# HARDY-SOBOLEV INEQUALITY WITH SINGULARITY A CURVE

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Abstract. We consider a bounded domain $\Omega$ of $\mathbb{R}^N$, $N \geq 3$, and $h$ a continuous function on $\Omega$. Let $\Gamma$ be a closed curve contained in $\Omega$. We study existence of positive solutions $u \in H^1_0(\Omega)$ to the equation

$$-\Delta u + hu = \rho_\Gamma^{-\sigma} u^{2^*_\sigma - 1} \qquad \text{in } \Omega$$

where $2^*_\sigma := \frac{2(N-\sigma)}{N-2}$, $\sigma \in (0, 2)$, and $\rho_\Gamma$ is the distance function to $\Gamma$. For $N \geq 4$, we find a sufficient condition, given by the local geometry of the curve, for the existence of a ground-state solution. In the case $N = 3$, we obtain existence of ground-state solution provided the trace of the regular part of the Green of $-\Delta + h$ is positive at a point of the curve.

## 1. Introduction

For $N \geq 3$, $0 \leq k \leq N-1$ and $\sigma \in [0, 2)$, we consider the Hardy-Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx \geq C \left( \int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} dx \right)^{2/2^*_\sigma} \qquad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.1}$$

where $x = (t, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $C = C(N, \sigma, k) > 0$ and $2^*_\sigma := \frac{2(N-\sigma)}{N-2}$. Here the Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is given by the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm $v \longmapsto \left( \int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{1/2}$. Inequality (1.1) interpolates between cylindrical Hardy inequality, which corresponds to the case $\sigma = 2$ and $k \neq N-2$, and the Sobolev inequality which is the case $\sigma = 0$. Moreover it is invariant under scaling on $\mathbb{R}^N$ and by translations in the $t$-direction. It is well known that in the case of Hardy inequality, $\sigma = 2$ and $k \neq N-2$, there is no positive constant $C$ and $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ for which equality holds in (1.1). For $\sigma \in [0, 2)$, the best positive constant $C$ in (1.1) is

$$S_{N,\sigma} := \inf \left\{ \int_{\mathbb{R}^N} |\nabla v|^2 dx, \ v \in \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} dx = 1 \right\}. \tag{1.2}$$

In the case $\sigma = 0$, $S_{N,0}$ is achieved by the standard bubble $(1 + |x|^2)^{\frac{2-N}{2}}$, which is unique up to scaling and translations, see e.g. Aubin [23] and Talenti [1]. For $k = 0$, (1.1) is a particular case of the Caffarelli-Kohn-Nirenberg inequality, see [6]. In this case, Lieb showed in [20] that the function $(1 + |x|^{2-\sigma})^{\frac{2-N}{2-\sigma}}$ achieves $S_{N,\sigma}$. When $k = N-1$, Musina proved in [21] that the support of the minimizer is contained in a half-space. Therefore (1.1) becomes the Hardy-Sobolev inequality with singularity all the boundary of the halfspace.

For $1 \leq k \leq N-2$ and $\sigma \in (0, 2)$, Badiale and Tarentello proved the existence of a minimizer $w$ for (1.2) in their paper [3], where they were motivated by questions from astrophysics. Moreover Mancini, Fabri and Sandeep shwoed decay and symmetry properties of $w$ in [10]. In particular, they prove that $w(t, z) = \theta(|t|, |z|)$, for some positive function $\theta$. An interesting classification result was also derived in [10] when $\sigma = 1$, that every minimizer is of the form $((1 + |z|)^2 + |t|^2)^{\frac{2-N}{2}}$, up to scaling in $\mathbb{R}^N$ and translations in the $t$-direction.

Since in this paper we are interested with Hardy-Sobolev inequality with weight singular at a given curve, our asymptotic energy level is given by $S_{N,\sigma}$ with $k = 1$ and $\sigma \in (0, 2)$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $h$ a continuous function on $\Omega$. Let $\Gamma \subset \Omega$ be a smooth closed curve. In this paper, we are concerned with the existence of minimizers for the infinimum

$$\mu_h(\Omega, \Gamma) := \inf_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Omega h u^2 dx}{\left(\displaystyle\int_\Omega \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dx\right)^{\frac{2}{2^*_\sigma}}}, \tag{1.3}$$

where $\sigma \in [0,2]$, $2^*_\sigma := \dfrac{2(N-\sigma)}{N-2}$ and $\rho_\Gamma(x) := \mathrm{dist}(x, \Gamma)$. Here and in the following, we assume that $-\Delta + h$ defines a coercive bilinear form on $H_0^1(\Omega)$ —which is a necessary condition for the existence of minimmizer for $\mu_h(\Omega, \Gamma)$. We are interested with the effect of the geometry and/or the location of the curve $\Gamma$ on the existence of minimmizer for $\mu_h(\Omega, \Gamma)$.
We not that for $\sigma = 0$, (1.3) reduces to the famous Brezis-Nirenberg problem [4]. In this case, for $N \geq 4$ it is enough that $h(y_0) < 0$ to get a minimizer, whereas for $N = 3$, the problem is no more local and existence of minimizers is guaranteed by the positiveness of a certain mass —the trace of the regular part of the Green function of the operator $-\Delta + h$ with zero Dirichlet data, see Druet [9]. For $\sigma = 2$, the problem reduces to a linear eigenvalue problem with Hardy potential, existence and nonexistence results were obtained by the second author in [25].
Here, we deal with the case $\sigma \in (0, 2)$. Our results exhibit similar local/global phenomenon as in [4] and [9], with the additional property that for $N \geq 4$, the curvature of the curve at a point $y_0$ tells how much $h(y_0)$ should be negative, while positive mass at a point $y_0 \in \Gamma$ is enough in dimension $N = 3$.

Our first main result is the following

**Theorem 1.1.** *Let $N \geq 4$, $\sigma \in (0, 2)$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Consider $\Gamma$ a smooth closed curve contained in $\Omega$. Let $h$ be a continuous function such that the linear operator $-\Delta + h$ is coercive. Then there exists a positive constant $C_{N,\sigma}$, only depending on $N$ and $\sigma$ with the property that if there exists $y_0 \in \Gamma$ such that*

$$h(y_0) < -C_{N,\sigma} |\kappa(y_0)|^2 \tag{1.4}$$

*then $\mu_h(\Omega, \Gamma) < S_{N,\sigma}$, and $\mu_h(\Omega, \Gamma)$ is achieved by a positive function. Here $\kappa : \Gamma \to \mathbb{R}^N$ is the curvature vector of $\Gamma$.*

Inequality (1.4) in Theorem 1.1 shows that the *sign* of the directional curvatures of $\Gamma$ is not important but the *size* of the curvature $\kappa$ at a point is.

For the explicit value of $C_{N,\sigma}$ appearing in (1.4), we refer the reader to Proposition 4.2 below. It is given by weighted integrals involving partial derivatives of $w$, a minimizer for $S_{N,\sigma}$. In the case $N = 4$, we have $C_{N,\sigma} = \frac{3}{2}$.

We now give a consequence of Theorem 1.4 in the case where $h \equiv \lambda$ a constant function. We denote by $\lambda_1(\Omega) > 0$ the first Dirichlet eigenvalue of $-\Delta$ in $\Omega$. It is easy to see that $-\Delta + \lambda$ is coercive for every $\lambda > -\lambda_1(\Omega)$. In our next result, we will consider a curve $\Gamma$ with curvature vanishing at a point. This is (trivially) the case when $\Gamma$ contains a segment.

**Corollary 1.2.** *Let $N \geq 4$, $\sigma \in (0, 2)$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Consider $\Gamma$ a smooth closed curve contained in $\Omega$. Suppose that the curvature $\kappa$ of $\Gamma$ vanishes at a point. Then for every $\lambda \in (-\lambda_1(\Omega), 0)$, we have $\mu_\lambda(\Omega, \Gamma) < S_{N,\sigma}$, and $\mu_\lambda(\Omega, \Gamma)$ is achieved by a positive function.*

We observe that if $\Gamma = S^1_R$ a circle of radius $R > 0$ and $h \equiv \lambda \in \mathbb{R}$ then condition (1.4) translates into

$$\lambda < -\frac{C_{N,\sigma}}{R^2}.$$

Therefore, provided $-\lambda_1(\Omega) < -\frac{C_{N,\sigma}}{R^2}$, we have that $\mu_\lambda(\Omega, S^1_R)$ is achieved for every $\lambda \in (-\lambda_1(\Omega), -\frac{C_{N,\sigma}}{R^2})$. One is thus led to find domains for which $-\lambda_1(\Omega) < -\frac{C_{N,\sigma}}{R^2}$. A particular example is given by the annulus $\Omega_\varepsilon = B_{R+\varepsilon} \setminus B_{R-\varepsilon}$, which contains $S^1_R$ for $\varepsilon > 0$. It is well known from e.g. the Faber-Krahn inequality that $\lambda_1(\Omega_\varepsilon) \geq \frac{c(N)}{\varepsilon^2}$, so that for sufficiently small $\varepsilon$, one always has $-\lambda_1(\Omega_\varepsilon) < -\frac{C_{N,\sigma}}{R^2}$.

We now turn to the 3-dimensional case. We let $G(x, y)$ be the Dirichlet Green function of the operator $-\Delta + h$, with zero Dirichlet data. It satisfies

$$\begin{cases} -\Delta_x G(x,y) + h(x)G(x,y) = 0 & \text{for every } x \in \Omega \setminus \{y\} \\ G(x,y) = 0 & \text{for every } x \in \partial\Omega. \end{cases} \tag{1.5}$$

In addition, for $N = 3$, there exists a continuous function $\mathbf{m} : \Omega \to \mathbb{R}$ and a positive constant $c > 0$ such that

$$G(x,y) = \frac{c}{|x-y|} + c\,\mathbf{m}(y) + o(1) \qquad \text{as } x \to y. \tag{1.6}$$

We call the function $\mathbf{m} : \Omega \to \mathbb{R}$ the *mass* of $-\Delta + h$ in $\Omega$. We note that $-\mathbf{m}$ is occasionally called the *Robin function* of $-\Delta + h$ in the literature. We now state our second main result.

**Theorem 1.3.** *Let $\sigma \in (0, 2)$ and $\Omega$ be a bounded domain of $\mathbb{R}^3$. Consider $\Gamma$ a smooth closed curve contained in $\Omega$. Let $h$ be a continuous function such that the linear operator $-\Delta + h$ is coercive. If $\boldsymbol{m}(y_0) > 0$, for some $y_0 \in \Gamma$, then $\mu_h(\Omega, \Gamma) < S_{3,\sigma}$, and $\mu_h(\Omega, \Gamma)$ is achieved by a positive function.*

Since the mass $\mathbf{m}$ is independent on the curve, Theorem 1.3 shows that the *location* of the curve in the domain $\Omega$ — so to intersect the positive part of $\mathbf{m}$— matters for the existence of solution in general. We note that there are situations in which the mass is a everywhere positive. This is the case four operator $-\Delta + \lambda$, provided $\lambda \in \left(-\lambda_1(B_1), -\frac{1}{4}\lambda_1(B_1)\right)$, as observed in Brezis-Nirenberg [4]. We therefore have the

**Corollary 1.4.** *Let $B_1$ the unit ball of $\mathbb{R}^3$ and let $\Gamma$ be any smooth closed curve contained in $B_1$. If $\lambda \in \left(-\lambda_1(B_1), -\frac{1}{4}\lambda_1(B_1)\right)$ then $\mu_\lambda(\Omega, \Gamma) < S_{3,\sigma}$, and $\mu_\lambda(\Omega, \Gamma)$ is achieved by a positive function.*

The effect of curvatures in the study of Hardy-Sobolev inequalities have been intensively studied in the recent years. For each of these works, the sign of the curvatures at the point of singularity plays important roles for the existence a solution. The first paper, to our knowledge, being the one of Ghoussoub and Kang [17] who considered the Hardy-Sobolev inequality with singularity at the boundary. For more results in this direction, see the works of Ghoussoub and Robert in [12–14, 16], Demyanov and Nazarov [8], Chern and Lin [7], Lin and Li [19], the authors and Minlend in [11] and the references there in. We point out that in the pure Hardy-Sobolev case, $\sigma \in (0, 2)$, with singularity at the boundary, one has existence of minimizers for every dimension $N \geq 3$ as long as the mean curvature of the boundary is negative at the point singularity, see [15].

The Hardy-Sobolev inequality with interior singularity on Riemannian manifolds have been studied by Jaber [18] and Thiam [24]. Here also the impact of the scalar curvature at the point singularity plays an important role for the existence of minimizers in higher dimensions $N \geq 4$. The paper [18] contains also existence result under positive mass condition for $N = 3$.

We expect that the arguments in this paper can be generalized for the case $\Gamma \subset \Omega$ a $k$-dimensional closed submanifold, with $2 \leq k \leq N - 2$. Here we believe that the norm of the second fundamental from of $\Gamma$ will play a crucial role for the existence of minimizers. An other problem of interest would be the case $\Gamma \subset \partial\Omega$ is a $k$-dimensional submanifold of $\partial\Omega$ with, $1 \leq k \leq N - 1$. In this situation, we suspect that the sign if the mean curvature of $\partial\Omega$ at a point might influence on the the existence of minimizers. Finally we note that Ghoussoub and Robert in [14] obtained several results for the case $\Gamma$ a subspace of dimension $k \geq 2$, and among other results, if $\Gamma$ intersects $\partial\Omega$ transversely, they obtain existence results under some negativity assumptions on the mean curvature.

