases}$$

From Remark 7.3 and Proposition 7.4, it follows that $w = \bar{P}(t,\cdot) * (\eta g)$, where $\bar{P}$ is the Bessel kernel for the conjugate problem given by

$$\bar{P}(z) = C'_{N,s}\, t^{2-2s} m^{\frac{N+2-2s}{2}} |z|^{-\frac{N+2-2s}{2}} K_{\frac{N+2-2s}{2}}(m|z|),$$

see (7.2); we refer to Section 6.1 for asymptotics of the Bessel function K_ν.

Claim: $w \in C^{0,\gamma}(\overline{\mathbb{R}^{N+1}_+})$ for every $R > 0$ and

$$\|w\|_{C^{0,\gamma}(\overline{B^+_R})} \le C_{N,s,m,R} \|\eta g\|_{C^{0,\gamma}(\mathbb{R}^N)}. \tag{4.3}$$

Indeed, by a change of variables, we have that

$$w(t,x) = C'_{N,s} \int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} \left((tm(1+|y|^2)^{1/2}\right)^{\frac{N_s}{2}} K_{\frac{N_s}{2}} \left((tm(1+|y|^2)^{1/2}\right) (\eta g)(x - ty) dy,$$

where $N_s = N + 2 - 2s$. Let us set $f(t,|y|) = (tm(1+|y|^2)^{1/2})^{N_s/2} K_{\frac{N_s}{2}} \left((tm(1+|y|^2)^{1/2}\right)$ and $u(x) = \eta(x) g(x)$. Letting $x_1, x_2 \in B'_R$ and $0 \le t_2 < t_1 < 1$, we have

$$\begin{aligned} w(t_1,x_1) - w(t_2,x_2) &= \int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} [f(t_1,|y|) - f(t_2,|y|)] u(x_1 - t_1 y) dy \\ &\quad + \int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} [u(x_1 - t_1 y) - u(x_2 - t_2 y)] f(t_2,|y|) dy. \end{aligned} \tag{4.4}$$

Using the fact that $K'_{\frac{N_s}{2}} = -\frac{N_s}{2r} K_{\frac{N_s}{2}} - K_{\frac{N_s}{2}-1}$, we infer that

$$\left| \left(r^{\frac{N_s}{2}} K_{\frac{N_s}{2}} \right)' \right| = \left| -r^{\frac{N_s}{2}} K_{\frac{N_s}{2}-1} \right| \le C_{N,s} m \quad \text{for } N > 2s.$$

It follows that

$$|f(t_1,|y|) - f(t_2,|y|)| \le C_{N,s,m} |t_1 - t_2| (1+|y|^2)^{1/2}.$$

We recall that $\operatorname{supp} u \subset B'_{3/2}$ and observe that $|y| \le \frac{1}{t_1}(3/2 + 2R) \le \frac{2}{t_1 - t_2}(3/2 + 2R)$ provided $|x_1 - t_1 y| \le \frac{3}{2}$. Therefore

$$\begin{aligned} &\int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} |f(t_1,|y|) - f(t_2,|y|)| |u(x_1 - t_1 y)| dy \\ &\qquad \le C_{N,s,m} |t_1 - t_2| \|u\|_{L^\infty(\mathbb{R}^N)} \int_{\{|y| \le \frac{1}{t_1 - t_2}(3/2+2R)\}} (1+|y|^2)^{-\frac{N_s - 1}{2}} dy \\ &\qquad \le C_{N,s,m,R} \|u\|_{L^\infty(\mathbb{R}^N)} |t_1 - t_2|^{2-2s}. \end{aligned} \tag{4.5}$$

Next we have, for $\gamma \in [0, 2-2s)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} |u(x_1 - t_1 y) - u(x_2 - t_2 y)| |f(t_2,|y|)| dy \\ &\le \|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \|f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)} \left(|t_1 - t_2|^\gamma \int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} |y|^\gamma dy + |x_1 - x_2|^\gamma \int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} dy \right). \end{aligned}$$

Hence, for every $\gamma \in [0, 2-2s)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} (1+|y|^2)^{-\frac{N_s}{2}} |u(x_1 - t_1 y) - u(x_2 - t_2 y)| |f(t_2,|y|)| dy \\ &\qquad \le C_{N,s,m} \|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \|f\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)} (|t_1 - t_2|^\gamma + |x_1 - x_2|^\gamma). \end{aligned}$$

This, together with (4.5) in (4.4), proves the claim.

We have that $U := v - W$ satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) + m^2 t^{1-2s} U = 0, & \text{in } B^+_1, \\ t^{1-2s} U_t = 0, & \text{on } B'_{1/2} \end{cases}$$

and $U := v - W \in H^1(B_1^+; t^{1-2s})$. We deduce that, for some positive constants C, β depending only on N, s, m,

$$\|\nabla_x^2 U\|_{C^{0,\beta}(\overline{B_{1/4}^+})} \le C_{s,N,m} \|U\|_{L^2(B_1^+;t^{1-2s})} \le C \left(\|v\|_{L^2(B_1^+;t^{1-2s})} + \|g\|_{L^\infty(B'_{1/2})} \right)$$

by Proposition 3.7. We also observe that

$$-t^{1-2s}\Delta_x U - (t^{1-2s}U_t)_t + m^2 t^{1-2s} U = 0, \quad \text{in } B_1^+.$$

Then, by integration, we obtain that, for every $x \in B'_{1/4}$ and $0 < t_0, t \le 1/4$,

$$t^{1-2s}U_t(t,x) = t_0^{1-2s}U_t(t_0,x) - \int_t^{t_0} \tau^{1-2s}\Delta_x U(\tau,x)d\tau + m^2 \int_t^{t_0} \tau^{1-2s}U(\tau,x)d\tau. \tag{4.6}$$

Therefore $t^{1-2s}U_t \in C^{0,\alpha}(\overline{B_{1/8}^+})$ and thus $t^{1-2s}v_t = t^{1-2s}U_t - w \in C^{0,\alpha}(\overline{B_{1/8}^+})$ from which we deduce that $\|t^{1-2s}v_t\|_{C^{0,\alpha}(\overline{B_{1/8}^+})} \le C_{s,N,m} \left(\|v\|_{L^2(B_1^+;t^{1-2s})} + \|g\|_{C^\gamma(B'_{1/2})} \right)$ by (4.3). □

Let V satisfy

$$V \in C^1(\mathbb{R}^N \setminus \{0\}), \quad |V(x)| + |x \cdot \nabla V(x)| \le C|x|^{-2s} \text{ as } |x| \to 0 \text{ for some } C > 0. \tag{4.7}$$

Let $w \in H^1(B_R^+; t^{1-2s})$ solve

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla w) + m^2 t^{1-2s} w = 0, & \text{in } B_R^+, \\ -\lim_{t\to 0^+} t^{1-2s} \frac{\partial w}{\partial t}(t,x) = \kappa_s V(x) w, & \text{on } B'_R, \end{cases} \tag{4.8}$$

in a weak sense, i.e., for all $\varphi \in C_c^\infty(B_R^+ \cup B'_R)$,

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla w \cdot \nabla\varphi \, dt\, dx + m^2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} w\varphi \, dt\, dx = \kappa_s \int_{B'_R} V(x) w\varphi \, dx. \tag{4.9}$$

The following Pohozaev-type identity holds.

Theorem 4.2. *Let w be a solution to* (4.8) *in sense of* (4.9), *with V satisfying* (4.7). *Then, for a.e. $r \in (0,R)$,*

$$\begin{aligned} -\frac{N-2s}{2}\int_{B_r^+} t^{1-2s}|\nabla w|^2 dz - \frac{m^2(N+2-2s)}{2}\int_{B_r^+} t^{1-2s}w^2 dz \\ + \frac{rm^2}{2}\int_{S_r^+} t^{1-2s}w^2 dS + \frac{r}{2}\int_{S_r^+} t^{1-2s}|\nabla w|^2 dS \\ = r\int_{S_r^+} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS - \frac{\kappa_s}{2}\int_{B'_r}(NV(x) + \nabla V(x)\cdot x)w^2\, dx + \frac{r\kappa_s}{2}\int_{\partial B'_r} V(x) w^2\, dS' \end{aligned} \tag{4.10}$$

and

$$\int_{B_r^+} t^{1-2s}|\nabla w|^2 dz + m^2\int_{B_r^+} t^{1-2s}w^2 dz = \int_{S_r^+} t^{1-2s}\frac{\partial w}{\partial\nu} w\, dS + \kappa_s \int_{B'_r} V(x) w^2(x)\, dx. \tag{4.11}$$

Proof. We have, on B_R^+, the formula

$$\operatorname{div}\left(\frac{1}{2}t^{1-2s}|\nabla w|^2 z - t^{1-2s}(z\cdot\nabla w)\nabla w\right) = \frac{N-2s}{2}t^{1-2s}|\nabla w|^2 - (z\cdot\nabla w)\operatorname{div}(t^{1-2s}\nabla w). \tag{4.12}$$

Let $\rho < r < R$. Integrating by parts (4.12) over the set

$$O_\varepsilon := (B_r^+ \setminus \overline{B_\rho^+}) \cap \{(t,x),\, t > \varepsilon\},$$

with $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{N-2s}{2}\int_{O_\varepsilon} t^{1-2s}|\nabla w(z)|^2 dz + m^2\frac{N+2-2s}{2}\int_{O_\varepsilon} t^{1-2s} w^2 dz - m^2\frac{r}{2}\int_{S_r^+\cap\{t>\varepsilon\}} t^{1-2s} w^2 dS \\
&\quad + m^2\frac{\varepsilon^{2-2s}}{2}\int_{B'_{\sqrt{r^2-\varepsilon^2}}\setminus B'_{\sqrt{\rho^2-\varepsilon^2}}} w^2(\varepsilon,x)dx + m^2\frac{\rho}{2}\int_{S_\rho^+\cap\{t>\varepsilon\}} t^{1-2s} w^2 dS \\
&\quad = -\frac{1}{2}\varepsilon^{2-2s}\int_{B'_{\sqrt{r^2-\varepsilon^2}}\setminus B'_{\sqrt{\rho^2-\varepsilon^2}}} |\nabla w|^2(\varepsilon,x)dx \\
&\quad + \varepsilon^{2-2s}\int_{B'_{\sqrt{r^2-\varepsilon^2}}\setminus B'_{\sqrt{\rho^2-\varepsilon^2}}} |w_t|^2(\varepsilon,x)dx \\
&\quad + \frac{r}{2}\int_{S_r^+\cap\{t>\varepsilon\}} t^{1-2s}|\nabla w|^2 dS - r\int_{S_r^+\cap\{t>\varepsilon\}} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS \\
&\quad - \frac{\rho}{2}\int_{S_\rho^+\cap\{t>\varepsilon\}} t^{1-2s}|\nabla w|^2 dS + \rho\int_{S_\rho^+\cap\{t>\varepsilon\}} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS \\
&\quad + \int_{B'_{\sqrt{r^2-\varepsilon^2}}\setminus B'_{\sqrt{\rho^2-\varepsilon^2}}} (x\cdot\nabla_x w(\varepsilon,x))\,\varepsilon^{1-2s} w_t(\varepsilon,x)\,dx.
\end{aligned}$$

We now claim that there exists a sequence $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty}\varepsilon_n^{2-2s}\left[\int_{B'_r}|\nabla w|^2(\varepsilon_n,x)dx + \int_{B'_r} w^2(\varepsilon_n,x)dx\right] = 0.$$

If no such sequence exists, we would have

$$\liminf_{\varepsilon\to 0}\varepsilon^{2-2s}\left[\int_{B'_r}|\nabla w|^2(\varepsilon,x)dx + \int_{B'_r} w^2(\varepsilon,x)dx\right] \ge C > 0$$

and thus there exists $\varepsilon_0 > 0$ such that

$$\varepsilon^{2-2s}\left[\int_{B'_r}|\nabla w|^2(\varepsilon,x)dx + \int_{B'_r} w^2(\varepsilon,x)dx\right] \ge \frac{C}{2} \quad \text{for all } \varepsilon\in(0,\varepsilon_0).$$

It follows that, for all $\varepsilon\in(0,\varepsilon_0)$,

$$\frac{1}{2}\varepsilon^{1-2s}\int_{B'_r}|\nabla w|^2(\varepsilon,x)dx + \varepsilon^{1-2s}\int_{B'_r} w^2(\varepsilon,x)dx \ge \frac{C}{2\varepsilon}$$

and so integrating the above inequality on $(0,\varepsilon_0)$ we contradict the fact that $w\in H^1(B_R^+;t^{1-2s})$.

Next, from the Dominated Convergence Theorem, Lemma 4.1, and Proposition 3.7, we have that

$$\lim_{\varepsilon\to 0}\int_{B'_{\sqrt{r^2-\varepsilon^2}}\setminus B'_{\sqrt{\rho^2-\varepsilon^2}}} (x\cdot\nabla_x w(\varepsilon,x))\,\varepsilon^{1-2s} w_t(\varepsilon,x)\,dx = -\kappa_s\int_{B'_r\setminus B'_\rho}(x\cdot\nabla_x w)\,V(x) w\,dx.$$

We conclude (replacing O_ε with O_{ε_n}, for a sequence $\varepsilon_n\to 0$) that

$$\begin{aligned}
&\frac{N-2s}{2}\int_{B_r^+\setminus B_\rho^+} t^{1-2s}|\nabla w(z)|^2 dz + m^2\frac{N+2-2s}{2}\int_{B_r^+\setminus B_\rho^+} t^{1-2s} w^2 dz \\
&- m^2\frac{r}{2}\int_{S_r^+} t^{1-2s} w^2 dS + m^2\frac{\rho}{2}\int_{S_\rho^+} t^{1-2s} w^2 dS = \frac{r}{2}\int_{S_r^+} t^{1-2s}|\nabla w|^2 dS - r\int_{S_r^+} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS \\
&\qquad - \frac{\rho}{2}\int_{S_\rho^+} t^{1-2s}|\nabla w|^2 dS - \rho\int_{S_\rho^+} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS - \kappa_s\int_{B'_r\setminus B'_\rho}(x\cdot\nabla_x w)\,V(x) w\,dx.
\end{aligned} \tag{4.13}$$

Furthermore, integration by parts yields

$$\int_{B'_r\setminus B'_\rho}(x\cdot\nabla_x w)\,V(x)w\,dx = -\frac{1}{2}\int_{B'_r\setminus B'_\rho}(NV(x)+\nabla V(x)\cdot x)w^2\,dx + \frac{r}{2}\int_{\partial B'_r}V(x)w^2\,dS' - \frac{\rho}{2}\int_{\partial B'_\rho}V(x)w^2\,dS'. \tag{4.14}$$

Since $w\in H^1(B_R^+;t^{1-2s})$, in view of Hardy and Sobolev inequalities, there exists a sequence $\rho_n\to 0$ such that

$$\lim_{n\to\infty}\rho_n\left[\int_{S^+_{\rho_n}}t^{1-2s}[|\nabla w|^2+w^2]dS+\int_{\partial B'_{\rho_n}}[|V(x)|+|x||\nabla V|]w^2dS'\right]=0.$$

Hence, taking $\rho=\rho_n$ and letting $n\to\infty$ in (4.13) and (4.14), we obtain (4.10). Finally (4.11) follows the proof in [9, Lemma 3.1]. □

5. The Almgren type frequency function

In this section, we introduce the *Almgren frequency function* at the origin 0 for the extended problem associated to the relativistic operator $(-\Delta+m^2)^s$ and study its limit as $r\to 0^+$. Let $R>0$ and $w\in H^1(B_R^+;t^{1-2s})$ be a nontrivial solution to

$$\begin{cases}-\operatorname{div}(t^{1-2s}\nabla w)+t^{1-2s}m^2w=0, & \text{in } B_R^+,\\ -\lim_{t\to 0^+}t^{1-2s}\frac{\partial w}{\partial t}(t,x)=\kappa_s\Big(\frac{a(x/|x|)}{|x|^{2s}}w+hw\Big), & \text{on } B'_R,\end{cases} \tag{5.1}$$

in the sense of (4.9). Arguing as in [9], it is easy to check that, for a.e. $r\in(0,R)$ and every $\widetilde{\varphi}\in C^\infty(\overline{B_r^+})$

$$\int_{B_r^+}t^{1-2s}\big(\nabla w\cdot\nabla\widetilde{\varphi}+m^2w\widetilde{\varphi}\big)\,dz=\int_{S_r^+}t^{1-2s}\frac{\partial w}{\partial\nu}\widetilde{\varphi}\,dS+\kappa_s\int_{B'_r}\left(\frac{a(\frac{x}{|x|})}{|x|^{2s}}w+hw\right)\widetilde{\varphi}\,dx. \tag{5.2}$$

The main result of this section is the existence of the limit as $r\to 0^+$ of the *Almgren's frequency* function (see [1] and [14]) associated to w

$$\mathcal{N}(r)=\frac{r\left[\displaystyle\int_{B_r^+}t^{1-2s}\big(|\nabla w|^2+m^2w^2\big)\,dt\,dx-\kappa_s\int_{B'_r}\big(\tfrac{a(x/|x|)}{|x|^{2s}}w^2+hw^2\big)\,dx\right]}{\displaystyle\int_{S_r^+}t^{1-2s}w^2\,dS}. \tag{5.3}$$

Theorem 5.1. *Let w satisfy* (5.1)*, with $s\in(0,1)$, $a\in C^1(\mathbb{S}^{N-1})$ satisfy* (1.10)*, and h as in assumption* (1.2)*. Then, letting $\mathcal{N}(r)$ as in (5.3), there there exists $k_0\in\mathbb{N}$, $k_0\geq 1$, such that*

$$\lim_{r\to 0^+}\mathcal{N}(r)=-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\mu_{k_0}(a)}. \tag{5.4}$$

Furthermore, if γ denotes the limit in (5.4), $M\geq 1$ is the multiplicity of the eigenvalue $\mu_{j_0}(a)=\mu_{j_0+1}(a)=\cdots=\mu_{j_0+M-1}(a)$ and $\{\psi_i\}_{i=j_0}^{j_0+M-1}$ ($j_0\leq k_0\leq j_0+M-1$) is an $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$-orthonormal basis for the eigenspace of problem (1.7) *associated to $\mu_{k_0}(a)$, then*

$$\tau^{-\gamma}w(\tau\theta)\to\sum_{i=j_0}^{j_0+M-1}\beta_i\psi_i(\theta)\quad\text{in } C^{0,\alpha}(\overline{\mathbb{S}^N_+})\quad\text{as }\tau\to 0^+,$$

$$\tau^{-\gamma}w(0,\tau\theta')\to\sum_{i=j_0}^{j_0+M-1}\beta_i\psi_i(0,\theta')\quad\text{in } C^{1,\alpha}(\mathbb{S}^{N-1})\quad\text{as }\tau\to 0^+,$$

for some $\alpha \in (0,1)$, *where*

$$\beta_i = R^{-\gamma} \int_{\mathbb{S}^N_+} \theta_1^{1-2s} w(R\,\theta)\psi_i(\theta)\, dS \tag{5.5}$$
$$- R^{-2\gamma-N+2s} \int_0^R \frac{\rho^{\gamma+N-1}}{2\gamma+N-2s}\bigg(\kappa_s \int_{\mathbb{S}^{N-1}} h(\rho\theta')w(0,\rho\theta')\psi_i(0,\theta')\, dS'$$
$$- m^2\rho^{2-2s}\int_{\mathbb{S}^N_+} \theta_1^{1-2s} w(\rho\theta)\psi_i(\theta)\, dS\bigg) d\rho$$
$$+ \int_0^R \frac{\rho^{2s-\gamma-1}}{2\gamma+N-2s}\bigg(\kappa_s \int_{\mathbb{S}^{N-1}} h(\rho\theta')w(0,\rho\theta')\psi_i(0,\theta')\, dS'$$
$$- m^2\rho^{2-2s}\int_{\mathbb{S}^N_+} \theta_1^{1-2s} w(\rho\theta)\psi_i(\theta)\, dS\bigg) d\rho$$

for all $R>0$ *such that* $\overline{B'_R}\subset\Omega$ *and* $(\beta_{j_0}, \beta_{j_0+1}, \dots, \beta_{j_0+M-1}) \neq (0,0,\dots,0)$.

From the Pohozaev-type identity (4.10) and (4.11) it follows that, for a.e. $r\in(0,R)$,

$$-\frac{N-2s}{2}\bigg[\int_{B_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2w^2\big)dz - \kappa_s\int_{B'_r} \frac{a(\frac{x}{|x|})}{|x|^{2s}} w^2 dx\bigg] \tag{5.6}$$
$$+ \frac{r}{2}\bigg[\int_{S_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2w^2\big)dS - \kappa_s\int_{\partial B'_r} \frac{a(\frac{x}{|x|})}{|x|^{2s}} w^2 dS'\bigg]$$
$$= r\int_{S_r^+} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS - \frac{\kappa_s}{2}\int_{B'_r}(Nh+\nabla h\cdot x)w^2\,dx + \frac{r\kappa_s}{2}\int_{\partial B'_r} hw^2\,dS' + m^2\int_{B_r^+} t^{1-2s}w^2dz$$

and

$$\int_{B_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2w^2\big)\,dz - \kappa_s\int_{B'_r}\frac{a(\frac{x}{|x|})}{|x|^{2s}}w^2\,dz = \int_{S_r^+} t^{1-2s}\frac{\partial w}{\partial\nu} w\,dS + \kappa_s\int_{B'_r} hw^2\,dx. \tag{5.7}$$

For every $r\in(0,R]$ we define

$$D(r) = \frac{1}{r^{N-2s}}\bigg[\int_{B_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2w^2\big)\,dt\,dx - \kappa_s\int_{B'_r}\bigg(\frac{a(x/|x|)}{|x|^{2s}}w^2 + h(x)w^2\bigg)\,dx\bigg] \tag{5.8}$$

and

$$H(r) = \frac{1}{r^{N+1-2s}}\int_{S_r^+} t^{1-2s}w^2\,dS = \int_{\mathbb{S}^N_+}\theta_1^{1-2s}w^2(r\theta)\,dS. \tag{5.9}$$

Lemma 5.2. $H\in C^1(0,R)$ *and*

$$H'(r) = \frac{2}{r^{N+1-2s}}\int_{S_r^+} t^{1-2s} w\frac{\partial w}{\partial\nu}\,dS, \quad \textit{for every } r\in(0,R), \tag{5.10}$$

$$H'(r) = \frac{2}{r}D(r), \quad \textit{for every } r\in(0,R). \tag{5.11}$$

Proof. The proof of (5.10) can be performed arguing as in [9, Lemma 3.8] and using the regularity results of Lemma 4.1 and Proposition 3.7. The continuity of H' on the interval $(0,R)$ follows by the representation of H' given above, Lemma 4.1, Propositions 3.5, 3.7 and the Dominated Convergence Theorem. Finally, (5.11) follows from (5.10), (5.8), and (5.7). □

The regularity of the function D is established in the following lemma.

Lemma 5.3. *The function* D *defined in (5.8) belongs to* $W^{1,1}_{\rm loc}(0,R)$ *and*

$$D'(r) = \frac{2}{r^{N+1-2s}}\bigg[r\int_{S_r^+} t^{1-2s}\left|\frac{\partial w}{\partial\nu}\right|^2 dS - \kappa_s\int_{B'_r}\Big(sh + \frac{1}{2}(\nabla h\cdot x)\Big)w^2\,dx + m^2\int_{B_r^+} t^{1-2s}w^2dz\bigg]$$

in a distributional sense and for a.e. $r\in(0,R)$.