The proof of Theorem 1.1 and Theorem 1.3 rely on test function methods. Namely to build appropriate test functions allowing to compare $\mu_h(\Omega,\Gamma)$ and $S_{N,\sigma}$. While it always holds that $\mu_h(\Omega,\Gamma)\leq S_{N,\sigma}$, our main task is to find a function for which $\mu_h(\Omega,\Gamma)<S_{N,\sigma}$. This then allows to recover compactness and thus every minimizing sequence for $\mu_h(\Omega,\Gamma)$ converges to a minimizer. Building these approximates solutions requires to have sharp decay estimates of a minimizer $w$ for $S_{N,\sigma}$, see Section 3. In Section 4, we treat the case $N=4$ in the spirit of Aubin [1]. Here we find a continuous familly of test functions $(u_\varepsilon)_{\varepsilon>0}$ concentrating at a point $y_0\in\Gamma$ which yields $\mu_h(\Omega,\Gamma)<S_{N,\sigma}$, as $\varepsilon\to 0$, provided (1.4) holds. In Section 5, we consider the case $N=3$, which is more difficult. Here we use the argument of Schoen [22] to build our test function. However we cannot adopt the method in [22] straightforwardly. In fact, in contrast to the case $N\geq 4$, we could only find a descrete family of test function $(\Psi_{\varepsilon_n})_{n\in\mathbb{N}}$ that leads to the inequality $\mu_h(\Omega,\Gamma)<S_{3,\sigma}$. This is due to the fact that the (flat) ground-state $w$ for $S_{3,\sigma}$, $\sigma\in(0,2)$, is not known explicitly, it is not radially symmetric, it is not smooth, and $S_{3,\sigma}$ is only invariant under translations in the $t-$direction. As in [22], we use some global test functions. These are similar to the test functions $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ in dimension $N\geq 4$ near the concentration point $y_0$, but away from it is substituted with the regular part of the Green function $G(x,y_0)$, which makes appear the mass $\mathbf{m}(y_0)$ in its first order Taylor expansion, see (1.6).

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## 2. Geometric Preliminaries

Let $\Gamma\subset\mathbb{R}^N$ be a smooth closed curve. Let $(E_1;\dots;E_N)$ be an orthonormal basis of $\mathbb{R}^N$. For $y_0\in\Gamma$ and $r>0$ small, we consider the curve $\gamma:(-r,r)\to\Gamma$, parameterized by arclength such that $\gamma(0)=y_0$. Up to a translation and a rotation, we may assume that $\gamma'(0)=E_1$. We choose a smooth orthonormal frame field $(E_2(t);...;E_N(t))$ on the normal bundle of $\Gamma$ such that $(\gamma'(t);E_2(t);...;E_N(t))$ is an oriented basis of $\mathbb{R}^N$ for every $t\in(-r,r)$, with $E_i(0)=E_i$.
We fix the following notation, that will be used a lot in the paper,

$$Q_r:=(-r,r)\times B_{\mathbb{R}^{N-1}}(0,r),$$

where $B_{\mathbb{R}^k}(0,r)$ denotes the ball in $\mathbb{R}^k$ with radius $r$ centered at the origin. Provided $r>0$ small, the map $F_{y_0}:Q_r\to\Omega$, given by

$$(t,z)\mapsto F_{y_0}(t,z):=\gamma(t)+\sum_{i=2}^{N}z_iE_i(t),$$

is smooth and parameterizes a neighborhood of $y_0=F_{y_0}(0,0)$. We consider $\rho_\Gamma:\Gamma\to\mathbb{R}$ the distance function to the curve given by

$$\rho_\Gamma(y)=\min_{\overline{y}\in\mathbb{R}^N}|y-\overline{y}|.$$

In the above coordinates, we have

$$\rho_\Gamma\left(F_{y_0}(x)\right)=|z|\qquad\text{for every } x=(t,z)\in Q_r. \tag{2.1}$$

Clearly, for every $t \in (-r,r)$ and $i = 2,\dots N$, there are real numbers $\kappa_i(t)$ and $\tau_i^j(t)$ such that

$$E_i'(t) = \kappa_i(t)\gamma'(t) + \sum_{j=2}^{N} \tau_i^j(t) E_j(t). \tag{2.2}$$

The quantity $\kappa_i(t)$ is the curvature in the $E_i(t)$-direction while $\tau_i^j(t)$ is the torsion from the osculating plane spanned by $\{\gamma'(t); E_j(t)\}$ in the direction $E_j$. We note that provided $r > 0$ small, $\kappa_i$ and $\tau_i^j$ are smooth functions on $(-r,r)$. Moreover, it is easy to see that

$$\tau_i^j(t) = -\tau_j^i(t) \qquad \text{for } i,j = 2,\dots,N. \tag{2.3}$$

Next, we derive the expansion of the metric induced by the parameterization $F_{y_0}$ defined above. For $x = (t,z) \in Q_r$, we define

$$g_{11}(x) = \partial_t F_{y_0}(x)\cdot\partial_t F_{y_0}(x), \qquad g_{1i}(x) = \partial_t F_{y_0}(x)\cdot\partial_{z_i} F_{y_0}(x), \qquad g_{ij}(x) = \partial_{z_j} F_{y_0}(x)\cdot\partial_{z_i} F_{y_0}(x).$$

We have the following result.

**Lemma 2.1.** *There exits $r > 0$, only depending on $\Gamma$ and $N$, such that for ever $x = (t,z) \in Q_r$*

$$\begin{cases} g_{11}(x) = 1 + 2\sum_{i=2}^{N} z_i\kappa_i(0) + 2t\sum_{i=2}^{N} z_i\kappa_i'(0) + \sum_{ij=2}^{N} z_iz_j\kappa_i(0)\kappa_j(0) + \sum_{ij=2}^{N} z_iz_j\beta_{ij}(0) + O\left(|x|^3\right) \\ g_{1i}(x) = \sum_{j=2}^{N} z_j\tau_j^i(0) + t\sum_{j=2}^{N} z_j\left(\tau_j^i\right)'(0) + O\left(|x|^3\right) \\ g_{ij}(x) = \delta_{ij}, \end{cases} \tag{2.4}$$

*where*

$$\beta_{ij}(t) := \sum_{l=2}^{N} \tau_i^l(t)\tau_j^l(t).$$

*Proof.* To alleviate the notations, we will write $F = F_{y_0}$. We have

$$\partial_t F(x) = \gamma'(t) + \sum_{j=2}^{N} z_j E_j'(t) \qquad \text{and} \qquad \partial_{z_i} F(x) = E_i(t). \tag{2.5}$$

Therefore

$$g_{ij}(x) = E_i(t)\cdot E_j(t) = \delta_{ij}. \tag{2.6}$$

By (2.2) and (2.5), we have

$$g_{1i}(x) = \sum_{l=2}^{N} z_l E_l'(t)\cdot E_i(t) = \sum_{j=2}^{N} z_j\tau_j^i(t) \tag{2.7}$$

and

$$g_{11}(x) = \partial_t F(x)\cdot\partial_t F(x) = 1 + 2\sum_{i=2}^{N} z_i\kappa_i(t) + \sum_{ij=2}^{N} z_iz_j\kappa_i(t)\kappa_j(t) + \sum_{ij=2}^{N} z_iz_j\left(\sum_{l=2}^{N}\tau_i^l(t)\tau_j^l(t)\right). \tag{2.8}$$

By Taylor expansions, we get

$$\kappa_i(t) = \kappa_i(0) + t\kappa_i'(0) + O\left(t^2\right) \qquad \text{and} \qquad \tau_i^k(t) = \tau_i^k(0) + t\left(\tau_i^k\right)'(0) + O\left(t^2\right).$$

Using these identities in (2.8) and (2.7), we get (2.4), thanks to (2.6). This ends the proof of the lemma. □

As a consequence we have the following result.

**Lemma 2.2.** *There exists $r > 0$ only depending on $\Gamma$ and $N$, such that for every $x \in Q_r$, we have*

$$\sqrt{|g|}(x) = 1 + \sum_{i=2}^{N} z_i \kappa_i(0) + t \sum_{i=2}^{N} z_i \kappa_i'(0) + \frac{1}{2} \sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O\left(|x|^3\right), \tag{2.9}$$

*where $|g|$ stands for the determinant of $g$. Moreover $g^{-1}(x)$, the matrix inverse of $g(x)$, has components given by*

$$\begin{cases} g^{11}(x) = 1 - 2\sum_{i=2}^{N} z_i \kappa_i(0) - 2t \sum_{i=2}^{N} z_i \kappa_i'(0) + 3 \sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O\left(|x|^3\right) \\ g^{i1}(x) = -\sum_{j=2}^{N} z_j \tau_j^i(0) - t \sum_{j=2}^{N} z_j \left(\tau_j^i\right)'(0) + 2 \sum_{j=2}^{N} z_l z_j \kappa_l(0) \tau_j^i(0) + O\left(|x|^3\right) \\ g^{ij}(x) = \delta_{ij} + \sum_{lm=2}^{N} z_l z_m \tau_l^j(0) \tau_m^i(0) + O\left(|x|^3\right). \end{cases} \tag{2.10}$$

*Proof.* We write

$$g(x) = id + H(x),$$

where $id$ denotes the identity matrix on $\mathbb{R}^N$ and $H$ is a symmetric matrix with components $H_{\alpha\beta}$, for $\alpha, \beta = 1, \dots, N$, given by

$$\begin{cases} H_{11}(x) = 2\sum_{i=2}^{N} z_i \kappa_i(0) + 2t \sum_{i=2}^{N} z_i \kappa_i'(0) + \sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + \sum_{ij=2}^{N} z_i z_j \beta_{ij}(0) + O\left(|x|^3\right) \\ H_{1i}(x) = \sum_{j=2}^{N} z_i \tau_j^i(0) + O\left(|x|^2\right) \\ H_{ij}(x) = 0. \end{cases} \tag{2.11}$$

We recall that as $|H| \to 0$,

$$\sqrt{|g|} = \sqrt{\det(I + H)} = 1 + \frac{\operatorname{tr} H}{2} + \frac{(\operatorname{tr} H)^2}{4} - \frac{\operatorname{tr}(H^2)}{4} + O\left(|H|^3\right). \tag{2.12}$$

Now by (2.11), as $|x| \to 0$, we have

$$\frac{\operatorname{tr} H}{2} = \sum_{i=2}^{N} z_i \kappa_i(0) + t \sum_{i=2}^{N} z_i \kappa_i'(0) + \frac{1}{2}\sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + \frac{1}{2}\sum_{ij=2}^{N} z_i z_j \beta_{ij}(0) + O\left(|x|^3\right), \tag{2.13}$$

so that

$$\frac{(\operatorname{tr} H)^2}{4} = \sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O\left(|x|^3\right). \tag{2.14}$$

Moreover, from (2.11), we deduce that

$$\operatorname{tr}(H^2)(x) = \sum_{\alpha=1}^{N} \left(H^2(x)\right)_{\alpha\alpha} = \sum_{\alpha\beta=1}^{N} H_{\alpha\beta}(x) H_{\beta\alpha}(x) = \sum_{\alpha\beta=1}^{N} H_{\alpha\beta}^2(x) = H_{11}^2(x) + 2\sum_{i=2}^{N} H_{i1}^2(x),$$

so that

$$-\frac{\operatorname{tr}(H^2)}{4} = -\sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) - \frac{1}{2}\sum_{ijl=2}^{N} z_i z_j \tau_i^l(0)\tau_j^l(0) + O\left(|x|^3\right). \tag{2.15}$$

Therefore plugging the expression from (2.13), (2.14) and (2.15) in (2.12), we get

$$\sqrt{|g|}(x) = 1 + \sum_{i=2}^{N} z_i \kappa_i(0) + t \sum_{i=2}^{N} z_i \kappa_i'(0) + \frac{1}{2} \sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O\left(|x|^3\right).$$

The proof of (2.9) is thus finished.

By Lemma 2.1 we can write

$$g(x) = id + A(x) + B(x) + O\left(|x|^3\right),$$

where $A$ and $B$ are symmetric matrix with components $(A_{\alpha\beta})$ and $(A_{\alpha\beta})$, $\alpha, \beta = 1, \dots, N$, given respectively by

$$A_{11}(x) = 2\sum_{i=2}^{N} z_i \kappa_i(0), \qquad A_{i1}(x) = \sum_{j=2}^{N} z_j \tau_j^i(0) \qquad \text{and} \qquad A_{ij}(x) = 0 \tag{2.16}$$

and

$$\begin{cases} B_{11}(x) = 2t \sum_{i=2}^{N} z_i \kappa'(0) + \sum_{i=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + \sum_{ij=2}^{N} z_i z_j \beta_{ij}(0) \\ B_{i1}(x) = t \sum_{j=2} z_j \left(\tau_j^i\right)'(0) \qquad \text{and} \qquad B_{ij}(x) = 0. \end{cases} \tag{2.17}$$

We observe that, as $|x| \to 0$,

$$g^{-1}(x) = id - A(x) - B(x) + A^2(x) + O\left(|x|^3\right).$$

We then deduce from (2.16) and (2.17) that

$$\begin{aligned} g^{11}(x) &= 1 - A_{11}(x) - B_{11}(x) + A_{11}^2(x) + \sum_{i=1}^{N} A_{1i}^2(x) + O\left(|x|^3\right) \\ &= 1 - 2\sum_{i=2}^{N} z_i \kappa_i(0) - 2t \sum_{i=2}^{N} z_i \kappa'(0) + 3\sum_{i=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + 3\sum_{ij=2}^{N} z_i z_j \beta_{ij}(0) + O\left(|x|^3\right), \end{aligned}$$

$$\begin{aligned} g^{i1}(x) &= -A_{1i}(x) - B_{1i}(x) + \sum_{\alpha=1}^{N} A_{i\alpha} A_{1\alpha} + O\left(|x|^3\right) \\ &= -A_{1i}(x) - B_{1i}(x) + A_{i1}(x)A_{11}(x) + \sum_{j=2}^{N} A_{ij}(x) A_{1j}(x) + O\left(|x|^3\right) \\ &= -\sum_{j=2}^{N} z_j \tau_j^i(0) - t\sum_{j=2} z_j \left(\tau_j^i\right)'(0) + 2\sum_{jl=2}^{N} z_l z_j \kappa_l(0) \tau_j^i(0) \end{aligned}$$

and

$$\begin{aligned} g^{ij}(x) &= \delta_{ij} - A_{ij}(x) - B_{ij}(x) + \left(A^2\right)_{ij}(x) + O\left(|x|^3\right) \\ &= \delta_{ij} - A_{ij}(x) - B_{ij}(x) + A_{1i}A_{1j} + \sum_{l=2}^{N} A_{il}(x) A_{jl}(x) + O\left(|x|^3\right) \\ &= \delta_{ij} + \sum_{lm=2}^{N} z_l z_m \tau_m^i(0) \tau_l^j(0) + O\left(|x|^3\right). \end{aligned}$$

This ends the proof. □

## 3. Some preliminary results

We consider the best constant for the cylindrical Hardy-Sobolev inequality

$$S_{N,\sigma} = \min \left\{ \int_{\mathbb{R}^N} |\nabla w|^2 dx \,:\, w \in \mathcal{D}^{1,2}(\mathbb{R}^N),\; \int_{\mathbb{R}^N} |z|^{-\sigma} |w|^{2^*_\sigma} dx = 1 \right\}.$$

As mentioned in the first section, it is attained by a positive function $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, satisfying

$$-\Delta w = S_{N,\sigma} |z|^{-\sigma} w^{2^*_\sigma - 1} \qquad \text{in } \mathbb{R}^N, \tag{3.1}$$

see e.g. [3]. Moreover from [10], we have

$$w(x) = w(t,z) = \theta\left(|t|, |z|\right) \qquad \text{for a function} \qquad \theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+. \tag{3.2}$$

Next we prove further decay properties of $w$ involving its higher derivatives. We start with the following results.

**Lemma 3.1.** *Let $\theta$ be given by* (3.2). *Then we have the following properties.*

(i) *The function $t \mapsto \theta(t,\rho)$ is of class $C^\infty$ with all its derivatives uniformly bounded with respect to $\rho$.*

(ii) *There exists a constant $C > 0$ such that for $|(t,\rho)| \le 1$, we have*

$$\theta_\rho(t,\rho) + \theta_{t\rho}(t,\rho) + \rho\theta_{\rho\rho}(t,\rho) \le C\rho^{1-\sigma}.$$

*Proof.* For the proof of $(i)$, see [10]. To prove $(ii)$, we first use polar coordinates to deduce that

$$\rho^{2-N}(\rho^{N-2}\theta_\rho)_\rho + \theta_{tt} = S_{N,\sigma}\rho^{-\sigma}\theta^{2^*_\sigma - 1} \qquad \text{for } t, \rho \in \mathbb{R}_+. \tag{3.3}$$

Integrating this identity in the $\rho$ variable, we therefore get, for every $\rho > 0$,

$$\theta_\rho(t,\rho) = \frac{-1}{\rho^{N-2}} \int_0^\rho r^{N-2}\theta_{tt}(t,r)dr + S_{N,\sigma}\frac{1}{\rho^{N-2}} \int_0^\rho r^{N-2} r^{-\sigma} \theta^{2^*_\sigma - 1}(t,r)dr.$$

Moreover, we have

$$\theta_{t\rho}(t,\rho) = \frac{-1}{\rho^{N-2}} \int_0^\rho r^{N-2}\theta_{ttt}(t,r)dr + S_{N,\sigma}\frac{1}{\rho^{N-2}} \int_0^\rho r^{N-2} r^{-\sigma} \partial_t\theta(t,r) \theta^{2^*_\sigma - 2}(t,r)dr.$$

By $(i)$ and the fact that $2^*_\sigma \ge 2$, we obtain

$$|\theta_\rho(t,\rho)| + |\theta_{t\rho}(t,\rho)| \le C\rho + C\rho^{1-\sigma} \le C\rho^{1-\sigma} \qquad \text{for } |(t,\rho)| \le 1.$$

Now using this in (3.3), we get $|\theta_{\rho\rho}| \le C\rho^{-\sigma}$, for $|(t,\rho)| \le 1$. The proof of $(ii)$ is completed. □

As a consequence we derive decay estimates of the derivatives of $w$ up to order two.

**Corollary 3.2.** *Let $w$ be a ground state for $S_{N,\sigma}$ then there exist positive constants $C_1, C_2$, only depending on $N$ and $\sigma$, such that*

(i) *For every $x \in \mathbb{R}^N$*

$$\frac{C_1}{1 + |x|^{N-2}} \le w(x) \le \frac{C_2}{1 + |x|^{N-2}}. \tag{3.4}$$

(ii) *For $|x| = |(t,z)| \le 1$*

$$|\nabla w(x)| + |x||D^2 w(x)| \le C_2 |z|^{1-\sigma}$$

(iii) *For $|x| = |(t,z)| \ge 1$*

$$|\nabla w(x)| + |x||D^2 w(x)| \le C_2 \max(1, |z|^{-\sigma})|x|^{1-N}.$$

*Proof.* For the proof of $(i)$, we refer to [10, Lemma 3.1]. The proof of $(ii)$ is an immediate consequence of Lemma 3.1(ii), recalling that $w(t,z) = \theta(|t|,|z|)$. Now $(iii)$ follows by Kelvin transform, using that the function $v : \mathbb{R}^N \to \mathbb{R}$, given by $v(t,z) = v(x) = \theta(|t||x|^{-2}, |z||x|^{-2})|x|^{2-N}$ is also a ground state for $S_{N,\sigma}$, thus it satisfies $(ii)$. □

We close this section with the following result.

**Lemma 3.3.** *Let* $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $N \geq 3$, *satisfy* $v(t,z) = \overline{\theta}(|t|,|z|)$, *for some some function* $\overline{\theta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. *Then for* $0 < r < R$, *we have*

$$\int_{Q_R \setminus Q_r} |\nabla v|_g^2 \sqrt{|g|} dx = \int_{Q_R \setminus Q_r} |\nabla v|^2 dx + \frac{|\kappa(x_0)|^2}{N-1} \int_{Q_R \setminus Q_r} |z|^2 \left|\partial_t v\right|^2 dx$$
$$+ \frac{|\kappa(x_0)|^2}{2(N-1)} \int_{Q_R \setminus Q_r} |z|^2 |\nabla v|^2 dx + O\left( \int_{Q_R \setminus Q_r} |x|^3 |\nabla v|^2 dx \right).$$

*Proof.* It is easy to see that

$$\int_{Q_R \setminus Q_r} |\nabla v|_g^2 \sqrt{|g|} dx = \int_{Q_R \setminus Q_r} |\nabla v|^2 dx + \int_{Q_R \setminus Q_r} (|\nabla v|_g^2 - |\nabla v|^2) \sqrt{|g|} dx$$
$$+ \int_{Q_R \setminus Q_r} |\nabla v|^2 (\sqrt{|g|} - 1) dx. \tag{3.5}$$

We recall that

$$|\nabla v|_g^2(x) - |\nabla v|^2(x) = \sum_{\alpha\beta=1}^{N} \left[ g^{\alpha\beta}(x) - \delta_{\alpha\beta} \right] \partial_{z_\alpha} v(x) \partial_{z_\beta} v(x).$$

It then follows that

$$\int_{Q_R \setminus Q_r} \left[ |\nabla v|_g^2 - |\nabla v|^2 \right] \sqrt{|g|} dx = \sum_{ij=2}^{N} \int_{Q_R \setminus Q_r} \left[ g^{ij} - \delta_{ij} \right] \partial_{z_i} v \partial_{z_j} v \sqrt{|g|} dx$$
$$+ \sum_{i=2}^{N} \int_{Q_R \setminus Q_r} g^{i1} \left( \partial_t v \partial_{z_i} v \right) \sqrt{|g|} dx \tag{3.6}$$
$$+ \sum_{i=2}^{N} \int_{Q_R \setminus Q_r} \left[ g^{11} - 1 \right] \left( \partial_t v \right)^2 \sqrt{|g|} dx.$$

We first use Lemma 2.2 and (2.3), to get

$$\sum_{ij=2}^{N} \int_{Q_R \setminus Q_r} \left[ g^{ij} - \delta_{ij} \right] \partial_{z_i} v \partial_{z_j} v \sqrt{|g|}\, dx$$
$$= \sum_{ij=2}^{N} \sum_{lm=2}^{N} \tau_m^i(0) \tau_l^j(0) \int_{Q_R \setminus Q_r} z_i z_j z_l z_m \frac{|\nabla_z v|^2}{|z|^2}\, dx + O\left( \int_{Q_R \setminus Q_r} |x|^3 |\nabla_z v|^2\, dx \right)$$
$$= O\left( \int_{Q_R \setminus Q_r} |x|^3 |\nabla_z w|^2 dx \right). \tag{3.7}$$

Next, we observe that

$$\sum_{i=2}^{N} \int_{Q_R \setminus Q_r} g^{i1} \left( \partial_t v \cdot \partial_i v \right) \sqrt{|g|}\, dx = \sum_{i=2}^{N} \int_{Q_R \setminus Q_r} \Upsilon(|t|,|z|) t z_i g^{i1}\, dx,$$

where $\Upsilon(|t|,|z|) = \overline{\theta}_t(|t|,|z|) \overline{\theta}_\rho(|t|,|z|) \frac{1}{|t|} \frac{1}{|z|}$. In addition, from (2.3), we see that

$$\sum_{ij=2}^{N} \tau_j^i(0) z_i z_j = \sum_{ij=2}^{N} (\tau_i^i)'(0) z_i z_j = 0.$$

Consequently, from (2.9) and (2.10), we obtain

$$\begin{aligned}
&\sum_{i=2}^{N}\int_{Q_R\setminus Q_r} g^{i1}\partial_t v\partial_{z_i} v\sqrt{|g|}\,dx=\int_{Q_R\setminus Q_r}\Upsilon(|t|,|z|)t\sum_{i=2}^{N} z_i g^{i1}\sqrt{|g|}\,dtdz\\
&=-\sum_{ij=2}^{N}\tau_j^i(0)\int_{Q_R\setminus Q_r}\Upsilon(|t|,|z|)tz_iz_j\,dtdz-\sum_{ij=2}^{N}\left(\tau_j^i\right)'(0)\int_{Q_R\setminus Q_r}\Upsilon(|t|,|z|)t^2z_iz_j\,dtdz\\
&+2\sum_{ijl=2}^{N}\kappa_l(0)\tau_i^j(0)\int_{Q_R\setminus Q_r}\Upsilon(|t|,|z|)tz_iz_jz_l\,dtdz-\sum_{ijl=2}^{N}\kappa'(0)\tau_j^i(0)\int_{Q_R\setminus Q_r}\Upsilon(|t|,|z|)z_lz_iz_jt^2\,dtdz\\
&-\sum_{ijl=2}^{N}\tau_j^i(0)\kappa_l(0)\int_{Q_R\setminus Q_r}\Upsilon(|t|,|z|)z_lz_iz_jt\,dtdz+O\left(\int_{Q_R\setminus Q_r}|x|^3|\nabla v|^2\,dx\right)\\
&=O\left(\int_{Q_R\setminus Q_r}|x|^3|\nabla v|^2\,dx\right). \qquad (3.8)
\end{aligned}$$

By (2.9) and (2.10), we have

$$\int_{Q_R\setminus Q_r}|\partial_t v|^2\left[g^{11}-1\right]\sqrt{|g|}\,dx=\frac{|\kappa(x_0)|^2}{N-1}\int_{Q_R\setminus Q_r}|z|^2\,|\partial_t v|^2\,dx+O\left(\int_{Q_R\setminus Q_r}|x|^3\,|\partial_t v|^2\,dx\right).$$

Using this, (3.7) and (3.8) in (3.6), we then deduce that

$$\int_{Q_R\setminus Q_r}\left[|\nabla v|_g^2-|\nabla v|^2\right]\sqrt{|g|}\,dx=\frac{|\kappa(x_0)|^2}{N-1}\int_{Q_R\setminus Q_r}|z|^2\,|\partial_t v|^2\,dx+O\left(\int_{Q_R\setminus Q_r}|x|^3|\nabla v|^2dx\right). \qquad (3.9)$$

Now by (2.9) and (2.10), we also have that

$$\int_{Q_R\setminus Q_r}|\nabla v|^2(\sqrt{|g|}-1)dx=\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_R\setminus Q_r}|z|^2|\nabla v|^2dx+O\left(\int_{Q_R}|x|^3|\nabla v|^2dx\right).$$

This with (3.9) and (3.5) give the desired result.