Proof. For any $r \in (0, r_0)$ let

$$I(r) = \int_{B_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2 w^2\big)\, dt\, dx - \kappa_s \int_{B'_r} \left(\frac{a(x/|x|)}{|x|^{2s}} w^2 + h(x) w^2\right) dx. \tag{5.12}$$

Since $w \in H^1(B_R^+; t^{1-2s})$, from [9, Lemma 2.5] we deduce that $I \in W^{1,1}(0,R)$ and

$$I'(r) = \int_{S_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2 w^2\big)\, dS - \kappa_s \int_{\partial B'_r} \left(\frac{a(x/|x|)}{|x|^{2s}} w^2 + h(x) w^2\right) dS' \tag{5.13}$$

for a.e. $r \in (0,R)$ and in the distributional sense. Therefore $D \in W^{1,1}_{\rm loc}(0,R)$ and, using (5.6), (5.12), and (5.13) into

$$D'(r) = r^{2s-1-N}[-(N-2s)I(r) + rI'(r)],$$

we obtain the conclusion. □

We prove now that, since $w \not\equiv 0$, $H(r)$ does not vanish for r sufficiently small.

Lemma 5.4. *There exists $R_0 \in (0,R)$ such that $H(r) > 0$ for any $r \in (0, R_0)$, where H is defined by (5.9).*

Proof. Let $R_0 \in (0,R)$ such that $1 - \kappa_s C_h R_0^{\chi}(C'_{a,N,s})^{-1} > 0$, with $C'_{a,N,s}$ as in Corollary 2.6. Suppose by contradiction that there exists $r_0 \in (0, R_0)$ such that $H(r_0) = 0$. Then $w = 0$ a.e. on $S^+_{r_0}$. From (5.7) it follows that

$$\int_{B^+_{r_0}} t^{1-2s}\big(|\nabla w|^2 + m^2 w^2\big)\, dz - \kappa_s \int_{B'_{r_0}} \frac{a(\frac{x}{|x|})}{|x|^{2s}} w^2\, dz - \kappa_s \int_{B'_{r_0}} h w^2\, dx = 0.$$

From (1.2), Corollaries 2.3 and 2.6, it follows that

$$\begin{aligned} 0 &= \int_{B^+_{r_0}} t^{1-2s}\big(|\nabla w|^2 + m^2 w^2\big)\, dz - \kappa_s \int_{B'_{r_0}} \frac{a(\frac{x}{|x|})}{|x|^{2s}} w^2\, dz - \kappa_s \int_{B'_{r_0}} h w^2\, dx \\ &\geq C_{a,N,s}\Big(1 - \kappa_s C_h r_0^{\chi}(C'_{a,N,s})^{-1}\Big) \int_{B^+_{r_0}} t^{1-2s} |\nabla w|^2 dz, \end{aligned}$$

which, being $1 - \kappa_s C_h r_0^{\chi}(C'_{a,N,s})^{-1} > 0$, implies $w \equiv 0$ in $B^+_{r_0}$ by Lemma 2.2. Classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [27]) allow to conclude that $w = 0$ a.e. in B_R^+, a contradiction. □

Letting R_0 be as in Lemma 5.4 and recalling (5.3), the *Almgren type frequency function*

$$\mathcal{N}(r) = \frac{D(r)}{H(r)} \tag{5.14}$$

is well defined in $(0, R_0)$.

Lemma 5.5. *The function $\mathcal{N}$ defined in (5.14) belongs to $W^{1,1}_{\rm loc}(0,R_0)$ and*

$$\mathcal{N}'(r) = \nu_1(r) + \nu_2(r) \tag{5.15}$$

in a distributional sense and for a.e. $r \in (0, R_0)$, where

$$\nu_1(r) = \frac{2r\Big[\Big(\int_{S_r^+} t^{1-2s}\left|\frac{\partial w}{\partial \nu}\right|^2 dS\Big)\Big(\int_{S_r^+} t^{1-2s} w^2\, dS\Big) - \Big(\int_{S_r^+} t^{1-2s} w \frac{\partial w}{\partial \nu}\, dS\Big)^2\Big]}{\Big(\int_{S_r^+} t^{1-2s} w^2\, dS\Big)^2} \tag{5.16}$$

and

$$\nu_2(r) = \frac{2m^2 \int_{B_r^+} t^{1-2s} w^2 dz - \kappa_s \int_{B'_r} (2sh + \nabla h \cdot x) w^2\, dx}{\int_{S_r^+} t^{1-2s} w^2\, dS}. \tag{5.17}$$

Proof. It follows from Lemmas 5.2, 5.4, and 5.3. □

Lemma 5.6. *Let $\mathcal{N}$ be the function defined in (5.14). There exist $\tilde{R} \in (0, R_0)$ and a constant $\overline{C} > 0$ such that*

$$\int_{B_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2 w^2\big)\, dt\, dx - \kappa_s \int_{B_r'} \left(\frac{a(x/|x|)}{|x|^{2s}} w^2 + h(x) w^2\right) dx \tag{5.18}$$

$$\geq -\left(\frac{N-2s}{2r}\right)\int_{S_r^+} t^{1-2s} w^2 dS + \overline{C}\left(\int_{B_r'} \frac{w^2}{|x|^{2s}}\, dx + \int_{B_r^+} t^{1-2s}|\nabla w|^2\, dt\, dx + \int_{B_r^+} t^{1-2s}\frac{w^2}{|z|^2}\, dt\, dx\right)$$

and

$$\mathcal{N}(r) > -\frac{N-2s}{2} \tag{5.19}$$

for every $r \in (0, \tilde{R})$.

Proof. From Lemma 2.2 and Corollary 2.6, it follows that

$$\int_{B_r^+} t^{1-2s}\big(|\nabla w|^2 + m^2 w^2\big)\, dt\, dx - \kappa_s \int_{B_r'} \left(\frac{a(x/|x|)}{|x|^{2s}} w^2 + h(x) w^2\right) dx + \left(\frac{N-2s}{2r}\right)\int_{S_r^+} t^{1-2s} w^2 dS$$
$$\geq \left(1 - \frac{m^2 r^2}{\mu_1(a) + (\frac{N-2s}{2})^2} - \frac{\kappa_s C_h r^\chi}{C'_{N,a,s}}\right)\left(\int_{B_r^+} t^{1-2s}|\nabla w|^2\, dt\, dx\right.$$
$$\left. - \kappa_s \int_{B_r'} \frac{a(x/|x|)}{|x|^{2s}} w^2\, dx + \left(\frac{N-2s}{2r}\right)\int_{S_r^+} t^{1-2s} w^2 dS\right)$$

for every $r \in (0, R_0)$. The conclusion follows from the above estimate, choosing r sufficiently small and using Lemma 2.2 and Corollaries 2.3, 2.6. □

Lemma 5.7. *Let $\tilde{R}$ be as in Lemma 5.6 and ν_2 as in (5.17). There exists $C_1 > 0$ such that*

$$|\nu_2(r)| \leq C_1 \left[\mathcal{N}(r) + \frac{N-2s}{2}\right] r^{-1+\chi}$$

for a.e. $r \in (0, \tilde{R})$.

Proof. From (1.2) and (5.18) we deduce that

$$\left|\int_{B_r'} (2sh(x) + \nabla h(x)\cdot x) w^2\, dx\right| \leq 2C_h r^\chi \int_{B_r'} \frac{|w|^2}{|x|^{2s}}\, dx \leq 2C_h \overline{C}^{-1} r^{\chi+N-2s}\left[D(r) + \tfrac{N-2s}{2} H(r)\right],$$

and, therefore, for any $r \in (0, \tilde{R})$, we have that

$$\left|\frac{\int_{B_r'} (2sh(x) + \nabla h(x)\cdot x) w^2\, dx}{\int_{S_r^+} t^{1-2s} w^2\, dS}\right| \leq 2C_h \overline{C}^{-1} r^{-1+\chi}\frac{D(r) + \frac{N-2s}{2} H(r)}{H(r)} \tag{5.20}$$
$$= 2C_h \overline{C}^{-1} r^{-1+\chi}\left[\mathcal{N}(r) + \frac{N-2s}{2}\right].$$

On the other hand, from (5.18) it also follows that

$$\left|\frac{\int_{B_r^+} t^{1-2s} w^2 dz}{\int_{S_r^+} t^{1-2s} w^2\, dS}\right| \leq \overline{C}^{-1} r\left[\mathcal{N}(r) + \frac{N-2s}{2}\right]. \tag{5.21}$$

Combining (5.20) with (5.21) we obtain the stated estimate. □

Lemma 5.8. *Let $\tilde{R}$ be as in Lemma 5.6, $\mathcal{N}$ as in (5.14) and H as in* (5.9)*. Then*

- (i) *there exist a positive constant $C_2 > 0$ such that $\mathcal{N}(r) \leq C_2$ for all $r \in (0, \tilde{R})$;*
- (ii) *the limit $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ exists and is finite;*
- (iii) *there exists a constant $K_1 > 0$ such that $H(r) \leq K_1 r^{2\gamma}$ for all $r \in (0, \tilde{R})$;*
- (iv) *for any $\sigma > 0$ there exists a constant $K_2(\sigma) > 0$ depending on σ such that $H(r) \geq K_2(\sigma)\, r^{2\gamma+\sigma}$ for all $r \in (0, \tilde{R})$.*

Proof. By Lemma 5.5, Schwarz's inequality, and Lemma 5.7, we obtain

$$\left(\mathcal{N}+\frac{N-2s}{2}\right)'(r)\geq \nu_2(r)\geq -C_1\left[\mathcal{N}(r)+\frac{N-2s}{2}\right]r^{-1+\chi} \tag{5.22}$$

for a.e. $r\in(0,\tilde R)$. Integration over $(r,\tilde R)$ yields

$$\mathcal{N}(r)\leq -\frac{N-2s}{2}+\left(\mathcal{N}(\tilde R)+\frac{N-2s}{2}\right)e^{\frac{C_1}{\chi}\tilde R^{\chi}}$$

for any $r\in(0,\tilde R)$, thus proving claim (i).

By Lemmas 5.7 and 5.8, the function ν_2 defined in (5.17) belongs to $L^1(0,\tilde R)$. Hence, by Lemma 5.5 and Schwarz's inequality, $\mathcal{N}'$ is the sum of a nonnegative function and of a L^1-function on $(0,\tilde R)$. Therefore

$$\mathcal{N}(r)=\mathcal{N}(\tilde R)-\int_r^{\tilde R}\mathcal{N}'(\rho)\,d\rho$$

admits a limit as $r\to 0^+$ which is necessarily finite in view of (5.19) and part (i). Claim (ii) is thereby proved.

By (ii) $\mathcal{N}'\in L^1(0,\tilde R)$ and, by (i), $\mathcal{N}$ is bounded, then from (5.22) and (i) it follows that

$$\mathcal{N}(r)-\gamma=\int_0^r\mathcal{N}'(\rho)\,d\rho\geq -C_3r^{\chi}$$

for all $r\in(0,\tilde R)$. Therefore by (5.11) and (5.14), we deduce that, for all $r\in(0,\tilde R)$,

$$\frac{H'(r)}{H(r)}=\frac{2\,\mathcal{N}(r)}{r}\geq\frac{2\gamma}{r}-2C_3r^{-1+\chi},$$

which, after integration over the interval $(r,\tilde R)$, yields (iii).