□

## 4. Existence of minimzers for $\mu_h(\Omega,\Gamma)$ in dimension $N\geq 4$

We consider $\Omega$ a bounded domain of $\mathbb{R}^N$, $N\geq 3$, and $\Gamma\subset\Omega$ be a smooth closed curve. For $u\in H_0^1(\Omega)\setminus\{0\}$, we define the ratio

$$J(u):=\frac{\displaystyle\int_\Omega|\nabla u|^2dy+\int_\Omega hu^2dy}{\left(\displaystyle\int_\Omega\rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dy\right)^{2/2^*_\sigma}}. \qquad (4.1)$$

We will construct a family of test function $(u_\varepsilon)_\varepsilon\in H_0^1(\Omega)$ and provide an expansion for $J(u_\varepsilon)$ as $\varepsilon\to 0$. We let $\eta\in\mathcal{C}_c^\infty\left(F_{y_0}\left(Q_{2r}\right)\right)$ be such that

$$0\leq\eta\leq 1\qquad\text{and}\qquad\eta\equiv 1\quad\text{in } Q_r.$$

For $\varepsilon>0$, we consider $u_\varepsilon:\Omega\to\mathbb{R}$ given by

$$u_\varepsilon(y):=\varepsilon^{\frac{2-N}{2}}\eta(F_{y_0}^{-1}(y))w\left(\frac{F_{y_0}^{-1}(y)}{\varepsilon}\right). \qquad (4.2)$$

In particular, for every $x=(t,z)\in\mathbb{R}\times\mathbb{R}^{N-1}$, we have

$$u_\varepsilon\left(F_{y_0}(x)\right):=\varepsilon^{\frac{2-N}{2}}\eta\left(x\right)\theta\left(\frac{|t|}{\varepsilon},\frac{|z|}{\varepsilon}\right). \tag{4.3}$$

It is clear that $u_\varepsilon\in H^1_0(\Omega)$. We have the following

**Lemma 4.1.** *For $J$ given by* (4.1) *and $u_\varepsilon$ given by* (4.2)*, as $\varepsilon\to 0$, we have*

$$\begin{aligned} J\left(u_\varepsilon\right)=&S_{N,\sigma}+\varepsilon^2\frac{|\kappa(x_0)|^2}{N-1}\int_{Q_{r/\varepsilon}}|z|^2\left|\partial_t w\right|^2dx+\varepsilon^2\frac{|\kappa(x_0)|^2}{2(N-1)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx\\ &-\frac{\varepsilon^2}{2^*_\sigma}\frac{|\kappa(y_0)|^2}{(N-1)}S_{N,\sigma}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx+\varepsilon^2h(y_0)\int_{Q_{r/\varepsilon}}w^2dx\\ &+O\left(\varepsilon^2\int_{Q_{r/\varepsilon}}|h(F_{y_0}(\varepsilon x))-h(y_0)|w^2dx\right)+O\left(\varepsilon^{N-2}\right). \end{aligned} \tag{4.4}$$

*Proof.* To simplify the notations, we will write $F$ in the place of $F_{y_0}$. Recalling (4.2), we write

$$u_\varepsilon(y)=\varepsilon^{\frac{2-N}{2}}\eta(F^{-1}(y))W_\varepsilon(y),$$

where $W_\varepsilon(y)=w\left(\frac{F^{-1}(y)}{\varepsilon}\right)$. Then

$$|\nabla u_\varepsilon|^2=\varepsilon^{2-N}\left(\eta^2|\nabla W_\varepsilon|^2+\eta^2|\nabla W_\varepsilon|^2+\frac{1}{2}\nabla W_\varepsilon^2\cdot\nabla\eta^2\right).$$

Integrating by parts, we have

$$\begin{aligned} \int_\Omega|\nabla u_\varepsilon|^2dy&=\varepsilon^{2-N}\int_{F(Q_{2r})}\eta^2|\nabla W_\varepsilon|^2dy+\varepsilon^{2-N}\int_{F(Q_{2r})\setminus F(Q_r)}W_\varepsilon^2\left(|\nabla\eta|^2-\frac{1}{2}\Delta\eta^2\right)dy\\ &=\varepsilon^{2-N}\int_{F(Q_{2r})}\eta^2|\nabla W_\varepsilon|^2dy-\varepsilon^{2-N}\int_{F(Q_{2r})\setminus F(Q_r)}W_\varepsilon^2\eta\Delta\eta dy\\ &=\varepsilon^{2-N}\int_{F(Q_{2r})}\eta^2|\nabla W_\varepsilon|^2dy+O\left(\varepsilon^{2-N}\int_{F(Q_{2r})\setminus F(Q_r)}W_\varepsilon^2dy\right). \end{aligned} \tag{4.5}$$

By the change of variable $y=\frac{F(x)}{\varepsilon}$ and (4.3), we can apply Lemma 3.3, to get

$$\begin{aligned} &\int_\Omega|\nabla u_\varepsilon|^2dy=\int_{Q_{r/\varepsilon}}|\nabla w|^2_{g_\varepsilon}\sqrt{|g_\varepsilon|}dx+O\left(\varepsilon^2\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}w^2dx+\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|\nabla w|^2dx\right)\\ &=\int_{\mathbb{R}^N}|\nabla w|^2dx+\varepsilon^2\frac{|\kappa(y_0)|^2}{N-1}\int_{Q_{r/\varepsilon}}|z|^2\left|\partial_t w\right|^2dx+\varepsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx\\ &+O\left(\varepsilon^3\int_{Q_{r/\varepsilon}}|x|^3|\nabla w|^2dx+\varepsilon^2\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|w|^2dx+\int_{\mathbb{R}^N\setminus Q_{r/\varepsilon}}|\nabla w|^2dx+\varepsilon^2\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx\right). \end{aligned}$$

Using Corollary 3.2, we find that

$$\int_\Omega|\nabla u_\varepsilon|^2dy=S_{N,\sigma}+\varepsilon^2\frac{|\kappa(y_0)|^2}{N-1}\int_{Q_{r/\varepsilon}}|z|^2\left|\partial_t w\right|^2dx+\varepsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx+O\left(\varepsilon^{N-2}\right).$$

By the change of variable $y=\frac{F(x)}{\varepsilon}$, (3.2), (2.1) and (2.9), we get

$$\begin{aligned}
&\int_\Omega \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma}dy = \int_{Q_{r/\varepsilon}} |z|^{-s} w^{2^*_\sigma}\sqrt{|g_\varepsilon|}dx + O\left(\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma}(\eta(\varepsilon x)w)^{2^*_\sigma}dx\right)\\
&= \int_{Q_{r/\varepsilon}} |z|^{-\sigma} w^{2^*_\sigma}dx + \varepsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx\\
&\quad + O\left(\varepsilon^3\int_{Q_{r/\varepsilon}}|x|^3|z|^{-\sigma}w^{2^*_\sigma}dx + \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^{-\sigma}w^{2^*_\sigma}dx\right)\\
&= 1+\varepsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx\\
&\quad + O\left(\varepsilon^3\int_{Q_{r/\varepsilon}}|x|^3|z|^{-\sigma}w^{2^*_\sigma}dx + \int_{\mathbb{R}^N\setminus Q_{r/\varepsilon}}|z|^{-\sigma}w^{2^*_\sigma}dx + \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^{-\sigma}w^{2^*_\sigma}dx\right).
\end{aligned}$$

Using (3.4), we have

$$\varepsilon^3\int_{Q_{r/\varepsilon}}|x|^3|z|^{-\sigma}w^{2^*_\sigma}dx + \int_{\mathbb{R}^N\setminus Q_{r/\varepsilon}}|z|^{-\sigma}w^{2^*_\sigma}dx + \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^{-\sigma}w^{2^*_\sigma}dx = O\left(\varepsilon^{N-\sigma}\right).$$

Hence by Taylor expanding, we get

$$\left(\int_\Omega \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma}dx\right)^{2/2^*_\sigma} = 1+\frac{\varepsilon^2}{2^*_\sigma}\frac{|\kappa(y_0)|^2}{(N-1)}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx + O\left(\varepsilon^{N-\sigma}\right).$$

Finally, by (4.5), we conclude that

$$\begin{aligned}
J\left(u_\varepsilon\right) =& S_{N,\sigma} + \varepsilon^2\frac{|\kappa(y_0)|^2}{N-1}\int_{Q_{r/\varepsilon}}|z|^2\left|\partial_t w\right|^2dx + \varepsilon^2\frac{|\kappa(y_0)|^2}{2(N-1)}\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx\\
&-\frac{\varepsilon^2}{2^*_\sigma}\frac{|\kappa(y_0)|^2}{(N-1)}S_{N,\sigma}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx + \varepsilon^2 h(y_0)\int_{Q_{r/\varepsilon}}w^2dx\\
&+O\left(\varepsilon^2\int_{Q_{r/\varepsilon}}|h(F_{y_0}(\varepsilon x)-h(y_0)|w^2dx\right) + O\left(\varepsilon^{N-2}\right).
\end{aligned}$$

We thus get the desired result.

□

**Proposition 4.2.** *For $N\geq 5$, we define*

$$A_{N,\sigma} := \frac{1}{N-1}\int_{\mathbb{R}^N}|z|^2\left|\partial_t w\right|^2dx + \left(\frac{1}{2}-\frac{1}{2^*_\sigma}\right)\frac{1}{N-1}\int_{\mathbb{R}^N}|z|^2|\nabla w|^2dx + \frac{1}{2^*_\sigma}\int_{\mathbb{R}^N}w^2dx > 0$$

*and*

$$B_{N,\sigma} := \int_{\mathbb{R}^N}w^2dx.$$

*Assume that, for some $y_0\in\Gamma$, there holds*

$$\begin{cases} h(y_0) < -\dfrac{A_{N,\sigma}}{B_{N,\sigma}}|\kappa(y_0)|^2 & \text{for } N\geq 5\\ h(y_0) < -\dfrac{3}{2}|\kappa(y_0)|^2 & \text{for } N=4.\end{cases}$$

*Then*

$$\mu_h\left(\Omega,\Gamma\right) < S_{N,\sigma}.$$

*Proof.* We claim that

$$S_{N,\sigma}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx=\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx-(N-1)\int_{Q_{r/\varepsilon}}w^2dx+O(\varepsilon^{N-2}). \tag{4.6}$$

To prove this claim, we let $\eta_\varepsilon(x)=\eta(\varepsilon x)$. We multiply (3.1) by $|z|^2\eta_\varepsilon w$ and integrate by parts to get

$$\begin{aligned}S_{N,\sigma}\int_{Q_{2r/\varepsilon}}\eta_\varepsilon|z|^{2-\sigma}w^{2^*_\sigma}dx&=\int_{Q_{2r/\varepsilon}}\nabla w\cdot\nabla\left(\eta_\varepsilon|z|^2w\right)dx\\&=\int_{Q_{2r/\varepsilon}}\eta_\varepsilon|z|^2|\nabla w|^2dx+\frac{1}{2}\int_{Q_{2r/\varepsilon}}\nabla w^2\cdot\nabla\left(|z|^2\eta_\varepsilon\right)dx\\&=\int_{Q_{2r/\varepsilon}}\eta_\varepsilon|z|^2|\nabla w|^2dx-\frac{1}{2}\int_{Q_{2r/\varepsilon}}w^2\Delta\left(|z|^2\eta_\varepsilon\right)dx\\&=\int_{Q_{2r/\varepsilon}}\eta_\varepsilon|z|^2|\nabla w|^2dx-(N-1)\int_{Q_{2r/\varepsilon}}w^2\eta_\varepsilon dx\\&\quad-\frac{1}{2}\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}w^2(|z|^2\Delta\eta_\varepsilon+4\nabla\eta_\varepsilon\cdot z)dx.\end{aligned}$$

We then deduce that

$$\begin{aligned}S_{N,\sigma}\int_{Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx=&\int_{Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx-(N-1)\int_{Q_{r/\varepsilon}}w^2dx\\&+O\left(\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^{2-\sigma}w^{2^*_\sigma}dx+\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx+\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}w^2dx\right)\\&+O\left(\varepsilon\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z||\nabla w|dx+\varepsilon^2\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|z|^2w^2dx\right).\end{aligned}$$

Thanks to Corollary 3.2, we get (4.6) as claimed.

Next, by the continuity of $h$, for $\delta>0$, we can find $r_\delta>0$ such that

$$|h(y)-h(y_0)|<\delta\qquad\text{for ever } y\in F\left(Q_{r_\delta}\right). \tag{4.7}$$

**Case** $N\geq 5$.
Using (4.6) and (4.7) in (4.4), we obtain, for every $r\in(0,r_\delta)$

$$\begin{aligned}J\left(u_\varepsilon\right)=&S_{N,\sigma}+\varepsilon^2\frac{|\kappa(y_0)|^2}{N-1}\int_{\mathbb{R}^N}|z|^2\left|\partial_t w\right|^2dx+\varepsilon^2\left(\frac{1}{2}-\frac{1}{2^*_\sigma}\right)\frac{|\kappa(y_0)|^2}{N-1}\int_{\mathbb{R}^N}|z|^2|\nabla w|^2dx\\&+\frac{\varepsilon^2}{2^*_\sigma}|\kappa(y_0)|^2\int_{\mathbb{R}^N}w^2dx+\varepsilon^2h(y_0)\int_{\mathbb{R}^N}w^2dx+O\left(\varepsilon^2\delta^2\int_{\mathbb{R}^N}w^2dx\right)+O\left(\varepsilon^{N-2}\right),\end{aligned}$$

where we have used Corollary 3.2 to get the estimates

$$\int_{\mathbb{R}^N\setminus Q_{r/\varepsilon}}|z|^2|\nabla w|^2dx+\int_{\mathbb{R}^N\setminus Q_{r/\varepsilon}}w^2dx=O(\varepsilon).$$

It follows that, for every $r\in(0,r_\delta)$,

$$J\left(u_\varepsilon\right)=S_{N,\sigma}+\varepsilon^2\left\{A_{N,\sigma}|\kappa(y_0)|^2+B_{N,\sigma}h(y_0)\right\}+O(\delta\varepsilon^2B_{N,\sigma})+O\left(\varepsilon^3\right).$$

Suppose now that

$$A_{N,\sigma}|\kappa(y_0)|^2+B_{N,\sigma}h(y_0)<0.$$

We can thus choose respectively $\delta>0$ small and $\varepsilon>0$ small so that $J(u_\varepsilon)<S_{N,\sigma}$. Hence we get

$$\mu_h\left(\Omega,\Gamma\right)<S_{N,\sigma}.$$

**Case** $N = 4$.
From (4.4) and (4.7), we estimate, for every $r \in (0, r_\delta)$

$$J\left(u_\varepsilon\right) \leq S_{N,\sigma} + \varepsilon^2 \frac{3|\kappa(y_0)|^2}{2(N-1)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx - \frac{\varepsilon^2}{2^*_\sigma} \frac{|\kappa(y_0)|^2}{(N-1)} S_{N,\sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2^*_\sigma} dx$$
$$+ \varepsilon^2 h(y_0) \int_{Q_{r/\varepsilon}} w^2 dx + O\left(\varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 dx\right) + O\left(\varepsilon^{N-2}\right).$$

This with (4.6) yield

$$J\left(u_\varepsilon\right) \leq S_{N,\sigma} + \varepsilon^2 \frac{3|\kappa(y_0)|^2}{2(N-1)} S_{N,\sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2^*_\sigma} dx - \frac{\varepsilon^2}{2^*_\sigma} \frac{|\kappa(y_0)|^2}{(N-1)} S_{N,\sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2^*_\sigma} dx$$
$$+ \varepsilon^2 \left\{\frac{3}{2}|\kappa(y_0)|^2 + h(y_0)\right\} \int_{Q_{r/\varepsilon}} w^2 dx + O\left(\varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 dx\right) + O\left(\varepsilon^{N-2}\right).$$