From (ii) it follows that, for any $\sigma>0$ there exists $r_\sigma>0$ such that $\mathcal{N}(r)<\gamma+\frac{\sigma}{2}$ for any $r\in(0,r_\sigma)$ and hence

$$\frac{H'(r)}{H(r)}=\frac{2\,\mathcal{N}(r)}{r}<\frac{2\gamma+\sigma}{r}\quad\text{for all } r\in(0,r_\sigma).$$

Integrating over the interval (r,r_σ) and by continuity of H outside 0, we obtain (iv). □

5.1. **The blow-up argument.**

Lemma 5.9. *Let w satisfy* (5.1)*, with $s\in(0,1)$, h as in assumption* (1.2) *and $a\in C^1(\mathbb{S}^{N-1})$ satisfy* (1.10)*. Let $\gamma:=\lim_{r\to0^+}\mathcal{N}(r)$ as in Lemma 5.8. Then*

(i) *there exists $k_0\in\mathbb{N}$, $k_0\geq 1$, such that $\gamma=-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\mu_{k_0}(a)}$;*

(ii) *for every sequence $\tau_n\to0^+$, there exist a subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction ψ of problem* (1.7) *associated to the eigenvalue $\mu_{k_0}(a)$ such that $\|\psi\|_{L^2(\mathbb{S}^N_+;\theta_1^{1-2s})}=1$ and*

$$\frac{w(\tau_{n_k}z)}{\sqrt{H(\tau_{n_k})}}\to|z|^{\gamma}\psi\Big(\frac{z}{|z|}\Big)$$

strongly in $H^1(B_r^+;t^{1-2s})$ and in $C^{0,\alpha}_{\rm loc}(\overline{B_r^+}\setminus\{0\})$ for some $\alpha\in(0,1)$ and all $r\in(0,1)$ and

$$\frac{w(0,\tau_{n_k}x)}{\sqrt{H(\tau_{n_k})}}\to|x|^{\gamma}\psi\Big(0,\frac{x}{|x|}\Big)$$

in $C^{1,\alpha}_{\rm loc}(B_1'\setminus\{0\})$.

Proof. Let us set

$$w^\tau(z)=\frac{w(\tau z)}{\sqrt{H(\tau)}}. \tag{5.23}$$

We notice that $\int_{S_1^+}t^{1-2s}|w^\tau|^2dS=1$. Moreover, by scaling and Lemma 5.8, part (i),

$$\int_{B_1^+}t^{1-2s}\Big(|\nabla w^\tau(z)|^2+m^2\tau^2|w^\tau(z)|^2\Big)dz-\kappa_s\int_{B_1'}\left(\frac{a(\frac{x}{|x|})}{|x|^{2s}}+\tau^{2s}h(\tau x)\right)|w^\tau|^2dx=\mathcal{N}(\tau)\leq C_2$$

for every $\tau \in (0, \tilde{R})$, whereas, from (5.18),

$$\mathcal{N}(\tau) \geq \frac{\tau^{-N+2s}}{H(\tau)}\left(-\left(\frac{N-2s}{2\tau}\right) \int_{S_\tau^+} t^{1-2s} w^2 dS + \overline{C} \int_{B_\tau^+} t^{1-2s} |\nabla w|^2 \, dt \, dx \right)$$
$$= -\frac{N-2s}{2} + \overline{C} \int_{B_1^+} t^{1-2s} |\nabla w^\tau(z)|^2 dz$$

for every $\tau \in (0, \tilde{R})$. From the above estimates, $\{w^\tau\}_{\tau \in (0,\tilde{R})}$ is bounded in $H^1(B_1^+; t^{1-2s})$. Therefore, for any given sequence $\tau_n \to 0^+$, there exists a subsequence $\tau_{n_k} \to 0^+$ such that $w^{\tau_{n_k}} \rightharpoonup \widetilde{w}$ weakly in $H^1(B_1^+; t^{1-2s})$ for some $\widetilde{w} \in H^1(B_1^+; t^{1-2s})$. Moreover, $\int_{S_1^+} t^{1-2s} |\widetilde{w}|^2 dS = 1$ due to compactness of the trace embedding $H^1(B_1^+; t^{1-2s}) \hookrightarrow\hookrightarrow L^2(S_1^+; t_1^{1-2s})$. In particular $\widetilde{w} \not\equiv 0$.

For every small $\tau \in (0, \tilde{R})$, w^τ satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla w^\tau) + \tau^2 t^{1-2s} m^2 w^\tau = 0, & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s} \frac{\partial w^\tau}{\partial t} = \kappa_s \Big(\frac{a(x/|x|)}{|x|^{2s}} w^\tau + \tau^{2s} h(\tau x) w^\tau \Big), & \text{on } B_1', \end{cases} \tag{5.24}$$

in a weak sense, i.e.

$$\int_{B_1^+} t^{1-2s} \big(\nabla w^\tau \cdot \nabla \widetilde{\varphi} + m^2 \tau^2 w^\tau \widetilde{\varphi} \big) \, dz = \kappa_s \int_{B_1'} \left(\frac{a(\frac{x}{|x|})}{|x|^{2s}} w^\tau + \tau^{2s} h(\tau x) w^\tau \right) \widetilde{\varphi}(0, x) \, dx$$

for all $\widetilde{\varphi} \in H^1(B_1^+; t^{1-2s})$ s.t. $\widetilde{\varphi} = 0$ on $\mathbb{S}_+^N$ and, for such $\widetilde{\varphi}$, by (1.2) and [9, Lemma 2.5],

$$\tau^{2s} \int_{B_1'} h(\tau x) w^\tau \widetilde{\varphi}(0, x) \, dx = o(1) \quad \text{as } \tau \to 0^+$$

and, by [9, Lemma 2.4],

$$\tau^2 \int_{B_1^+} t^{1-2s} w^\tau \widetilde{\varphi} \, dz = o(1) \quad \text{as } \tau \to 0^+.$$

From weak convergence $w^{\tau_{n_k}} \rightharpoonup \widetilde{w}$ in $H^1(B_1^+; t^{1-2s})$, we can pass to the limit in (5.24) along the sequence τ_{n_k} and obtain that $\widetilde{w}$ weakly solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla \widetilde{w}) = 0, & \text{in } B_1^+, \\ -\lim_{t\to 0^+} t^{1-2s} \frac{\partial \widetilde{w}}{\partial t} = \kappa_s \frac{a(x/|x|)}{|x|^{2s}} \widetilde{w}, & \text{on } B_1'. \end{cases} \tag{5.25}$$

From Proposition 3.5, we have that

$$w^{\tau_{n_k}} \to \widetilde{w} \quad \text{in } C^{0,\alpha}_{\text{loc}}(\overline{B_r^+} \setminus \{0\}),$$

while Proposition 3.7 and Lemma 4.1 imply that

$$\nabla_x w^{\tau_{n_k}} \to \nabla_x \widetilde{w}, \quad \text{and} \quad t^{1-2s} \frac{\partial w^{\tau_{n_k}}}{\partial t} \to t^{1-2s} \frac{\partial \widetilde{w}}{\partial t} \quad \text{in } C^{0,\alpha}_{\text{loc}}(\overline{B_r^+} \setminus \{0\}) \tag{5.26}$$

for some $\alpha \in (0, 1)$ and all $r \in (0, 1)$. By (1.2), [9, Lemma 2.5], and boundedness of $\{w^\tau\}_{\tau \in (0,\tilde{R})}$ in $H^1(B_1^+; t^{1-2s})$, it follows that

$$\tau^{2s} \int_{B_1'} h(\tau x) |w^\tau|^2 \, dx = o(1) \quad \text{as } \tau \to 0^+, \tag{5.27}$$

and, by [9, Lemma 2.4],

$$\tau^2 \int_{B_1^+} t^{1-2s} |w^\tau|^2 \, dz = o(1) \quad \text{as } \tau \to 0^+. \tag{5.28}$$

Multiplying equation (5.24) with w^τ, integrating in B_r^+, and using (5.26), (5.27), (5.28), and Corollary 2.3, we easily obtain that $\|w^{\tau_{n_k}}\|_{H^1(B_r^+; t^{1-2s})} \to \|\widetilde{w}\|_{H^1(B_r^+; t^{1-2s})}$ for all $r \in (0, 1)$, and hence

$$w^{\tau_{n_k}} \to \widetilde{w} \quad \text{in } H^1(B_r^+; t^{1-2s}) \tag{5.29}$$

for any $r \in (0,1)$. For any $r \in (0,1)$ and $k \in \mathbb{N}$, let us define the functions

$$D_k(r) = \frac{1}{r^{N-2s}} \bigg[\int_{B_r^+} t^{1-2s} \big(|\nabla w^{\tau_{n_k}}|^2 + m^2 \tau_{n_k}^2 |w^{\tau_{n_k}}|^2 \big) \, dt \, dx \\ - \kappa_s \int_{B_r'} \bigg(\frac{a(\frac{x}{|x|})}{|x|^{2s}} + \tau_{n_k}^{2s} h(\tau_{n_k} x) \bigg) |w^{\tau_{n_k}}|^2 dx \bigg]$$

and

$$H_k(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} |w^{\tau_{n_k}}|^2 \, dS.$$

Direct calculations show that $\mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\tau_{n_k} r)}{H(\tau_{n_k} r)} = \mathcal{N}(\tau_{n_k} r)$ for all $r \in (0,1)$. From (5.29), (5.27), and (5.28), it follows that, for any fixed $r \in (0,1)$, $D_k(r) \to \widetilde{D}(r)$, where

$$\widetilde{D}(r) = \frac{1}{r^{N-2s}} \bigg[\int_{B_r^+} t^{1-2s} |\nabla \widetilde{w}|^2 \, dt \, dx - \kappa_s \int_{B_r'} \frac{a(\frac{x}{|x|})}{|x|^{2s}} \widetilde{w}^2 \, dx \bigg] \quad \text{for all } r \in (0,1).$$

The compactness of the trace embedding $H^1(B_r^+; t^{1-2s}) \hookrightarrow\hookrightarrow L^2(S_r^+; t_1^{1-2s})$ ensures that, for every $r \in (0,1)$, $H_k(r) \to \widetilde{H}(r)$, where

$$\widetilde{H}(r) = \frac{1}{r^{N+1-2s}} \int_{S_r^+} t^{1-2s} \widetilde{w}^2 \, dS.$$

Arguing as in Lemma 5.4, we can easily prove that $\widetilde{H}(r) > 0$ for all $r \in (0,1)$ and the function

$$\widetilde{\mathcal{N}}(r) := \frac{\widetilde{D}(r)}{\widetilde{H}(r)} \tag{5.30}$$

is well defined for $r \in (0,1)$. From Lemma 5.8 part (ii), we deduce that

$$\widetilde{\mathcal{N}}(r) = \lim_{k \to \infty} \mathcal{N}(\tau_{n_k} r) = \gamma$$

for all $r \in (0,1)$. In particular, $\widetilde{\mathcal{N}}$ is constant in $(0,1)$ and hence $\widetilde{\mathcal{N}}'(r) = 0$ for any $r \in (0,1)$. By (5.25) and Lemma 5.5 with $h \equiv 0$ and $m = 0$, we obtain

$$\bigg(\int_{S_r^+} t^{1-2s} \left| \frac{\partial \widetilde{w}}{\partial \nu} \right|^2 dS \bigg) \cdot \bigg(\int_{S_r^+} t^{1-2s} \widetilde{w}^2 \, dS \bigg) - \bigg(\int_{S_r^+} t^{1-2s} \widetilde{w} \frac{\partial \widetilde{w}}{\partial \nu} \, dS \bigg)^2 = 0$$

for all $r \in (0,1)$, which implies that $\widetilde{w}$ and $\frac{\partial \widetilde{w}}{\partial \nu}$ have the same direction as vectors in $L^2(S_r^+; t^{1-2s})$ and hence there exists a function $\eta = \eta(r)$ such that $\frac{\partial \widetilde{w}}{\partial \nu}(r,\theta) = \eta(r) \widetilde{w}(r,\theta)$ for all $r \in (0,1)$ and $\theta \in \mathbb{S}^N_+$. After integration we obtain

$$\widetilde{w}(r,\theta) = e^{\int_1^r \eta(s) ds} \widetilde{w}(1,\theta) = \varphi(r) \psi(\theta), \quad r \in (0,1), \ \theta \in \mathbb{S}^N_+, \tag{5.31}$$

where $\varphi(r) = e^{\int_1^r \eta(s) ds}$ and $\psi(\theta) = \widetilde{w}(1,\theta)$. From (5.25) and (5.31), it follows that

$$\begin{cases} \frac{1}{r^N} \big(r^{N+1-2s} \varphi' \big)' \theta_1^{1-2s} \psi(\theta) + r^{-1-2s} \varphi(r) \operatorname{div}_{\mathbb{S}^N} (\theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta)) = 0, \\ -\lim_{\theta_1 \to 0^+} \theta_1^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta) \cdot \mathbf{e}_1 = \kappa_s a(\theta') \psi(0,\theta'). \end{cases}$$

Taking r fixed we deduce that ψ is an eigenfunction of the eigenvalue problem (1.7). If $\mu_{k_0}(a)$ is the corresponding eigenvalue then $\varphi(r)$ solves the equation

$$\varphi''(r) + \frac{N+1-2s}{r} \varphi' - \frac{\mu_{k_0}(a)}{r^2} \varphi(r) = 0$$

and hence $\varphi(r)$ is of the form

$$\varphi(r) = c_1 r^{\sigma_{k_0}^+} + c_2 r^{\sigma_{k_0}^-}$$

for some $c_1, c_2 \in \mathbb{R}$, where

$$\sigma_{k_0}^+ = -\tfrac{N-2s}{2} + \sqrt{\big(\tfrac{N-2s}{2}\big)^2 + \mu_{k_0}(a)} \quad \text{and} \quad \sigma_{k_0}^- = -\tfrac{N-2s}{2} - \sqrt{\big(\tfrac{N-2s}{2}\big)^2 + \mu_{k_0}(a)}.$$

Since the function $|x|^{\sigma^-_{k_0}}\psi(\frac{x}{|x|}) \notin L^2(B_1'; |x|^{-2s})$ and hence $|z|^{\sigma^-_{k_0}}\psi(\frac{z}{|z|}) \notin H^1(B_1^+; t^{1-2s})$ in virtue of Lemma [9, Lemma 2.5], we deduce that $c_2 = 0$ and $\varphi(r) = c_1 r^{\sigma^+_{k_0}}$. Moreover, from $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

$$\widetilde{w}(r,\theta) = r^{\sigma^+_{k_0}}\psi(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in \mathbb{S}^N_+. \tag{5.32}$$

Substituting (5.32) into (5.30), we obtain that $\gamma = \widetilde{\mathcal{N}}(r) = \frac{\widetilde{D}(r)}{\widetilde{H}(r)} = \sigma^+_{k_0}$. This completes the proof of the lemma. □

Lemma 5.10. *If w satisfies* (5.1)*, H is defined in* (5.9)*, and $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ is as in Lemma 5.8, then the limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ exists and it is finite.*

Proof. In view of Lemma 5.8, part (iii), it is sufficient to prove that the limit exists. From (5.9), (5.11), and Lemma 5.8 it follows that

$$\frac{d}{dr}\frac{H(r)}{r^{2\gamma}} = 2r^{-2\gamma-1}(D(r) - \gamma H(r)) = 2r^{-2\gamma-1}H(r)\int_0^r \mathcal{N}'(\rho)d\rho,$$

which, by integration over $(r, \tilde{R})$, yields

$$\frac{H(\tilde{R})}{\tilde{R}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\tilde{R}} f_1(\rho)\, d\rho + \int_r^{\tilde{R}} f_2(\rho) d\rho \tag{5.33}$$

where $f_i(\rho) = 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \nu_i(t)dt\right)$, $i = 1, 2$, and ν_1 and ν_2 are as in (5.16) and (5.17). Since, by Schwarz's inequality, $\nu_1 \geq 0$, we have that $\lim_{r\to 0^+}\int_r^{\tilde{R}} f_1(\rho)d\rho$ exists. On the other hand, by Lemmas 5.7 and 5.8, we have that

$$|f_2(\rho)| \leq \frac{2K_1C_1}{\chi}\Big(C_2 + \frac{N-2s}{2}\Big)\rho^{-1+\chi}$$

for all $\rho \in (0, \widetilde{R})$, which proves that $f_2 \in L^1(0, \widetilde{R})$. Hence both terms at the right hand side of (5.33) admit a limit as $r \to 0^+$ thus completing the proof. □

From Lemma 5.9, the following point-wise estimate for solutions to (5.1) follow.

Lemma 5.11. *Let w satisfy* (5.1)*. Then there exists $C_4 > 0$ and $\bar{r} \in (0, \tilde{R})$ such that $|w(z)| \leq C_4|z|^\gamma$ for all $z \in \overline{B^+_{\bar{r}}}$, where $\gamma := \lim_{r\to 0^+}\mathcal{N}(r)$ is as in Lemma 5.8.*

Proof. We first claim that

$$\sup_{S_r^+}|w|^2 = O(H(r)) \quad \text{as } r \to 0^+. \tag{5.34}$$

In order to prove (5.34), we argue by contradiction and assume that there exists a sequence $\tau_n \to 0^+$ such that

$$\sup_{\theta\in\mathbb{S}^N_+}\Big|w\Big(\frac{\tau_n}{2}\theta\Big)\Big|^2 > nH\Big(\frac{\tau_n}{2}\Big),$$

i.e., defining w^τ as in (5.23),

$$\sup_{x\in S^+_{1/2}}|w^{\tau_n}(z)|^2 > 2^{N+1-2s}n\int_{S^+_{1/2}} t^{1-2s}|w^{\tau_n}(z)|^2 dS. \tag{5.35}$$

From Lemma 5.9, along a subsequence τ_{n_k} we have that $w^{\tau_{n_k}} \to |z|^\gamma\psi\big(\frac{z}{|z|}\big)$ in $C^{0,\alpha}_{\rm loc}(\overline{S^+_{1/2}})$, for some ψ eigenfunction of problem (1.7), hence passing to the limit in (5.35) gives rise to a contradiction and claim (5.34) is proved. The conclusion follows from combination of (5.34) and part (iii) of Lemma 5.8. □

We will now prove that $\lim_{r\to 0^+} r^{-2\gamma} H(r)$ is strictly positive.

Lemma 5.12. *Under the same assumptions of Lemma 5.10,* $\lim_{r\to 0^+} r^{-2\gamma} H(r) > 0$.

Proof. For all $k \geq 1$, let ψ_k be as in (2.3), i.e. ψ_k is a $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$-normalized eigenfunction of problem (1.7) associated to the eigenvalue $\mu_k(a)$ and $\{\psi_k\}_k$ is an orthonormal basis of $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$. From Lemma 5.9 there exist $j_0, M \in \mathbb{N}\setminus\{0\}$, such that M is the multiplicity of the eigenvalue $\mu_{j_0}(a) = \mu_{j_0+1}(a) = \cdots = \mu_{j_0+M-1}(a)$ and

$$\gamma = \lim_{r\to 0^+} \mathcal{N}(r) = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_i(a)}, \quad i = j_0, \dots, j_0+M-1.$$

Let us expand w as $w(z) = w(\tau\theta) = \sum_{k=1}^{\infty} \varphi_k(\tau)\psi_k(\theta)$, where $\tau = |z| \in (0,R]$, $\theta = z/|z| \in \mathbb{S}^N_+$, and

$$\varphi_k(\tau) = \int_{\mathbb{S}^N_+} \theta_1^{1-2s} w(\tau\,\theta)\psi_k(\theta)\,dS.$$

The Parseval identity yields

$$H(\tau) = \int_{\mathbb{S}^N_+} \theta_1^{1-2s} w^2(\tau\theta)\,dS = \sum_{k=1}^{\infty} \varphi_k^2(\tau), \quad \text{for all } 0 < \tau \leq R. \tag{5.36}$$

In particular, from Lemma 5.8 (iii) and (5.36) it follows that, for all $k \geq 1$,

$$\varphi_k(\tau) = O(\tau^{\gamma}) \quad \text{as } \tau \to 0^+. \tag{5.37}$$

Equations (5.1) and (2.3) imply that, for every k,

$$-\varphi_k''(\tau) - \frac{N+1-2s}{\tau}\varphi_k'(\tau) + \frac{\mu_k(a)}{\tau^2}\varphi_k(\tau) = \zeta_k(\tau), \quad \text{in } (0,R),$$

where

$$\zeta_k(\tau) = \frac{\kappa_s}{\tau^{2-2s}} \int_{\mathbb{S}^{N-1}} h(\tau\theta')w(0,\tau\theta')\psi_k(0,\theta')\,dS' - m^2\varphi_k(\tau). \tag{5.38}$$

A direct calculation shows that, for some $c_1^k, c_2^k \in \mathbb{R}$,

$$\varphi_k(\tau) = \tau^{\sigma_k^+}\left(c_1^k + \int_\tau^R \frac{t^{-\sigma_k^+ + 1}}{\sigma_k^+ - \sigma_k^-}\zeta_k(t)\,dt\right) + \tau^{\sigma_k^-}\left(c_2^k + \int_\tau^R \frac{t^{-\sigma_k^- + 1}}{\sigma_k^- - \sigma_k^+}\zeta_k(t)\,dt\right), \tag{5.39}$$

with $\sigma_k^{\pm} = -\frac{N-2s}{2} \pm \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_k(a)}$. From (1.2), (5.37), and Lemma 5.11, we deduce that, for all $i = j_0, \dots, j_0+M-1$,

$$\zeta_i(\tau) = O(\tau^{-2+\chi+\sigma_i^+}) \quad \text{as } \tau \to 0^+. \tag{5.40}$$

Consequently, the functions $t \mapsto t^{-\sigma_i^+ + 1}\zeta_i(t)$, $t \mapsto t^{-\sigma_i^- + 1}\zeta_i(t)$ belong to $L^1(0,R)$. Hence

$$\tau^{\sigma_i^+}\left(c_1^i + \int_\tau^R \frac{\rho^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(\rho)\,d\rho\right) = o(\tau^{\sigma_i^-}) \quad \text{as } \tau \to 0^+,$$

and then, by (5.37), there must be

$$c_2^i = -\int_0^R \frac{t^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+}\,\zeta_i(t)\,dt.$$

Using (5.40), we then deduce that

$$\tau^{\sigma_i^-}\left(c_2^i + \int_\tau^R \frac{t^{-\sigma_i^- + 1}}{\sigma_i^- - \sigma_i^+}\zeta_i(t)\,dt\right) = \tau^{\sigma_i^-}\left(\int_0^\tau \frac{t^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt\right) = O(\tau^{\sigma_i^+ + \chi}) \tag{5.41}$$

as $\tau \to 0^+$. Combining (5.39) and (5.41), we obtain that, for all $i = j_0, \dots, j_0+M-1$,

$$\varphi_i(\tau) = \tau^{\sigma_i^+}\left(c_1^i + \int_\tau^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt + O(\tau^{\chi})\right) \quad \text{as } \tau \to 0^+. \tag{5.42}$$