Since, by (3.4),

$$\int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2^*_\sigma} dx = O(1),$$

we therefore have

$$J\left(u_\varepsilon\right) \leq S_{4,\sigma} + \varepsilon^2 \left\{\frac{3|\kappa(y_0)|^2}{2} + h(y_0)\right\} \int_{Q_{r/\varepsilon}} w^2 dx + O\left(\varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 dx\right) + C\varepsilon^2, .$$

for some positive constant $C$ independent on $\varepsilon$. By (3.4), we have that

$$\int_{Q_{r/\varepsilon}} \frac{C_1^2}{1+|x|^2} dx \leq \int_{Q_{r/\varepsilon}} w^2 dx \leq \int_{Q_{r/\varepsilon}} \frac{C_2^2}{1+|x|^2} dx,$$

so that

$$\int_{B_{\mathbb{R}^4}(0,r/\varepsilon)} \frac{C_1^2}{\left(1+|x|^2\right)^2} dx \leq \int_{Q_{r/\varepsilon}} w^2 dx \leq \int_{B_{\mathbb{R}^4}(0,2r/\varepsilon)} \frac{C_2^2}{\left(1+|x|^2\right)^2} dx. \tag{4.8}$$

Using polar coordinates and a change of variable, for $R > 0$, we have

$$\begin{aligned} \int_{B_{\mathbb{R}^4}(0,R)} \frac{dx}{\left(1+|x|^2\right)^2} dx &= |S^3| \int_0^R \frac{t^3}{\left(1+t^2\right)^2} dt \\ &= |S^3| \int_0^{\sqrt{R}} \frac{s}{2\left(1+s\right)^2} ds \\ &= \frac{|S^3|}{2} \left(\log\left(1+\sqrt{R}\right) - \frac{\sqrt{R}}{1+\sqrt{R}}\right). \end{aligned}$$

Therefore, there exist numerical constants $c, \overline{c} > 0$ such that for every $\varepsilon > 0$ small, we have

$$c|\log\varepsilon| \leq \int_{Q_{r/\varepsilon}} w^2 dx \leq \overline{c}|\log\varepsilon|. \tag{4.9}$$

Now we assume that $\frac{3|\kappa(y_0)|^2}{2} + h(y_0) < 0$. Therefore by Lemma 4.1 and (4.9), we get

$$J\left(u_\varepsilon\right) \leq S_{4,s} + c\left\{\frac{3}{2}|\kappa(y_0)|^2 + h(y_0)\right\} \varepsilon^2 |\log\varepsilon| + \overline{c}\delta\varepsilon^2|\log\varepsilon| + C\varepsilon^2.$$

Then choosing $\delta > 0$ small and $\varepsilon$ small, respectively, we deduce that $\mu_h\left(\Omega, \Gamma\right) \leq J(u_\varepsilon) < S_{4,\sigma}$. This ends the proof of the proposition. □

*Proof of Theorem 1.1 (completed).* Thanks to Proposition 4.2, we can apply Proposition 6.2 to get the result with $C_{N,\sigma} = \frac{A_{N,\sigma}}{B_{N,\sigma}}$ for $N \geq 5$ and $C_{4,\sigma} = \frac{3}{2}$.

□

## 5. Existence of minimizer for $\mu_h(\Omega, \Gamma)$ in dimension three

We consider the function

$$\mathcal{R} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad x \mapsto \mathcal{R}(x) = \frac{1}{|x|}$$

which satisfies

$$-\Delta \mathcal{R} = 0 \qquad \text{in } \mathbb{R}^3 \setminus \{0\}. \tag{5.1}$$

We denote by $G$ the solution to the equation

$$\begin{cases} -\Delta_x G(y, \cdot) + hG(y, \cdot) = 0 & \text{in } \Omega \setminus \{y\}. \\ G(y, \cdot) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

and satisfying

$$G(x, y) = \mathcal{R}(x - y) + O(1) \qquad \text{for } x, y \in \Omega \text{ and } x \neq y. \tag{5.3}$$

We note that $G$ is proportional to the Green function of $-\Delta + h$ with zero Dirichlet data.
We let $\chi \in C_c^\infty(-2, 2)$ with $\chi \equiv 1$ on $(-1, 1)$ and $0 \leq \chi < 1$. For $r > 0$, we consider the cylindrical symmetric cut-off function

$$\eta_r(t, z) = \chi\left(\frac{|t| + |z|}{r}\right) \qquad \text{for every } (t, z) \in \mathbb{R} \times \mathbb{R}^2. \tag{5.4}$$

It is clear that

$$\eta_r \equiv 1 \quad \text{in } Q_r, \qquad \eta_r \in H_0^1(Q_{2r}), \qquad |\nabla \eta_r| \leq \frac{C}{r} \quad \text{in } \mathbb{R}^3.$$

For $y_0 \in \Omega$, we let $r_0 \in (0, 1)$ such that

$$y_0 + Q_{2r_0} \subset \Omega. \tag{5.5}$$

We define the function $M_{y_0} : Q_{2r_0} \to \mathbb{R}$ given by

$$M_{y_0}(x) := G(y_0, x + y_0) - \eta_r(x)\frac{1}{|x|} \qquad \text{for every } x \in Q_{2r_0}. \tag{5.6}$$

It follows from (5.3) that $M_{y_0} \in L^\infty(Q_{r_0})$. By (5.2) and (5.1),

$$|-\Delta M_{y_0}(x) + h(x)M_{y_0}(x)| \leq \frac{C}{|x|} = C\mathcal{R}(x) \qquad \text{for every } x \in Q_{r_0},$$

whereas $\mathcal{R} \in L^p(Q_{r_0})$ for every $p \in (1, 3)$. Hence by elliptic regularity theory, $M_{y_0} \in W^{2,p}(Q_{r_0/2})$ for every $p \in (1, 3)$. Therefore by Morrey's embdding theorem, we deduce that

$$\|M_{y_0}\|_{C^{1,\varrho}(Q_{r_0/2})} \leq C \qquad \text{for every } \varrho \in (0, 1). \tag{5.7}$$

In view of (1.6), the mass of the operator $-\Delta + h$ in $\Omega$ at the point $y_0 \in \Omega$ is given by

$$\mathbf{m}(y_0) = M_{y_0}(0). \tag{5.8}$$

We recall that the positive ground state solution $w$ satisfies

$$-\Delta w = S_{3,\sigma}|z|^{-\sigma}w^{2^*_\sigma - 1} \qquad \text{in } \mathbb{R}^3, \qquad \int_{\mathbb{R}^3} |z|^{-\sigma}w^{2^*_\sigma}\,dx = 1, \tag{5.9}$$

where $x = (t, z) \in \mathbb{R} \times \mathbb{R}^2$. In addition by (3.4), we have

$$\frac{C_1}{1 + |x|} \leq w(x) \leq \frac{C_2}{1 + |x|} \qquad \text{for every } x \in \mathbb{R}^3. \tag{5.10}$$

The following result will be crucial in the following of this section.

**Lemma 5.1.** *Consider the function* $v_\varepsilon : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ *given by*

$$v_\varepsilon(x) = \varepsilon^{-1} w\left(\frac{x}{\varepsilon}\right).$$

*Then there exists a constant* $\boldsymbol{c} > 0$ *and a sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *(still denoted by* $\varepsilon$*) such that*

$$v_\varepsilon(x) \to \frac{\boldsymbol{c}}{|x|} \qquad \text{and} \qquad \nabla v_\varepsilon(x) \to -\boldsymbol{c}\frac{x}{|x|^3} \qquad \text{for all most every } x \in \mathbb{R}^3$$

*and*

$$v_\varepsilon(x) \to \frac{\boldsymbol{c}}{|x|} \qquad \text{and} \qquad \nabla v_\varepsilon(x) \to -\boldsymbol{c}\frac{x}{|x|^3} \qquad \text{for every } x \in \mathbb{R}^3 \setminus \{z = 0\}. \tag{5.11}$$

*Proof.* By Corollary 3.2, we have that $(v_\varepsilon)$ is bounded in $C^2_{loc}(\mathbb{R}^3 \setminus \{z = 0\})$. Therefore by Arzelá-Ascolli's theorem $v_\varepsilon$ converges to $v$ in $C^1_{loc}(\mathbb{R}^3 \setminus \{z = 0\})$. In particular,

$$v_\varepsilon \to v \qquad \text{and} \qquad \nabla v_\varepsilon \to \nabla v \qquad \text{almost every where on } \mathbb{R}^3.$$

It is plain, from (5.10), that

$$0 < \frac{C_1}{\varepsilon + |x|} \le v_\varepsilon(x) \le \frac{C_2}{\varepsilon + |x|} \qquad \text{for almost every } x \in \mathbb{R}^3. \tag{5.12}$$

By (5.9), we have

$$-\Delta v_\varepsilon(x) = \varepsilon^{2-\sigma} f_\varepsilon(x) \qquad \text{in } \mathbb{R}^3, \tag{5.13}$$

where

$$f_\varepsilon(x) = S_{3,\sigma}|z|^{-\sigma} v_\varepsilon^{2^*_\sigma - 1}(x) \le C|z|^{-\sigma}|x|^{-5+2\sigma} \qquad \text{for almost every } x = (t, z) \in \mathbb{R}^3.$$

We let $\varphi \in C^\infty_c\left(\mathbb{R}^3 \setminus \{0\}\right)$. We multiply (5.13) by $\varphi$ and integrate by parts to get

$$-\int_{\mathbb{R}^3} v_\varepsilon \Delta\varphi dx = \varepsilon^{2-\sigma}\int_{\mathbb{R}^3} f_\varepsilon(x)\varphi(x)dx.$$

By (5.12) and the dominated convergence theorem, we can pass to the limit in the above identity and deduce that

$$\Delta v = 0 \qquad \text{in } \mathcal{D}'\left(\mathbb{R}^3 \setminus \{0\}\right).$$

In particular $v$ is equivalent to a function of class $C^\infty\left(\mathbb{R}^3 \setminus \{0\}\right)$ which is still denoted by $v$. Thanks to (5.12), by Bôcher's theorem, there exists a constant $\mathbf{c} > 0$ such that

$$v(x) = \frac{\mathbf{c}}{|x|}.$$

The proof of the lemma is thus finished. □

We start by recording some useful estimates.

**Lemma 5.2.** *There exists a constant* $C > 0$ *such that for every* $\varepsilon, r \in (0, r_0/2)$*, we have*

$$\int_{Q_{r/\varepsilon}} |\nabla w|^2 dx \le C\max\left(1, \frac{\varepsilon}{r}\right), \qquad \int_{Q_{r/\varepsilon}} |w|^2 dx \le C\max\left(1, \frac{r}{\varepsilon}\right), \tag{5.14}$$

$$\int_{Q_{r/\varepsilon}} w|\nabla w| dx \le C\max\left(1, \log\frac{r}{\varepsilon}\right), \tag{5.15}$$

$$\int_{Q_{r/\varepsilon}} |\nabla w| dx \le C\max\left(1, \frac{r}{\varepsilon}\right), \qquad \int_{Q_{r/\varepsilon}} |w| dx \le C\max\left(1, \frac{r^2}{\varepsilon^2}\right) \tag{5.16}$$

*and*

$$\varepsilon^2\int_{Q_{r/\varepsilon}} |z|^{-\sigma}|x|^2 w^{2^*_\sigma} dx + \varepsilon\int_{Q_{4r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma} w^{2^*_\sigma - 1} dx + \int_{\mathbb{R}^3\setminus Q_{r/\varepsilon}} |z|^{-\sigma} w^{2^*_\sigma} dx \le Cr^{\sigma-3}\varepsilon^{3-\sigma}. \tag{5.17}$$

*Proof.* The proof of this lemma is not difficult and uses only the estimates in Corollary 3.2. We therefore skip the details.

□

5.1. **Proof of Theorem 1.3.** Given $y_0 \in \Gamma \subset \Omega \subset \mathbb{R}^3$, we let $r_0$ as defined in (5.5). For $r \in (0, r_0/2)$, we consider $F_{y_0} : Q_r \to \Omega$ (see Section 2) parameterizing a neighborhood of $y_0$ in $\Omega$, with the property that $F_{y_0}(0) = y_0$. For $\varepsilon > 0$, we consider $u_\varepsilon : \Omega \to \mathbb{R}$ given by

$$u_\varepsilon(y) := \varepsilon^{-1/2}\eta_r(F_{y_0}^{-1}(y))w\left(\frac{F_{y_0}^{-1}(y)}{\varepsilon}\right).$$

We can now define the test function $\Psi_\varepsilon : \Omega \to \mathbb{R}$ by

$$\Psi_\varepsilon(y) = u_\varepsilon(y) + \varepsilon^{1/2}\mathbf{c}\,\eta_{2r}(F_{y_0}^{-1}(y))M_{y_0}(F_{y_0}^{-1}(y)). \tag{5.18}$$

It is plain that $\Psi_\varepsilon \in H_0^1(\Omega)$ and

$$\Psi_\varepsilon(F_{y_0}(x)) = \varepsilon^{-1/2}\eta_r(x)w\left(\frac{x}{\varepsilon}\right) + \varepsilon^{1/2}\mathbf{c}\,\eta_{2r}(x)M_{y_0}(x) \qquad \text{for every } x \in \mathbb{R}^N.$$

The main result of this section is contained in the following

**Proposition 5.3.** *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\boldsymbol{c}$ be the sequence and the number given by Lemma 5.1. Then there exists $r_0, n_0 > 0$ such that for every $r \in (0, r_0)$ and $n \geq n_0$*

$$J(\Psi_\varepsilon) := \frac{\displaystyle\int_\Omega |\nabla\Psi_{\varepsilon_n}|^2dy + \int_\Omega h|\Psi_{\varepsilon_n}|^2dy}{\left(\displaystyle\int_\Omega \rho_\Gamma^{-\sigma}|\Psi_{\varepsilon_n}|^{2^*_\sigma}dy\right)^{\frac{2}{2^*_\sigma}}} = S_{3,\sigma} - \varepsilon_n\pi^2\boldsymbol{m}(y_0)\boldsymbol{c}^2 + \mathcal{O}_r(\varepsilon_n),$$

*for some numbers $\mathcal{O}_r(\varepsilon_n)$ satisfying*

$$\lim_{r\to 0}\lim_{n\to\infty}\varepsilon_n^{-1}\mathcal{O}_r(\varepsilon_n) = 0.$$

The proof of this proposition will be separated into two steps given by Lemma 5.4 and Lemma 5.5 below. To alleviate the notations, we will write $\varepsilon$ instead of $\varepsilon_n$ and we will remove the subscript $y_0$, by writing $M$ and $F$ in the place of $M_{y_0}$ and $F_{y_0}$ respectively. We define

$$\widetilde{\eta}_r(y) := \eta_r(F^{-1}(y)), \qquad V_\varepsilon(y) := v_\varepsilon(F^{-1}(y)) \qquad \text{and} \qquad \widetilde{M}_{2r}(y) := \eta_{2r}(F^{-1}(y))M(F^{-1}(y)),$$

where $v_\varepsilon(x) = \varepsilon^{-1}w\left(\frac{x}{\varepsilon}\right)$. With these notations, (5.18) becomes

$$\Psi_\varepsilon(y) = u_\varepsilon(y) + \varepsilon^{\frac{1}{2}}\mathbf{c}\,\widetilde{M}_{2r}(y) = \varepsilon^{\frac{1}{2}}V_\varepsilon(y) + \varepsilon^{\frac{1}{2}}\mathbf{c}\,\widetilde{M}_{2r}(y). \tag{5.19}$$

We first consider the numerator in (5.3).