Let us assume by contradiction that $\lim_{\lambda\to 0^+}\lambda^{-2\gamma}H(\lambda)=0$. Then, (5.36) would imply that $\lim_{\tau\to 0^+}\tau^{-\sigma_i^+}\varphi_i(\tau)=0$ for all $i\in\{j_0,\dots,j_0+M-1\}$. Hence, in view of (5.42),

$$c_1^i+\int_0^R\frac{t^{-\sigma_i^+ +1}}{\sigma_i^+-\sigma_i^-}\zeta_i(t)\,dt=0,$$

which, together with (5.40), implies

$$\tau^{\sigma_i^+}\left(c_1^i+\int_\tau^R\frac{t^{-\sigma_i^+ +1}}{\sigma_i^+-\sigma_i^-}\zeta_i(t)\,dt\right)=\tau^{\sigma_i^+}\int_0^\tau\frac{t^{-\sigma_i^+ +1}}{\sigma_i^- -\sigma_i^+}\zeta_i(t)\,dt=O(\tau^{\sigma_i^+ +\chi}) \tag{5.43}$$

as $\tau\to 0^+$. Collecting (5.39), (5.41), and (5.43), we conclude that $\varphi_i(\tau)=O(\tau^{\sigma_i^+ +\chi})$ as $\tau\to 0^+$ for every $i\in\{j_0,\dots,j_0+M-1\}$, namely,

$$\sqrt{H(\tau)}\,(w^\tau,\psi)_{L^2(\mathbb{S}^N_+;\theta_1^{1-2s})}=O(\tau^{\gamma+\chi})\quad\text{as }\tau\to 0^+$$

for every $\psi\in\mathcal{A}_0=\operatorname{span}\{\psi_i\}_{i=j_0}^{j_0+M-1}$, where $\mathcal{A}_0$ is the eigenspace of problem (1.7) associated to the eigenvalue $\mu_{j_0}(a)=\mu_{j_0+1}(a)=\cdots=\mu_{j_0+M-1}(a)$. From Lemma 5.8 part (iv), there exists $C(\chi)>0$ such that $\sqrt{H(\tau)}\ge C(\chi)\tau^{\gamma+\frac{\chi}{2}}$ for τ small, and therefore

$$(w^\tau,\psi)_{L^2(\mathbb{S}^N_+;\theta_1^{1-2s})}=O(\tau^{\frac{\chi}{2}})\quad\text{as }\tau\to 0^+ \tag{5.44}$$

for every $\psi\in\mathcal{A}_0$. From Lemma 5.9, for every sequence $\tau_n\to 0^+$, there exist a subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction $\widetilde{\psi}\in\mathcal{A}_0$

$$\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\widetilde{\psi}^2(\theta)dS=1\quad\text{and}\quad w^{\tau_{n_k}}\to\widetilde{\psi}\quad\text{in }L^2(\mathbb{S}^N_+;\theta_1^{1-2s}). \tag{5.45}$$

From (5.44) and (5.45), we infer that

$$0=\lim_{k\to+\infty}(w^{\tau_{n_k}},\widetilde{\psi})_{L^2(\mathbb{S}^N_+;\theta_1^{1-2s})}=\|\widetilde{\psi}\|^2_{L^2(\mathbb{S}^N_+;\theta_1^{1-2s})}=1,$$

thus reaching a contradiction. □

5.2. **Proof of Theorem 5.1.** Identity (5.4) follows from part (i) of Lemma 5.9, thus there exists $k_0\in\mathbb{N}$, $k_0\ge 1$, such that $\gamma=\lim_{r\to 0^+}\mathcal{N}(r)=-\frac{N-2s}{2}+\sqrt{\left(\frac{N-2s}{2}\right)^2+\mu_{k_0}(a)}$. Let us denote as M the multiplicity of $\mu_{j_0}(a)$ so that, for some $j_0\in\mathbb{N}$, $j_0\ge 1$, $j_0\le k_0\le j_0+M-1$, $\mu_{j_0}(a)=\mu_{j_0+1}(a)=\cdots=\mu_{j_0+M-1}(a)$ and let $\{\psi_i\}_{i=j_0}^{j_0+m-1}$ be an $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$-orthonormal basis for the eigenspace associated to $\mu_{k_0}(a)$.

Let $\{\tau_n\}_{n\in\mathbb{N}}\subset(0,+\infty)$ such that $\lim_{n\to+\infty}\tau_n=0$. Then, by Lemma 5.9 part (ii) and Lemmas 5.10 and 5.12, there exist a subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ and M real numbers $\beta_{j_0},\dots,\beta_{j_0+M-1}\in\mathbb{R}$ such that $(\beta_{j_0},\beta_{j_0+1},\dots,\beta_{j_0+M-1})\neq(0,0,\dots,0)$ and

$$\tau_{n_k}^{-\gamma}w(\tau_{n_k}\theta)\to\sum_{i=j_0}^{j_0+M-1}\beta_i\psi_i(\theta)\quad\text{in }C^{0,\alpha}(\mathbb{S}^N_+)\quad\text{as }k\to+\infty, \tag{5.46}$$

$$\tau_{n_k}^{-\gamma}w(0,\tau_{n_k}\theta')\to\sum_{i=j_0}^{j_0+M-1}\beta_i\psi_i(0,\theta')\quad\text{in }C^{1,\alpha}(\mathbb{S}^{N-1})\quad\text{as }k\to+\infty, \tag{5.47}$$

for some $\alpha\in(0,1)$. We now prove that the β_i's depend neither on the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$.

Defining φ_i and ζ_i as in (5.37) and (5.38), from (5.46) it follows that, for any $i=j_0,\dots,j_0+M-1$,

$$\tau_{n_k}^{-\gamma}\varphi_i(\tau_{n_k})\to\beta_i \tag{5.48}$$

as $k \to +\infty$. As deduced in the proof of Lemma 5.12, for any $i = j_0, \dots, j_0 + M - 1$ and $\tau \in (0, R]$ there holds

$$\begin{aligned}\varphi_i(\tau) &= \tau^{\sigma_i^+}\left(c_1^i + \int_\tau^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt\right) + \tau^{\sigma_i^-}\left(\int_0^\tau \frac{t^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt\right) \\ &= \tau^{\sigma_i^+}\left(c_1^i + \int_\tau^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt + O(\tau^\chi)\right) \quad \text{as } \tau \to 0^+,\end{aligned} \tag{5.49}$$

for some $c_1^i \in \mathbb{R}$. Choosing $\tau = R$ in the first line of (5.49), we obtain

$$c_1^i = R^{-\sigma_i^+}\varphi_i(R) - R^{\sigma_i^- - \sigma_i^+}\int_0^R \frac{s^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(s)\,ds.$$

Hence (5.49) yields

$$\tau^{-\gamma}\varphi_i(\tau) \to R^{-\sigma_i^+}\varphi_i(R) - R^{\sigma_i^- - \sigma_i^+}\int_0^R \frac{t^{-\sigma_i^- + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt + \int_0^R \frac{t^{-\sigma_i^+ + 1}}{\sigma_i^+ - \sigma_i^-}\zeta_i(t)\,dt$$

as $\tau \to 0^+$, and therefore from (5.48) we deduce that (5.5) holds; in particular the β_i's depend neither on the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$, thus implying that the convergences in (5.46) and (5.47) actually hold as $\tau \to 0^+$ and proving the theorem. □

We are now in position to prove Theorem 1.1 and its corollaries.

Proof of Theorem 1.1. Let $u \in H^s(\mathbb{R}^N)$ be a nontrivial weak solution to $Hu = 0$ in Ω. By Theorems 6.1 and 7.1 in the appendices there exists a unique $w = \mathcal{H}(u) \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ weakly solving

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla w) + m^2 t^{1-2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ w = u, & \text{on } \partial\mathbb{R}^{N+1}_+ = \{0\} \times \mathbb{R}^N, \end{cases}$$

which also satisfies

$$-\lim_{t\to 0^+} t^{1-2s}\frac{\partial w}{\partial t}(x) = \kappa_s(-\Delta + m^2)^s u(x), \quad \text{in } H^{-s}(\mathbb{R}^N)$$

in a weak sense. Therefore w solves (5.1) in the sense of (4.9). Then Theorem 1.1 follows from Theorem 5.1. □

Proof of Corollary 1.2. It follows as a particular case of Theorem 1.1 in the case $a \equiv 0$. □

Proof of Theorem 1.3. It follows from Theorem 1.1, observing that if, by contradiction, $u \not\equiv 0$, then convergences stated Theorem 1.1 would hold, thus contradicting that $u(x) = o(|x|^n)$ as $|x| \to 0$ if $n > -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_{k_0}(a)}$. □

Proof of Theorem 1.4. The proof follows from Corollary 1.2 arguing as in the proof of [9, Theorem 1.4]. □

6. Appendix A: Extension theorem

Let $s \in (0,1)$ and $N \in \mathbb{N}^*$. Throughout this section $\mathbb{R}^{N+1}_+ := \{z = (t,x) : t > 0, x \in \mathbb{R}^N\}$. Let $P(D) = P(D_x)$ be a pseudo-differential operator with constant coefficients and Fourier transform (symbol) $P(\xi) \geq 0$ with order $\ell \in \mathbb{R}$. We mean $|P(\xi)| \leq C(1 + |\xi|)^\ell$, for some positive constant C. For every $s \in (0,1)$, define the s-power of $P(D)$ as

$$\widehat{P(D)^s} u(\xi) = P(\xi)^s \widehat{u}(\xi).$$

Assume that the bilinear form

$$(u,v) \mapsto \int_{\mathbb{R}^N} (P(\xi))^s \widehat{u}\overline{\widehat{v}}\,d\xi = \int_{\mathbb{R}^N} u(P(D))^s v\,dx$$

defines a scalar product in $C_c^\infty(\mathbb{R}^N)$ for every $s \in (0,1]$. Let $\dot{H}^s_D(\mathbb{R}^N)$ be the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the above scalar product. Next we define the space $\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ to be the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s} wP(D)w\, dxdt + \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left(\frac{\partial w}{\partial t}\right)^2 dxdt \right)^{1/2}.$$

Scalar products in the above spaces are denoted as $\langle \cdot, \cdot \rangle_{\dot{H}^s_D(\mathbb{R}^N)}$ and $\langle \cdot, \cdot \rangle_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})}$.
Under the above setting and assumptions, the following result holds.

Theorem 6.1. *Let $s \in (0,1)$ and $u \in \dot{H}^s_D(\mathbb{R}^N)$. Then there exists a unique $w \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ solution to the problem*

$$\begin{cases} t^{1-2s} P(D)w - (t^{1-2s} w_t)_t = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ w = u, & \text{on } \mathbb{R}^N, \end{cases} \tag{6.1}$$

where the subscript t means derivatives with respect to t. In addition

$$-\lim_{t\to 0} t^{1-2s} \frac{\partial w}{\partial t} = \kappa_s\, (P(D))^s u \quad \text{in } \dot{H}^{-s}_D(\mathbb{R}^N), \tag{6.2}$$

in the sense that: for any $\Psi \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$

$$\langle w, \Psi \rangle_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} = \kappa_s \langle u, \Psi \rangle_{\dot{H}^s_D(\mathbb{R}^N)}.$$

Here $\dot{H}^{-s}_D(\mathbb{R}^N)$ denotes the dual of $\dot{H}^s_D(\mathbb{R}^N)$ while

$$\kappa_s = 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)} \tag{6.3}$$

and Γ is the usual Gamma function.

Extension theorems found useful applications in the study of fractional partial differential equations. For $P(D) = -\Delta$, see [4]. We also quote [23] with $P(D)$ a second order differential operator with possibly non constant coefficients, see also [5]. A main point in our result is that the function space is explicitly given.