**Lemma 5.4.** *We have*

$$\int_\Omega |\nabla\Psi_\varepsilon|^2dy + \int_\Omega h\Psi_\varepsilon^2dy = S_{3,\sigma} - \varepsilon\boldsymbol{m}(y_0)\boldsymbol{c}^2\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x) + \mathcal{O}_r(\varepsilon),$$

*where $\nu$ is the unit outer normal of $Q_r$.*

*Proof.* Recalling (5.19), direct computations give

$$\begin{aligned}\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\Psi_\varepsilon|^2dy &= \int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\left(\widetilde{\eta}_r u_\varepsilon\right)|^2dy+\varepsilon\mathbf{c}^2\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\widetilde{M}_{2r}|^2dy\\ &+2\varepsilon^{1/2}\mathbf{c}\int_{F(Q_{2r})\setminus F(Q_r)}\nabla\left(\widetilde{\eta}_r u_\varepsilon\right)\cdot\nabla\widetilde{M}_{2r}dy\\ &=\varepsilon\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\left(\widetilde{\eta}_r V_\varepsilon\right)|^2dy+\varepsilon\mathbf{c}^2\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\widetilde{M}_{2r}|^2dy\\ &+2\varepsilon\mathbf{c}\int_{F(Q_{2r})\setminus F(Q_r)}\nabla\left(\widetilde{\eta}_r V_\varepsilon\right)\cdot\nabla\widetilde{M}_{2r}dy. \end{aligned}\tag{5.20}$$

By (5.4), $\eta_r v_\varepsilon=\eta_r\varepsilon^{-1}w(\cdot/\varepsilon)$ is cylindrically symmetric. Therefore by the change variable $y=F(x)$ and using Lemma 3.3, we get

$$\begin{aligned}\varepsilon\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\left(\widetilde{\eta}_r V_\varepsilon\right)|^2dy &= \varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla\left(\eta_r v_\varepsilon\right)|_g^2\sqrt{g}dx\\ &=\varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla\left(\eta_r v_\varepsilon\right)|^2dx+O\left(\varepsilon r^2\int_{Q_{2r}\setminus Q_r}|\nabla\left(\eta_r v_\varepsilon\right)|^2dx\right).\end{aligned}\tag{5.21}$$

By computing, we find that

$$\begin{aligned}\varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla\left(\eta_r v_\varepsilon\right)|^2dx &\le \varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla v_\varepsilon|^2dx+\varepsilon\int_{Q_{2r}\setminus Q_r}v_\varepsilon^2|\nabla\eta_r|^2dx+2\varepsilon\int_{Q_{2r}\setminus Q_r}v_\varepsilon|\nabla v_\varepsilon||\nabla\eta_r|dx\\ &\le \varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla v_\varepsilon|^2dx+\frac{C}{r^2}\varepsilon\int_{Q_{2r}\setminus Q_r}v_\varepsilon^2dx+\frac{C}{r}\varepsilon\int_{Q_{2r}\setminus Q_r}v_\varepsilon|\nabla v_\varepsilon|dx\\ &=\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}|\nabla w|^2dx+C\frac{\varepsilon}{r^2}\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}w^2dx+\frac{C}{r}\varepsilon\int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}}w|\nabla w|dx.\end{aligned}$$

From this and (5.14) and (5.15), we get

$$O\left(\varepsilon r^2\int_{Q_{2r}\setminus Q_r}|\nabla\left(\eta_r v_\varepsilon\right)|^2dx\right)=\mathcal{O}_r(\varepsilon).$$

We replace this in (5.21) to have

$$\varepsilon\int_{F(Q_{2r})\setminus F(Q_r)}|\nabla\left(\widetilde{\eta}_r V_\varepsilon\right)|^2dy=\varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla(\eta_r v_\varepsilon)|^2dx+\mathcal{O}_r(\varepsilon).\tag{5.22}$$

We have the following estimates

$$0\le v_\varepsilon\le C|x|^{-1}\quad\text{for } x\in\mathbb{R}^3\setminus\{0\}\qquad\text{and}\qquad|\nabla v_\varepsilon(x)|\le C|x|^{-2}\quad\text{for } |x|\ge\varepsilon,\tag{5.23}$$

which easily follows from (5.10) and Corollary 3.2. By these estimates, Lemma 2.2 and (5.7) together with the change of variable $y=F(x)$, we have

$$\begin{aligned}\varepsilon\int_{F(Q_{2r})\setminus F(Q_r)}\nabla\left(\widetilde{\eta}_r V_\varepsilon\right)\cdot\nabla\widetilde{M}_{2r}dy=&\varepsilon\int_{Q_{2r}\setminus Q_r}\nabla\left(\eta_r v_\varepsilon\right)\cdot\nabla M dx\\ &+O\left(\varepsilon\int_{Q_{2r}\setminus Q_r}|\nabla v_\varepsilon|dx+\frac{\varepsilon}{r}\int_{Q_{2r}\setminus Q_r}v_\varepsilon dx\right)\\ =&\varepsilon\int_{Q_{2r}\setminus Q_r}\nabla\left(\eta_r v_\varepsilon\right)\cdot\nabla M dx+\mathcal{O}_r(\varepsilon).\end{aligned}$$

This with (5.22), (5.7) and (5.20) give

$$\int_{F(Q_{2r})\setminus F(Q_r)} |\nabla \Psi_\varepsilon|^2 dy = \varepsilon \int_{Q_{2r}\setminus Q_r} |\nabla \left(\eta_r v_\varepsilon\right)|^2 dx + \varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_{2r} M)|^2 dx$$
$$+ 2\varepsilon \mathbf{c} \int_{Q_{2r}\setminus Q_r} \nabla \left(\eta_r v_\varepsilon\right) \cdot \nabla M dx + \mathcal{O}_r(\varepsilon).$$

Thanks to Lemma 5.1 and (5.23), we can thus use the dominated convergence theorem to deduce that, as $\varepsilon \to 0$,

$$\int_{Q_{2r}\setminus Q_r} |\nabla \left(\eta_r v_\varepsilon\right)|^2 dx = \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} |\nabla \left(\eta_r \mathcal{R}\right)|^2 dx + o(1). \tag{5.24}$$

Similarly, we easily see that

$$\int_{Q_{2r}\setminus Q_r} \nabla \left(\eta_r v_\varepsilon\right) \cdot \nabla M dx = \mathbf{c} \int_{Q_{2r}\setminus Q_r} \nabla \left(\eta_r \mathcal{R}\right) \cdot \nabla M dx + o(1) \qquad \text{as } \varepsilon \to 0.$$

This and (5.24), then give

$$\int_{F(Q_{2r})\setminus F(Q_r)} |\nabla \Psi_\varepsilon|^2 dy = \varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} |\nabla \left(\eta_r \mathcal{R}\right)|^2 dx + \varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} |\nabla M|^2 dx$$
$$+ 2\varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} \nabla \left(\eta_r \mathcal{R}\right) \cdot \nabla M dx + \mathcal{O}_r(\varepsilon)$$
$$= \varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r \mathcal{R} + M)|^2 dx + \mathcal{O}_r(\varepsilon). \tag{5.25}$$

Since the support of $\Psi_\varepsilon$ is contained in $Q_{4r}$ while the one of $\eta_r$ is in $Q_{2r}$, it is easy to deduce from (5.7) that

$$\int_{\Omega\setminus F(Q_{2r})} |\nabla \Psi_\varepsilon|^2 dy = \varepsilon \mathbf{c}^2 \int_{F(Q_{4r})\setminus F(Q_{2r})} |\nabla \widetilde{M}_{2r}|^2 dy = \mathcal{O}_r(\varepsilon)$$

and from Lemma 5.2, that

$$\int_{\Omega\setminus F(Q_r)} h |\Psi_\varepsilon|^2 dy = \varepsilon \mathbf{c}^2 \int_{F(Q_{4r})\setminus F(Q_r)} h |\eta_r V_\varepsilon + \widetilde{M}_{2r}|^2 dy = \mathcal{O}_r(\varepsilon).$$

Therefore by (5.25), we conclude that

$$\int_{\Omega\setminus F(Q_r)} |\nabla \Psi_\varepsilon|^2 dy + \int_{\Omega\setminus F(Q_r)} h |\Psi_\varepsilon|^2 dy$$
$$= \varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} |\nabla(\eta_r \mathcal{R} + M)|^2 dx + \varepsilon \mathbf{c}^2 \int_{Q_{2r}\setminus Q_r} h(\cdot + y_0) |\eta_r \mathcal{R} + M|^2 dx + \mathcal{O}_r(\varepsilon).$$

Recall that $G(x + y_0, y_0) = \eta_r(x)\mathcal{R}(x) + M(x)$ for ever $x \in Q_{2r}$ and that by (5.2),

$$-\Delta_x G(x + y_0, y_0) + h(x + y_0) G(x + y_0, y_0) = 0 \qquad \text{for every } x \in Q_{2r} \setminus Q_r.$$

Therefore, by integration by parts, we find that

$$\int_{\Omega\setminus F(Q_r)} |\nabla \Psi_\varepsilon|^2 dy + \int_{\Omega\setminus F(Q_r)} h |\Psi_\varepsilon|^2 dy = \mathbf{c}^2 \int_{\partial(Q_{2r}\setminus Q_r)} (\eta_r \mathcal{R} + M) \frac{\partial(\eta_r \mathcal{R} + M)}{\partial \overline{\nu}} \sigma(x) + \mathcal{O}_r(\varepsilon),$$

where $\overline{\nu}$ is the exterior normal vectorfield to $Q_{2r} \setminus Q_r$. Thanks to (5.7), we finally get

$$\int_{\Omega\setminus F(Q_r)} |\nabla \Psi_\varepsilon|^2 dy + \int_{\Omega\setminus F(Q_r)} h |\Psi_\varepsilon|^2 dy = -\varepsilon \mathbf{c}^2 \int_{\partial Q_r} \mathcal{R} \frac{\partial \mathcal{R}}{\partial \nu} d\sigma(x) - \varepsilon \mathbf{c}^2 \int_{\partial Q_r} M \frac{\partial \mathcal{R}}{\partial \nu} d\sigma(x) + \mathcal{O}_r(\varepsilon), \tag{5.26}$$

where $\nu$ is the exterior normal vectorfield to $Q_r$.
Next we make the expansion of $\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy$ for $r$ and $\varepsilon$ small. First, we observe that, by Lemma 5.2 and (5.7), we have

$$\begin{aligned}\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy &= \int_{F(Q_r)}|\nabla u_\varepsilon|^2dy+\varepsilon\mathbf{c}^2\int_{F(Q_r)}|\nabla M|^2dy+2\varepsilon^{1/2}\mathbf{c}\int_{F(Q_r)}\nabla u_\varepsilon\cdot\nabla\widetilde{M}_{2r}dy\\ &=\int_{Q_{r/\varepsilon}}|\nabla w|^2dx+O\left(\varepsilon^2\int_{Q_{r/\varepsilon}}|x|^2|\nabla w|^2dx+\varepsilon^2\int_{Q_{r/\varepsilon}}|\nabla w|dx\right)+\mathcal{O}_r(\varepsilon)\\ &=\int_{Q_{r/\varepsilon}}|\nabla w|^2dx+\mathcal{O}_r(\varepsilon).\end{aligned}$$

By integration by parts and using (5.17), we deduce that

$$\begin{aligned}\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy &= S_{3,\sigma}\int_{Q_{r/\varepsilon}}|z|^{-\sigma}w^{2^*_\sigma}dx+\int_{\partial Q_{r/\varepsilon}}w\frac{\partial w}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon)\\ &=S_{3,\sigma}+\varepsilon\int_{\partial Q_r}v_\varepsilon\frac{\partial v_\varepsilon}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon).\end{aligned}\tag{5.27}$$

Now (5.23), (5.11) and the dominated convergence theorem yield, for fixed $r>0$ and $\varepsilon\to0$,

$$\begin{aligned}\int_{\partial Q_r}v_\varepsilon\frac{\partial v_\varepsilon}{\partial\nu}d\sigma(x) &= \int_{\partial B^2_{\mathbb{R}^2}(0,r)}\int_{-r}^{r}v_\varepsilon(t,z)\nabla v_\varepsilon(t,z)\cdot\frac{z}{|z|}d\sigma(z)dt+2\int_{B^2_{\mathbb{R}^2}}v_\varepsilon(r,z)\partial_t v_\varepsilon(r,z)dz\\ &=\mathbf{c}^2\int_{\partial B^2_{\mathbb{R}^2}(0,r)}\int_{-r}^{r}\mathcal{R}(t,z)\nabla\mathcal{R}(t,z)\cdot\frac{z}{|z|}d\sigma(z)dt+2\mathbf{c}^2\int_{B^2_{\mathbb{R}^2}}\mathcal{R}(r,z)\partial_t\mathcal{R}(r,z)dz+o(1)\\ &=\mathbf{c}^2\int_{\partial Q_r}\mathcal{R}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+o(1).\end{aligned}\tag{5.28}$$

Moreover (5.16) implies that

$$\int_{F(Q_r)}h\Psi_\varepsilon^2dy=\mathcal{O}_r(\varepsilon).$$

From this together with (5.27) and (5.28), we obtain

$$\int_{F(Q_r)}|\nabla\Psi_\varepsilon|^2dy+\int_{F(Q_r)}h\Psi_\varepsilon^2dy=S_{3,\sigma}+\mathbf{c}^2\varepsilon\int_{\partial Q_r}\mathcal{R}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon).$$

Combining this with (5.26), we then have

$$\int_\Omega|\nabla\Psi_\varepsilon|^2dy+\int_\Omega h\Psi_\varepsilon^2dy=S_{3,\sigma}-\varepsilon\mathbf{c}^2\int_{\partial Q_r}M\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon)+o\left(\varepsilon\right).\tag{5.29}$$

Since (recalling (5.8)) $M(y)=M(0)+O(r)=\mathbf{m}(y_0)+O(r)$ in $Q_{2r}$, we get the claimed result in the statement of the lemma. □

The following result together with the previous lemma provides the proof of Proposition5.3.

**Lemma 5.5.** *We have*

$$\left(\int_\Omega\rho_\Gamma^{-\sigma}|\Psi_\varepsilon|^{2^*_\sigma}dy\right)^{\frac{2}{2^*_\sigma}}=1-\frac{2}{S_{3,\sigma}}\varepsilon\boldsymbol{m}(y_0)\boldsymbol{c}^2\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon).$$

*Proof.* Since $2^*_\sigma>2$, there exists a positive constant $C(\sigma)$ such that

$$||a+b|^{2^*_\sigma}-|a|^{2^*_\sigma}-2^*_\sigma ab|a|^{2^*_\sigma-2}|\le C(\sigma)\left(|a|^{2^*_\sigma-2}b^2+|b|^{2^*_\sigma}\right)\qquad\text{for all } a,b\in\mathbb{R}.$$

As a consequence, we obtain

$$\begin{aligned}\int_\Omega \rho_\Gamma^{-\sigma}|\Psi_\varepsilon|^{2^*_\sigma}dy &= \int_{F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon + \varepsilon^{\frac{1}{2}}\widetilde{M}_{2r}|^{2^*_\sigma}dy + \int_{F(Q_{4r})\setminus F(Q_r)} \rho_\Gamma^{-\sigma}|W_\varepsilon + \varepsilon^{\frac{1}{2}}\widetilde{M}_{2r}|^{2^*_\sigma}dy \\ &= \int_{F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma}dy + 2^*_\sigma \mathbf{c}\varepsilon^{1/2}\int_{F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma-1}\widetilde{M}_{2r}dy \\ &+ O\left(\int_{F(Q_{4r})} \rho_\Gamma^{-\sigma}|\eta_r u_\varepsilon|^{2^*_\sigma-2}\left(\varepsilon^{1/2}\widetilde{M}_{2r}\right)^2 dy + \int_{F(Q_{4r})} \rho_\Gamma^{-\sigma}|\varepsilon^{1/2}\widetilde{M}_{2r}|^{2^*_\sigma}dy\right) \\ &+ O\left(\int_{F(Q_{4r})\setminus F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma}dy + 2^*_\sigma \mathbf{c}\varepsilon^{1/2}\int_{F(Q_{4r})\setminus F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma-1}\widetilde{M}_{2r}dy\right). \end{aligned} \tag{5.30}$$

By Hölder's inequality and (2.9), we have

$$\begin{aligned}\int_{F(Q_{4r})} \rho_\Gamma^{-\sigma}|\eta u_\varepsilon|^{2^*_\sigma-2}\left(\varepsilon^{1/2}\widetilde{\beta}_r\right)^2 dy &\le \varepsilon\|u_\varepsilon\|^{2^*_\sigma-2}_{L^{2^*_\sigma}(F(Q_{4r});\rho^{-\sigma})}\|\widetilde{M}_{2r}\|^2_{L^{2^*_\sigma}(F(Q_{4r});\rho_\Gamma^{-\sigma})} \\ &= \varepsilon\|w\|^{2^*_\sigma-2}_{L^{2^*_\sigma}(Q_{4r};|z|^{-\sigma}\sqrt{|g|})}\|\widetilde{M}_{2r}\|^2_{L^{2^*_\sigma}(F(Q_{4r});\rho_\Gamma^{-\sigma})} \\ &\le \varepsilon(1+Cr)\|\widetilde{M}_{2r}\|^2_{L^{2^*_\sigma}(F(Q_{4r});\rho_\Gamma^{-\sigma})} = \mathcal{O}_r(\varepsilon), \end{aligned} \tag{5.31}$$

recalling that $\|w\|_{L^{2^*_\sigma}(\mathbb{R}^3;|z|^{-\sigma})} = 1$. Furthermore, since $2^*_\sigma > 2$, by (5.7), we easily get

$$\int_{F(Q_{4r})} \rho_\Gamma^{-\sigma}|\varepsilon^{1/2}\widetilde{M}_{2r}|^{2^*_\sigma}dy = o(\varepsilon). \tag{5.32}$$

Moreover by change of variables and (5.17), we also have

$$\begin{aligned}&\int_{F(Q_{4r})\setminus F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma}dy + 2^*_\sigma \mathbf{c}\varepsilon^{1/2}\int_{F(Q_{4r})\setminus F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma-1}\widetilde{M}_{2r}dy \\ &\qquad \le C\int_{Q_{4r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma}|w|^{2^*_\sigma}dx + C\varepsilon\int_{Q_{4r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma}|w|^{2^*_\sigma-1}dx \\ &\qquad = o(\varepsilon).\end{aligned}$$

By this, (5.30), (5.32) and (5.31), it results

$$\int_\Omega \rho_\Gamma^{-\sigma}|\Psi_\varepsilon|^{2^*_\sigma}dy = \int_{F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma}dy + 2^*_\sigma \mathbf{c}\varepsilon^{1/2}\int_{F(Q_r)} \rho_\Gamma^{-\sigma}|u_\varepsilon|^{2^*_\sigma-1}\widetilde{M}_{2r}dy + \mathcal{O}_r(\varepsilon).$$

We define $B_\varepsilon(x) := M(\varepsilon x)\sqrt{|g_\varepsilon|}(x) = M(\varepsilon x)\sqrt{|g|}(\varepsilon x)$. Then by the change of variable $y = \frac{F(x)}{\varepsilon}$ in the above identity and recalling (2.9), then by oddness, we have

$$\begin{aligned}\int_\Omega \rho_\Gamma^{-\sigma}|\Psi_\varepsilon|^{2^*_\sigma}dy &= \int_{Q_{r/\varepsilon}} |z|^{-\sigma}w^{2^*_\sigma}\sqrt{|g_\varepsilon|}dx + 2^*_\sigma\varepsilon\mathbf{c}\int_{Q_{r/\varepsilon}} |z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon dx + \mathcal{O}_r(\varepsilon) \\ &= \int_{Q_{r/\varepsilon}} |z|^{-\sigma}w^{2^*_\sigma}dx + 2^*_\sigma\varepsilon\mathbf{c}\int_{Q_{r/\varepsilon}} |z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon dx + \mathcal{O}_r(\varepsilon) \\ &+ O\left(\varepsilon^2\int_{Q_{r/\varepsilon}} |z|^{-\sigma}|x|^2 w^{2^*_\sigma}dx\right) \\ &= 1 + 2^*_\sigma\varepsilon\mathbf{c}\int_{Q_{r/\varepsilon}} |z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon dx \\ &+ O\left(\int_{\mathbb{R}^3\setminus Q_{r/\varepsilon}} |z|^{-\sigma}w^{2^*_\sigma}dx + \varepsilon^2\int_{Q_{r/\varepsilon}} |z|^{-\sigma}|x|^2 w^{2^*_\sigma}dx\right) + \mathcal{O}_r(\varepsilon).\end{aligned}$$

Therefore by (5.17) we then have

$$\left(\int_\Omega \rho_\Gamma^{-\sigma}|\Psi_\varepsilon|^{2^*_\sigma}dy\right)^{\frac{2}{2^*_\sigma}} = 1+2\varepsilon\mathbf{c}\int_{Q_{r/\varepsilon}}|z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon(x)dx+\mathcal{O}_r(\varepsilon). \tag{5.33}$$

Multiply (5.9) by $B_\varepsilon \in \mathcal{C}^1(\overline{Q_r})$ and integrate by parts to get

$$\begin{aligned}S_{3,\sigma}\int_{Q_{r/\varepsilon}}|z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon dx &= \int_{Q_{r/\varepsilon}}\nabla w\cdot\nabla B_\varepsilon dx-\int_{\partial Q_{r/\varepsilon}}B_\varepsilon\frac{\partial w}{\partial\nu}d\sigma(x)\\ &= \int_{Q_{r/\varepsilon}}\nabla w\cdot\nabla B_\varepsilon dx-\int_{\partial Q_r}B_1\frac{\partial v_\varepsilon}{\partial\nu}d\sigma(x).\end{aligned}$$

Since $|\nabla B_\varepsilon|\le C\varepsilon$, by Lemma 5.1 and (5.7), we then have

$$\varepsilon\int_{Q_{r/\varepsilon}}\nabla w\cdot\nabla B_\varepsilon dx = O\left(\varepsilon^2\int_{Q_{r/\varepsilon}}|\nabla w|dx\right)=\mathcal{O}_r(\varepsilon).$$

Consequently

$$S_{3,\sigma}\varepsilon\int_{Q_{r/\varepsilon}}|z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon dx=-\varepsilon\int_{\partial Q_r}B_1\frac{\partial v_\varepsilon}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon),$$

on the one hand. On the other hand by Lemma 5.1, (5.7) and the dominated convergence theorem, we get

$$\int_{\partial Q_r}B_1\frac{\partial v_\varepsilon}{\partial\nu}d\sigma(x)=\mathbf{c}\int_{\partial Q_r}B_1\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+o(1)=\mathbf{c}M(0)\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+O(r)+o(1),$$

so that

$$\varepsilon\mathbf{c}\int_{Q_{r/\varepsilon}}|z|^{-\sigma}|w|^{2^*_\sigma-1}B_\varepsilon dx=-\varepsilon\mathbf{c}^2\frac{1}{S_{3,\sigma}}M(0)\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon).$$

It then follows from (5.33) that

$$\left(\int_\Omega \rho_\Gamma^{-\sigma}|\Psi_\varepsilon|^{2^*_\sigma}dy\right)^{\frac{2}{2^*_\sigma}} = 1-\frac{2}{S_{3,\sigma}}\varepsilon\mathbf{c}^2M(0)\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}d\sigma(x)+\mathcal{O}_r(\varepsilon).$$

Since $M(0)=\mathbf{m}(y_0)$, see (5.8), the proof of the lemma is thus finished. □

*Proof of Proposition 5.3 (completed).* By Lemma 5.4 and Lemma 5.5, we have

$$J(\Psi_\varepsilon)=S_{3,\sigma}-\varepsilon\mathbf{c}^2\mathbf{m}(y_0)\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}\,d\sigma(x)+\mathcal{O}_r(\varepsilon). \tag{5.34}$$

Finally, recalling that $\mathcal{R}(x)=\frac{1}{|x|}$, we can compute

$$\begin{aligned}\int_{\partial Q_r}\frac{\partial\mathcal{R}}{\partial\nu}\,d\sigma(x) &= -\int_{\partial Q_r}\frac{x\cdot\nu(x)}{|x|^3}\,d\sigma(x)\\ &= -2r\int_{B_{\mathbb{R}^2}(0,r)}\frac{1}{r^2+|z|^2}\,dz-2\pi\int_{-r}^{r}\frac{r^3}{r^2+t^2}dt\\ &= -\pi^2(1+r^2).\end{aligned}$$

From this and (5.34), we then have

$$J(\Psi_\varepsilon)=S_{3,\sigma}-\varepsilon\pi^2\mathbf{c}^2\mathbf{m}(y_0)+\mathcal{O}_r(\varepsilon),$$

which finishes the proof. □

As a consequence of Proposition 5.3, we can now complete the

*Proof of Theorem 1.3 (completed).* The proof follows from Lemma 5.3 that if $\mathbf{m}(y_0) > 0$ for some $y_0 \in \Gamma$ then $\mu_h(\Omega, \Gamma) < S_{3,\sigma}$. The fact that $\mu_h(\Omega, \Gamma)$ is attained by a positive function is an immediate consequence of Proposition 6.2 belwo. □

## 6. Appendix: Existence of minimizer for $\mu_h(\Omega, \Gamma)$

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $h$ a continuous function on $\Omega$. Let $\Gamma$ be a smooth closed curve $\Gamma$ contained in $\Omega$. We consider

$$\mu_h(\Omega, \Gamma) := \inf_{u \in H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Omega h u^2 dy}{\left(\displaystyle\int_\Omega \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dy\right)^{\frac{2}{2^*_\sigma}}}. \tag{6.1}$$

We also recall that

$$S_{N,\sigma} = \inf_{v \in \mathcal{D}^{1,2}(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} dx\right)^{\frac{2}{2^*_\sigma}}}, \tag{6.2}$$

with $x = (t, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$. Our aim in this section is to show that if $\mu_h(\Omega, \Gamma) < S_{N,\sigma}$ then the best constant $\mu_h(\Omega, \Gamma)$ is achieved. The argument of proof is standard. However, for sake of completeness, we add the proof. We start with the following

**Lemma 6.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$, with $N \geq 3$, and let $\Gamma \subset \Omega$ be a closed smooth curve. Then for every $r > 0$, there exist positive constants $c_r > 0$, only depending on $\Omega, \Gamma, N, \sigma$ and $r$, such that for every $u \in H^1_0(\Omega)$*

$$S_{N,\sigma} \left(\int_\Omega \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} \leq (1 + r) \int_\Omega |\nabla u|^2 dy + c_r \left[\int_\Omega u^2 dy + \left(\int_\Omega |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma}\right],$$