6.1. **Proof of Theorem 6.1.** We start with some preliminaries. For any $v \in C_c^\infty(\mathbb{R}^N)$, we define $\mathcal{H}(v)$ via its Fourier transform with respect to the variable x as $\widehat{\mathcal{H}(v)}(t,\xi) = \widehat{v}(\xi)\vartheta(\sqrt{P(\xi)}t)$, where $\vartheta \in H^1(\mathbb{R}_+; t^{1-2s})$ solves the ordinary differential equation:

$$\begin{cases} \vartheta'' + \frac{(1-2s)}{t}\vartheta' - \vartheta = 0, \\ \vartheta(0) = 1. \end{cases} \tag{6.4}$$

We note that ϑ is a given by a Bessel function:

$$\vartheta(r) = \frac{2}{\Gamma(s)} \left(\frac{r}{2}\right)^s K_s(r), \tag{6.5}$$

where, K_ν denotes the modified Bessel function of the second kind with order ν. It solves the equation

$$r^2 K_\nu'' + rK_\nu' - (r^2 + \nu^2) K_\nu = 0. \tag{6.6}$$

We have, see [8], for $\nu > 0$,

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2} \left(\frac{r}{2}\right)^{-\nu} \tag{6.7}$$

as $r \to 0$ and $K_{-\nu} = K_\nu$ for $\nu < 0$, while

$$K_\nu(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-1/2} e^{-r} \tag{6.8}$$

as $r \to +\infty$. By using the identity

$$K_\nu'(r) = -\frac{\nu}{r} K_\nu - K_{\nu-1},$$

we get

$$\kappa_s = \int_0^\infty t^{1-2s}(|\vartheta'(t)|^2 + |\vartheta(t)|^2)\,dt = -\lim_{t\to 0} t^{1-2s}\vartheta'(t) = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{6.9}$$

Since $v \in C_c^\infty(\mathbb{R}^N)$, $\widehat{v}$ decays faster than any polynomial. Then $\mathcal{H}(v) \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ and in addition it satisfies the equation

$$t^{1-2s}P(D)\mathcal{H}(v) - (t^{1-2s}\mathcal{H}(v)_t)_t = 0 \quad \text{in } \mathbb{R}^{N+1}_+. \tag{6.10}$$

We start by showing that $P(D)$ satisfies the *trace property* that any $w \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ has a trace which belongs to $\dot{H}^s_D(\mathbb{R}^N)$.

Proposition 6.2. *There exists a (unique) linear trace operator*

$$T : \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \dot{H}^s_D(\mathbb{R}^N)$$

such that $T(w)(t,x) = w(0,x)$ *for any* $w \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ *and moreover*

$$\kappa_s \|T(w)\|^2_{\dot{H}^s_D(\mathbb{R}^N)} \le \|w\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} \quad \text{for all } w \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s}), \tag{6.11}$$

where κ_s *is given by* (6.9)*. Equality holds in* (6.11) *for some function* $w \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ *if and only if* $t^{1-2s}P(D)w - (t^{1-2s}w_t)_t = 0$ *in* $\mathbb{R}^{N+1}_+$.

Proof. Let $v \in C_c^\infty(\mathbb{R}^N)$. By (6.10), we have that any $w \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ such that $w(0,\cdot) = v$ on $\mathbb{R}^N$ satisfies

$$\|\mathcal{H}(v)\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} \le \|w\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})}. \tag{6.12}$$

Thanks to Parseval identity, we have

$$\begin{aligned}
\|\mathcal{H}(v)\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} &= \int_{\mathbb{R}^{N+1}_+} t^{1-2s}P(\xi)\widehat{\mathcal{H}(v)}^2\,d\xi dt + \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\left(\frac{\partial\widehat{\mathcal{H}(v)}}{\partial t}\right)^2 d\xi dt \\
&= \int_{\mathbb{R}^{N+1}_+} t^{1-2s}P(\xi)\widehat{v}^2(\xi)|\vartheta(\sqrt{P(\xi)}t)|^2 d\xi dt + \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\widehat{v}^2(\xi)\left(\frac{\partial}{\partial t}\vartheta(\sqrt{P(\xi)}t)\right)^2 d\xi dt \\
&= \int_{\mathbb{R}^{N+1}_+} t^{1-2s}P(\xi)\widehat{v}^2(\xi)|\vartheta(\sqrt{P(\xi)}t)|^2 d\xi dt + \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\widehat{v}^2(\xi)P(\xi)|\vartheta'(\sqrt{P(\xi)}t)|^2\,d\xi dt \\
&= \left(\int_{\mathbb{R}^N}(P(\xi))^s\widehat{v}^2(\xi)\,d\xi\right)\left(\int_0^\infty t^{1-2s}(|\vartheta'(t)|^2 + |\vartheta(t)|^2)dt\right).
\end{aligned}$$

We conclude that, for any $v \in C_c^\infty(\mathbb{R}^N)$,

$$\|\mathcal{H}(v)\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} = \kappa_s\|v\|^2_{\dot{H}^s_D(\mathbb{R}^N)}. \tag{6.13}$$

This with (6.12) implies that

$$\kappa_s\|w(0,\cdot)\|^2_{\dot{H}^s_D(\mathbb{R}^N)} \le \|w\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} \quad \text{for all } w \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+}).$$

The operator T is now defined as the unique extension of the operator $w \mapsto w(0,\cdot)$.

□

For sake of simplicity, in this paper, we have denoted the trace of a function $w \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ with the same letter w.

6.1.1. *Proof of Theorem 6.1.* We first consider $u \in C_c^\infty(\mathbb{R}^N)$. In this case $w = \mathcal{H}(u)$ and it is, of course, unique in $\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$.

Now we observe that $-\lim_{t\to 0} t^{1-2s}\frac{\partial\widehat{\mathcal{H}(u)}(t,\xi)}{\partial t} = \kappa_s(P(\xi))^s\widehat{u}(\xi)$ so that

$$-\lim_{t\to 0} t^{1-2s}\frac{\partial\mathcal{H}(u)}{\partial t} = \kappa_s(P(D))^s u \quad \text{in } \dot{H}^{-s}_D(\mathbb{R}^N).$$

By (6.10) and Proposition 6.2, we deduce, after integration by parts, that

$$\langle\mathcal{H}(u), \Psi\rangle_{H^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} = \kappa_s\langle u, \Psi\rangle_{\dot{H}^s_D(\mathbb{R}^N)} \tag{6.14}$$

for any $\Psi \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$, and this proves the theorem in this case.

For the general case $u \in \dot{H}^s_D(\mathbb{R}^N)$, there exists a sequence $u_n \in C^\infty_c(\mathbb{R}^N)$ such that $u_n \to u$ in $\dot{H}^s_D(\mathbb{R}^N)$. It turns out that $\mathcal{H}(u_n) \rightharpoonup \widetilde{w}$ in $\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $Tr(\mathcal{H}(u_n)) \rightharpoonup Tr(\widetilde{w}) = u$ in $\dot{H}^s_D(\mathbb{R}^N)$. In particular for every $\psi \in C^\infty_c(\mathbb{R}^{N+1}_+)$

$$\langle \widetilde{w}, \psi \rangle_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} = 0.$$

This implies that $\widetilde{w} = w$ and it is unique in $\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$. By (6.14)

$$\langle \mathcal{H}(u_n), \Psi \rangle_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} = \kappa_s \langle u_n, \Psi \rangle_{\dot{H}^s_D(\mathbb{R}^N)}$$

for any $\Psi \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$. Taking the limit as $n \to \infty$, we get the desired result. □

Remark 6.3. *We note that the trace operator T defined in Proposition 6.2 is surjective. To see that, we argue by density. Let $v \in \dot{H}^s_D(\mathbb{R}^N)$. There exists a sequence $v_n \in C^\infty_c(\mathbb{R}^N)$ such that $v_n \to v$ in $\dot{H}^s_D(\mathbb{R}^N)$. By (6.13) $\mathcal{H}(v_n)$ is bounded and thus converges (up to subsequences) weakly to some function $w \in \dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})$ and Theorem 6.1 implies that the convergence is strong and thus*

$$\|w\|^2_{\dot{H}^1_D(\mathbb{R}^{N+1}_+; t^{1-2s})} = \kappa_s \|T(w)\|^2_{\dot{H}^s_D(\mathbb{R}^N)} = \kappa_s \|v\|^2_{\dot{H}^s_D(\mathbb{R}^N)}.$$

7. Appendix B: The relativistic Schrödinger operator $(-\Delta + m^2)^s$

Given $m > 0$, letting $P(D) = -\Delta + m^2$, we have $H^1_D(\mathbb{R}^{N+1}_+; t^{1-2s}) = H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $H^s_D(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. Applying Theorem 6.1, we have the following result.

Theorem 7.1. *Let $u \in H^s(\mathbb{R}^N)$ and let $w \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ be the unique solution to the problem*

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w) + m^2 t^{1-2s} w = 0, & \textit{in } \mathbb{R}^{N+1}_+, \\ w = u, & \textit{on } \mathbb{R}^N. \end{cases} \tag{7.1}$$

Then

$$-\lim_{t\to 0} t^{1-2s}\frac{\partial w}{\partial t} = \kappa_s(-\Delta + m^2)^s u \quad \textit{in } H^{-s}(\mathbb{R}^N).$$

7.1. **Bessel Kernel.** We can observe that the Bessel kernel $P_m(t,x)$ is given by the Fourier transform of the mapping $\xi \mapsto \vartheta(\sqrt{|\xi|^2 + m^2}\, t)$, where ϑ is the Bessel function solving the differential equation (6.4) and yet we can determine it explicitly.

Let U satisfy

$$-\mathrm{div}(t^{1-2s}\nabla U) + m^2 t^{1-2s} U = 0, \quad \text{in } \mathbb{R}^{N+1}_+.$$

We have that $V = t^{1-2s}\frac{\partial U}{\partial t}$ solves the conjugate problem:

$$-\mathrm{div}(t^{-1+2s}\nabla V) + m^2 t^{-1+2s} V = 0, \quad \text{in } \mathbb{R}^{N+1}_+.$$

We look for F (the fundamental solution) which satisfies

$$-\mathrm{div}(|t|^{-1+2s}\nabla F) + m^2 |t|^{-1+2s} F = \delta_0, \quad \text{in } \mathbb{R}^{N+1}.$$

By direct computations we have

$$F(z) = C_{N,s}\, m^{(N+2s-2)/2} |z|^{\frac{-2s-N+2}{2}} K_{\frac{N+2s-2}{2}}(m|z|),$$

where $C_{N,s}$ is a normalizing constant and K_ν denotes the modified Bessel function of the second kind with order ν solving (6.6). Hence the choice of the Bessel Kernel in $\mathbb{R}^{N+1}_+$ is

$$P_m(t,x) = -t^{-1+2s}\frac{\partial F(t,x)}{\partial t}.$$

Using the identity $K'_\nu(r) = \frac{\nu}{r}K_\nu - K_{\nu+1}$, we obtain

$$P_m(z) = C'_{N,s}\, t^{2s} m^{\frac{N+2s}{2}} |z|^{-\frac{N+2s}{2}} K_{\frac{N+2s}{2}}(m|z|). \tag{7.2}$$

By using (6.7) we deduce that $C'_{N,s}$ is given by

$$C'_{N,s} = p_{N,s}\frac{2^{(N+2s)/2-1}}{\Gamma((N+2s)/2)},$$

where $p_{N,s}$ is the constant for the (normalized) Poisson Kernel with $m = 0$, see [3]. We refer to [2], [6] for some Green function estimates for relativistic killed process. We also refer to [22] for estimates of the Bessel Kernel.

We notice that, since $P_m(t, x)$ is the Fourier transform of $\xi \mapsto \vartheta(\sqrt{|\xi|^2 + m^2}t)$, we have

$$\int_{\mathbb{R}^N} P_m(t, x)dx = \vartheta(mt). \tag{7.3}$$

Now we deduce the norm from the Dirichlet form associated to $(-\Delta + m^2)^s - m^{2s}$ via the Bessel kernel P_m. For $s = 1/2$ and $N = 3$, it was determined in [19, Theorem 7.12].