*where $2^*_\sigma = \frac{2(N-\sigma)}{N-2}$ and $\sigma \in (0, 2)$.*

*Proof.* We let $r > 0$ small. We can cover a tubular neighborhood of $\Gamma$ by a finite number of sets $(T_r^{y_i})_{1 \leq i \leq m}$ given by

$$T_r^{y_i} := F_{y_i}(Q_r), \qquad \text{with } y_i \in \Gamma.$$

We refer to Section 2 for the parameterization $F_{y_i} : Q_r \to \Omega$. See e.g. [2, Section 2.27], there exists $(\varphi_i)_{1 \leq i \leq m}$ a partition of unity subordinated to this covering such that

$$\sum_i^m \varphi_i = 1 \qquad \text{and} \qquad |\nabla \varphi_i^{\frac{1}{2^*_\sigma}}| \leq K \qquad \text{in } U := \cup_{i=1}^m T_r^{y_i}, \tag{6.3}$$

for some constant $K > 0$. We define

$$\psi_i(y) := \varphi_i^{\frac{1}{2^*_\sigma}}(y) u(y) \qquad \text{and} \qquad \widetilde{\psi}_i(x) = \psi_i(F_{y_i}(x)). \tag{6.4}$$

Recall that that $\rho_\Gamma \geq C > 0$ on $\Omega \setminus U$, for some positive constant $C > 0$. Therefore, since $\frac{2}{2^*_\sigma} < 1$, by (6.4) we get

$$\begin{aligned}
\left(\int_\Omega \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} &\leq \left(\int_U \rho_\Gamma^{-\sigma}\, |u|^{2^*_\sigma}\, dy\right)^{2/2^*_\sigma} + \left(\int_{\Omega\setminus U} \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} \\
&\leq \left(\sum_i^m \int_{T_r^{y_i}} \rho_\Gamma^{-\sigma} |\psi_i|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} + c_r \left(\int_\Omega |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} \\
&\leq \sum_i^m \left(\int_{T_r^{y_i}} \rho_\Gamma^{-\sigma} |\psi_i|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} + c_r \left(\int_\Omega |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} \qquad (6.5)
\end{aligned}$$

By change of variables and Lemma 2.2, we have

$$\begin{aligned}
\left(\int_{T_r^{y_i}} \rho_\Gamma^{-\sigma} |\psi_i|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} &= \left(\int_{Q_r} |z|^{-\sigma} |\widetilde{\psi}_i|^{2^*_\sigma} \sqrt{|g|}(x) dx\right)^{2/2^*_\sigma} \\
&\leq (1 + cr) \left(\int_{Q_r} |z|^{-\sigma} |\widetilde{\psi}_i|^{2^*_\sigma} dx\right)^{2/2^*_\sigma}.
\end{aligned}$$

In addition the Hardy-Sobolev inequality (1.2) yields

$$S_{N,\sigma} \left(\int_{Q_r} |z|^{-\sigma} |\widetilde{\psi}_i|^{2^*_\sigma} dx\right)^{2/2^*_\sigma} \leq \left(\int_{Q_r} |\nabla \widetilde{\psi}_i|^2 dx\right)^{2/2}.$$

Therefore by change of variables and Lemma 2.2, we get

$$\begin{aligned}
S_{N,\sigma} \left(\int_{T_r^{y_i}} \rho_\Gamma^{-\sigma} |\psi_i|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} &\leq (1 + cr) \int_{Q_r} |\nabla \widetilde{\psi}_i|^2 dx \\
&\leq (1 + c'r) \int_{T_r^{y_i}} |\nabla(\varphi_i^{\frac{1}{2^*_\sigma}} u)|^2 dy = (1 + c'r) \int_{T_r^{y_i}} |\varphi_i^{\frac{1}{2^*_\sigma}} \nabla u + u \nabla \varphi_i^{\frac{1}{2^*_\sigma}}|^2 dy + c_r \int_\Omega |u|^2 dy.
\end{aligned}$$

Applying Young's inequality using (6.3) and (6.4), we find that

$$\begin{aligned}
S_{N,\sigma} \left(\int_{T_r^{y_i}} \rho_\Gamma^{-\sigma} |\psi_i|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} &\leq (1 + c'r)(1 + \varepsilon) \int_{T_r^{y_i}} \varphi_i^{\frac{2}{2^*_\sigma}} |\nabla u|^2 dy + c_r(\varepsilon) \int_\Omega |u|^2 dy \\
&\leq (1 + c'r)(1 + \varepsilon) \int_{T_r^{y_i}} |\nabla u|^2 dy + c_r(\varepsilon) \int_\Omega |u|^2 dy.
\end{aligned}$$

Summing for $i$ equal 1 to $m$, we get

$$S_{N,\sigma} \sum_{i=1}^m \left(\int_{T_r^{y_i}} \rho_\Gamma^{-\sigma} |\psi_i|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} \leq (1 + c'r)(1 + \varepsilon) \left(\int_\Omega |\nabla u|^2 dy\right)^{1/2} + c_r(\varepsilon) \left(\int_\Omega |u|^2 dy\right)^{1/2}.$$

This together with (6.5) give

$$S_{N,\sigma} \left(\int_\Omega \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma} \leq (1 + c'r)(1 + \varepsilon) \int_\Omega |\nabla u|^2 dy + c_r(\varepsilon) \int_\Omega |u|^2 dy + c_r \left(\int_\Omega |u|^{2^*_\sigma} dy\right)^{2/2^*_\sigma}.$$

Since $\varepsilon$ and $r$ can be chosen arbitrarily small, we get the desired result. □

We can now prove the following existence result.

**Proposition 6.2.** *Consider $\mu_h(\Omega, \Gamma)$ and $S_{N,\sigma}$ given by (6.1) and (6.2) respectively. Suppose that*

$$\mu_h(\Omega, \Gamma) < S_{N,\sigma}. \qquad (6.6)$$

*Then $\mu_h(\Omega, \Gamma)$ is achieved by a positive function.*

*Proof.* Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $\mu_h(\Omega,\Gamma)$ normalized so that

$$\int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx = 1 \quad \text{and} \quad \mu_h(\Omega,\Gamma) = \int_\Omega |\nabla u_n|^2 dx + \int_\Omega h u_n^2 dx + o(1). \tag{6.7}$$

By coercivity of $-\Delta + h$, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$ and thus , up to a subsequence,

$$u_n \rightharpoonup u \qquad \text{weakly in } H^1_0(\Omega),$$

and

$$u_n \to u \qquad \text{strongly in } L^p(\Omega) \quad \text{for} \quad 1 \le p < 2^*_0 := \frac{2N}{N-2}. \tag{6.8}$$

The weak convergence in $H^1_0(\Omega)$ implies that

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla (u_n - u)|^2 dx + \int_\Omega |\nabla u|^2 dx + o(1). \tag{6.9}$$

By Brezis-Lieb lemma [5] and the strong convergence in the Lebesgue spaces $L^p(\Omega)$, we have

$$1 = \int_\Omega \rho_\Gamma^{-\sigma}|u_n|^{2^*_\sigma}dx = \int_\Omega \rho_\Gamma^{-\sigma}|u-u_n|^{2^*_\sigma}dx + \int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx + o(1). \tag{6.10}$$

By Lemma 6.1, (6.8) —note that $2^*_\sigma < 2^*_0$ , we then deduce that

$$S_{N,\sigma}\left(\int_\Omega \rho_\Gamma^{-\sigma}|u-u_n|^{2^*_\sigma}dx\right)^{2/2^*_\sigma} \le (1+r)\int_\Omega |\nabla(u-u_n)|^2 dx + o(1). \tag{6.11}$$

Using (6.9), (6.10) and (6.11), we have

$$\begin{aligned} S_{N,\sigma}\left(1 - \int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx\right)^{2/2^*_\sigma} &\le (1+r)\left(\int_\Omega |\nabla u_n|^2 dx - \int_\Omega |\nabla u|^2 dx\right) + o(1) \\ &= (1+r)\left(\mu_h(\Omega,\Gamma) - \int_\Omega h u_n^2 dx - \int_\Omega |\nabla u|^2 dx\right) + o(1) \\ &= (1+r)\left(\mu_h(\Omega,\Gamma) - \int_\Omega h u^2 dx - \int_\Omega |\nabla u|^2 dx\right) + o(1) \\ &\le (1+r)\,\mu_h(\Omega,\Gamma)\left(1 - \left(\int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx\right)^{2/2^*_\sigma}\right) + o(1). \end{aligned} \tag{6.12}$$

By the concavity of the map $t \mapsto t^{2/2^*_\sigma}$ on $[0,1]$, we have

$$1 \le \left(1 - \int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx\right)^{2/2^*_\sigma} + \left(\int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx\right)^{2/2^*_\sigma}.$$

From this, then taking the limits respectively as $n \to +\infty$ and as $r \to 0$ in (6.12), we find that

$$[S_{N,\sigma} - \mu_h(\Omega,\Gamma)]\left(1 - \left(\int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx\right)^{2/2^*_\sigma}\right) \le 0.$$

Thanks to (6.6), we then get

$$1 \le \int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx.$$

Since by (6.7) and Fatou's lemma,

$$1 = \int_\Omega \rho_\Gamma^{-\sigma}|u_n|^{2^*_\sigma}dx \ge \int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx,$$

we conclude that

$$\int_\Omega \rho_\Gamma^{-\sigma}|u|^{2^*_\sigma}dx = 1.$$

It then follows from (6.7) that $u_n \to u$ in $L^{2^*_\sigma}(\Omega;\rho_\Gamma^{-\sigma})$ and thus $u_n \to u$ in $H^1_0(\Omega)$. Therefore $u$ is a minimizer for $\mu_h(\Omega,\Gamma)$. Since $|u|$ is also a minimizer for $\mu_h(\Omega,\Gamma)$, we may assume that $u \gneqq 0$. Therefore $u > 0$ by the maximum principle. □

## References


[1] T. Aubin, *Problémes isopérimétriques de Sobolev*, J. Differential Geom. 11(1976) 573-598.

[2] T. Aubin, *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. ISBN: 3-540-60752-8.

[3] M. Badiale, G. Tarantello, *A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics*, Arch. Rational Mech. Anal. 163(4)(2002) 259-293.

[4] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical exponents*, Comm. Pure Appl. Math 36 (1983), 437-477.

[5] H. Brezis, E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., 88, 486-490, 1983.

[6] L. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weights*, Composit. Math. 53(3), 259-275(1984).

[7] J-L. Chern and C-S Lin, *Minimizers of Caffarelli-Kohn-Nirenberg Inequalities with the singularity on the boundary.* Arch. Rational Mech. Anal. 197, Number 2 (2010), 401-432.

[8] A. V. Demyanov, I.A. Nazarov, *On the solvability of the Dirichlet problem for the semilinear Schrödinger equation with a singular potential*, (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov, (POMI) 336 (2006), Kraev. Zadachi Mat. Fiz.i Smezh. Vopr. Teor. Funkts. 37, 25–45, 274; translation in J. Math. Sci. (N.Y.) 143 (2007), no. 2, 2857–2868.

[9] O. Druet, *Elliptic equations with critical Sobolev exponents in dimension 3*, Ann. Inst. H. Poincaré Anal. Non Lináire 19 (2002), no. 2, 125- 142.

[10] I. Fabbri, G. Mancini, K. Sandeep,*Classification of solutions of a critical Hardy Sobolev operator.* J. Differential equations 224(2006) 258-276.

[11] M. M. Fall, I. A. Minlend and E. H. A. Thiam, *The role of the mean curvature in a Hardy-Sobolev trace inequality.* NoDEA Nonlinear Differential Equations Appl. 22(2015), no.5, 1047-1066.

[12] N. Ghoussoub and F. Robert, *On the Hardy-Schrödinger operator with a boundary singularity*, Preprint 2014. https://arxiv.org/abs/1410.1913.

[13] N. Ghoussoub and F. Robert, *Sobolev inequalities for the Hardy-Schrödinger operator: extremals and critical dimensions*, Bull. Math. Sci. 6 (2016), no. 1, 89-144.

[14] N. Ghoussoub and F. Robert, *Elliptic Equations with Critical Growth and a Large Set of Boundary Singularities.* Trans. Amer. Math. Soc., Vol. 361, No. 9 (Sep., 2009), pp. 4843-4870.

[15] N. Ghoussoub and F. Robert, *Concentration estimates for Emden-Fowler equations with boundary singularities and critical growth.* IMRP Int. Math. Res. Pap. (2006), 21867, 1-85.

[16] N. Ghoussoub and F. Robert, *The effect of curvature on the best constant in the Hardy Sobolev inequalities*, Geom. Funct. Anal. 16(6), 1201-1245(2006).

[17] N. Ghoussoub, X. S. Kang, *Hardy-Sobolev critical elliptic equations with boundary singularities*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 6, 767–793.

[18] H. Jaber, *Hardy-Sobolev equations on compact Riemannian manifolds*, Nonlinear Anal. 103(2014), 39-54.

[19] Y. Li and C. Lin, *A Nonlinear Elliptic PDE with Two Sobolev-Hardy Critical Exponents*, published online september 28, 2011, Springer-Verlag (2011).

[20] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. 118(1983) 349-374.

[21] R. Musina, *Existence of extremals for the Maziya and for the Caffarelli-Kohn-Nirenberg inequalities*, Nonlinear Anal., 70 (2009), no. 8, 3002-3007.

[22] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential geometry, 20:479-495, 1984.

[23] G. Talenti, *Best constant in Sobolev inequality.* Ann. Mat. Pura Appli. 110(1976) 353-372.

[24] E. H. A. Thiam, *Hardy and Hardy-Sobolev inequalities on Riemannian manifold.* To appear in IMHOTEP.

[25] E. H. A. Thiam, *Weighted Hardy inequality on Riemannian manifolds.* Commun. Contemp. Math. 18 (2016), no. 6, 1550072, 25 pp.

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