Proposition 7.2. *For every $u \in H^s(\mathbb{R}^N)$, we have that*

$$\int_{\mathbb{R}^N} \left[\left|(-\Delta + m^2)^{\frac{s}{2}} u\right|^2 - m^{2s}u^2 \right] dx = \frac{c_{N,s}}{2} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x - y|)\, dxdy,$$

with

$$c_{N,s} = \frac{2^{-(N+2s)/2+1}}{\Gamma((N + 2s)/2)} \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\xi_1)}{|\xi|^{N+2s}} \right)^{-1} = \frac{2^{-(N+2s)/2+1}}{\Gamma((N + 2s)/2)} \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(2 - s)} s(1 - s). \tag{7.4}$$

Proof. We know that

$$\begin{aligned} -\int_{\mathbb{R}^N} \lim_{t\to 0} t^{1-2s} \frac{\partial\vartheta(\sqrt{|\xi|^2 + m^2}t)}{\partial t} \widehat{u}^2(\xi)\, d\xi &= \kappa_s \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s \widehat{u}^2(\xi)\, d\xi \\ &= \kappa_s \int_{\mathbb{R}^N} \left|(-\Delta + m^2)^{\frac{s}{2}} u(x)\right|^2 dx. \end{aligned}$$

Or equivalently

$$-2s \int_{\mathbb{R}^N} \lim_{t\to 0} \frac{\vartheta(\sqrt{|\xi|^2 + m^2}t) - 1}{t^{2s}} \widehat{u}^2(\xi)\, d\xi = \kappa_s \int_{\mathbb{R}^N} \left|(-\Delta + m^2)^{\frac{s}{2}} u(x)\right|^2 dx. \tag{7.5}$$

We now compute the left hand side of the above equality using the Bessel Kernel P_m. Given $t > 0$, again by Parseval identity, we have

$$\int_{\mathbb{R}^N} \frac{\vartheta(\sqrt{|\xi|^2 + m^2}t) - 1}{t^{2s}} \widehat{u}^2(\xi)\, d\xi = \frac{1}{t^{2s}} \int_{\mathbb{R}^N} \Big(u(x)P_m(t, \cdot) * u(x) - u^2(x) \Big)\, dx,$$

where $P_m(t, \cdot) * u(x) = \int_{\mathbb{R}^N} u(y)P_m(t, x-y)dy$. We normalize P_m by putting $\widetilde{P}_m(t, x) = \frac{1}{\vartheta(tm)} P_m(t, x)$ so that $\int_{\mathbb{R}^N} \widetilde{P}_m(t, x)dx = 1$. We therefore have for $t > 0$

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{\vartheta(\sqrt{|\xi|^2 + m^2}t) - 1}{t^{2s}} \widehat{u}^2(\xi)\, d\xi \\ &= \frac{\vartheta(tm)}{t^{2s}} \int_{\mathbb{R}^N} \Big(u(x)\widetilde{P}_m(t, \cdot) * u(x) - u^2(x) \Big)\, dx + \frac{1}{t^{2s}} \int_{\mathbb{R}^N} u^2(x)(\vartheta(tm) - 1)\, dx \\ &= -\frac{\vartheta(tm)}{2t^{2s}} \int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 \widetilde{P}_m(t, x - y)\, dydx + \frac{1}{t^{2s}} \int_{\mathbb{R}^N} u^2(x)(\vartheta(tm) - 1)\, dx \\ &= -\frac{1}{2t^{2s}} \int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 P_m(t, x - y)\, dydx + \frac{1}{t^{2s}} \int_{\mathbb{R}^N} u^2(x)(\vartheta(tm) - 1)\, dx. \end{aligned}$$

We conclude that for every $t > 0$

$$\int_{\mathbb{R}^N} \frac{\vartheta(\sqrt{|\xi|^2 + m^2}t) - 1}{t^{2s}} \widehat{u}^2(\xi)\, d\xi = \frac{1}{t^{2s}} \int_{\mathbb{R}^N} u^2(x)(\vartheta(tm) - 1)\, dx \tag{7.6}$$

$$- C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{(t^2 + |x - y|^2)^{\frac{N+2s}{4}}} K_{\frac{N+2s}{2}}(m(t^2 + |x - y|^2)^{1/2})\, dxdy. \tag{7.7}$$

We now have to check that we can pass to the limit as $t \to 0$ under all the above three integrals. Firstly, we observe that the function $r \mapsto \frac{\vartheta(r)-1}{r^{2s}}$ is decreasing because K_s is decreasing and thus since $u \in H^s(\mathbb{R}^N)$, we deduce from (7.5) that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \frac{\vartheta(\sqrt{|\xi|^2 + m^2}t) - 1}{t^{2s}} \widehat{u}^2(\xi)\, d\xi = -\frac{\kappa_s}{2s} \int_{\mathbb{R}^N} \left|(-\Delta + m^2)^{\frac{s}{2}} u(x)\right|^2 dx. \tag{7.8}$$

For the same reason, we have that

$$\lim_{t\to 0}\frac{1}{t^{2s}}\int_{\mathbb{R}^N} u^2(x)(\vartheta(tm)-1)\,dx = -m^{2s}\frac{\kappa_s}{2s}\int_{\mathbb{R}^N} u^2(x)\,dx. \tag{7.9}$$

Secondly, thanks to the asymptotics of K_ν, we have that there exist $r, R > 0$ such that

$$|z|^{-\nu}K_\nu(m|z|) \le \begin{cases} C|z|^{-2\nu}, & \text{for } |x-y| < r,\\ C, & \text{for } R \ge |x-y| \ge r,\\ C|z|^{-2\nu}, & \text{for } |x-y| > R,\end{cases}$$

where C is a positive constant depending only on N, s, r, R and m. Since $u \in H^s(\mathbb{R}^N)$, we can pass the limit as $t \to 0$ under the integral in (7.7). This with (7.8) and (7.9) in (7.6) yields the result. Finally, to prove (7.4) we use the precise estimate (6.7) and comparing with the Dirichlet form in the case $m = 0$, see [7].

Remark 7.3. *We first remark from the above result that for every $u \in C_c^2(\mathbb{R}^N)$*

$$(-\Delta+m^2)^s u(x) = c_{N,s} m^{\frac{N+2s}{2}} P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}}K_{\frac{N+2s}{2}}(m|x-y|)\,dy + m^{2s}u(x).$$

We observe, using similar arguments as in the the proof of Proposition 7.2, that, for

$$w(t,x) = P_m(t,\cdot) * u,$$

with $u \in C_c^2(\mathbb{R}^N)$, we have that

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla w)(t,x) + m^2 t^{1-2s} w(t,x) = 0, & \text{for all } (t,x) \in \mathbb{R}^{N+1}_+,\\ -\lim_{t\to 0} t^{1-2s}\frac{\partial w}{\partial t}(t,x) = \kappa_s(-\Delta+m^2)^s u(x), & \text{for all } x \in \mathbb{R}^N.\end{cases}$$

We now prove the following result.

Proposition 7.4. *Let $u \in C(\Omega)$ such that $\int_{\mathbb{R}^N}(1+|x|^{N+2s})^{-1}|u(x)|dx < +\infty$. Let*

$$w(t,x) = \big(P_m(t,\cdot) * u\big)(x).$$

Then

$$\lim_{t\to 0^+} w(t,x) = u(x)$$

for every $x \in \Omega$.

Proof. We recall from (7.3) that $\int_{\mathbb{R}^N} P_m(t,x)dx = \vartheta(tm)$ for all $t > 0$. Let $x_0 \in \Omega$; by continuity, for every $\varepsilon > 0$, there exist $t_\varepsilon, r_\varepsilon > 0$ such that $|u(y) - \vartheta(tm)u(x_0)| < \varepsilon$ for every $y \in B'_{r_\varepsilon}(x_0)$ and $0 < t < t_\varepsilon$. Then

$$\begin{aligned}\Big|w(t,x_0) - \vartheta(tm)u(x_0)\Big| &= \Big|\int_{\mathbb{R}^N}(u(y)-u(x_0))P_m(t,x_0-y)dy\Big|\\ &\le \int_{|y-x_0|<r_\varepsilon}|u(y)-u(x_0)|P_m(t,x_0-y)dy + \int_{|y-x_0|\ge r_\varepsilon}|u(y)-u(x_0)|P_m(t,x_0-y)dy\\ &\le \varepsilon\vartheta(tm) + \int_{|y-x_0|\ge r_\varepsilon}|u(y)-u(x_0)|P_m(t,x_0-y)dy.\end{aligned}$$

Using the fact that

$$|z|^{-\nu}K_\nu(m|z|) \le C_\nu|z|^{-2\nu},$$

we have that

$$\begin{aligned}\int_{|y-x_0|\ge r_\varepsilon} P_m(t,x_0-y)dy &\le Ct^{2s}\int_{|y-x_0|\ge r_\varepsilon}\frac{1}{(t^2+|y-x_0|^2)^{\frac{N+2s}{2}}}dy\\ &\le Ct^{2s}\int_{|y-x_0|\ge r_\varepsilon}\frac{1}{(|y-x_0|)^{N+2s}}dy.\end{aligned}$$

Hence we have

$$|w(t,x_0) - \vartheta(tm)u(x_0)| \le \varepsilon\vartheta(tm) + t^{2s}C_{r_\varepsilon}\Big[\int_{\mathbb{R}^N}\frac{|u(x)|}{1+|x|^{N+2s}}dx + |u(x_0)|\Big].$$

Sending $t \to 0^+$ and $\varepsilon \to 0$ respectively, we get the result.

□

7.2. **Harnack and local Schauder estimates for the relativistic Schrödinger equation.** Let $f \in L^1_{loc}(B_1')$. We recall that a solution (resp. subsolution, supersolution) $u \in H^s(\mathbb{R}^N)$ to the equation

$$(-\Delta + m^2)^s u = (\text{resp. } \leq, \geq)\ f \quad \text{in } B_1' \tag{7.10}$$

satisfies

$$\int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s \widehat{u}\overline{\widehat{\varphi}} d\xi = (\text{resp. } \leq, \geq) \int_{B_1'} f\varphi dx \quad \text{for all } \varphi \in H^s(\mathbb{R}^N),$$

or, equivalently, thanks to Proposition 7.2,

$$\frac{c_{N,s}}{2} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|)\, dxdy$$
$$+ m^{2s} \int_{\mathbb{R}^N} u\varphi dx = (\text{resp. } \leq, \geq) \int_{B_1'} f\varphi dx.$$

The following regularity result holds.

Proposition 7.5. *Let $a, b \in L^p(B_1')$, for some $p > \frac{N}{2s}$.*

(1) If $u \in H^s(\mathbb{R}^N)$ satisfies $(-\Delta + m^2)^s u \leq a(x)u + b(x)$ in B_1' then $u \in L^\infty_{loc}(B_1')$.

(2) If $u \in H^s(\mathbb{R}^N)$ is nonnegative and satisfies $(-\Delta + m^2)^s u \geq a(x)u + b(x)$ in B_1', then

$$\inf_{B'_{1/2}} u + \|b\|_{L^p(B_1')} \geq C \sup_{B'_{1/2}} u.$$

(3) If $u \in H^s(\mathbb{R}^N)$ satisfies $(-\Delta + m^2)^s u = a(x)u + b(x)$ in B_1', then $u \in C^{0,\alpha}_{loc}(B_1')$ and

$$\|u\|_{C^{0,\alpha}(\overline{B'_{1/2}})} \leq C(\|u\|_{L^\infty(B'_{3/2})} + \|b\|_{L^p(B_1')}),$$

where $C > 0$ depends only on $N, s, m, \|a\|_{L^p(B_1')}$.

Proof. By Theorem 6.1, if $u \in H^s(\mathbb{R}^N)$ satisfies

$$(-\Delta + m^2)^s u = (\text{resp. } \leq, \geq)\ a(x)u + b(x) \quad \text{in } B_1'$$

then there exist a unique $w \in H^1(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that

$$\begin{cases} -\text{div}(t^{1-2s}\nabla w) + m^2 t^{1-2s} w = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ w = u, & \text{on } \mathbb{R}^N, \\ -\frac{1}{\kappa_s} \lim_{t\to 0} t^{1-2s} \frac{\partial w}{\partial t} = (\text{resp. } \leq, \geq)\ a(x)w + b(x), & \text{on } B_1', \end{cases}$$

weakly. The result then follows from Propositions 3.3, 3.4 and 3.5. □

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