# Monotonicity and nonexistence results for some fractional elliptic problems in the half space 

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#### Abstract

We study a class of fractional elliptic problems of the form $(-\Delta)^{s} u=f(u)$ in the half space $\mathbb{R}_{+}^{N}:=\left\{x \in \mathbb{R}^{N}: x_{1}>0\right\}$ with the complementary Dirichlet condition $u \equiv 0$ in $\mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}$. Under mild assumptions on the nonlinearity $f$, we show that bounded positive solutions are increasing in $x_{1}$. For the special case $f(u)=u^{q}$, we deduce nonexistence of positive bounded solutions in the case where $q \geq 1$ and $q<\frac{N-1+2 s}{N-1-2 s}$ if $N \geq 1+2 s$. We do not require integrability assumptions on the solutions we study.


## 1 Introduction

In the present paper, we are concerned with solutions $u \in L^{\infty}\left(\mathbb{R}^{N}\right)$ of the semilinear fractional problem

$$
\begin{cases}(-\Delta)^{s} u=f(u), \quad u \geq 0 & \text { in } \mathbb{R}_{+}^{N}  \tag{1.1}\\ u=0 & \text { in } \mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}\end{cases}
$$

Here $s \in(0,1), N \in \mathbb{N}, \mathbb{R}_{+}^{N}:=\left\{x \in \mathbb{R}^{N}: x_{1}>0\right\}$ and $f:[0, \infty) \rightarrow[0, \infty)$ is a nonnegative, nondecreasing and locally Lipschitz continuous nonlinearity. Special attention will be given to the case $f(t)=t^{q}$ with $q>1$. Due to applications in physics, biology and finance, linear and nonlinear equations involving the fractional Laplacian $(-\Delta)^{s}$ have received growing attention in recent years (see e.g. [34, Introduction] for various references), while they are are still much less understood than their non-fractional counterparts.
We briefly explain in which sense we consider problems of type (1.1). For functions $u \in C_{c}^{2}\left(\mathbb{R}^{N}\right)$, the fractional Laplacian $(-\Delta)^{s}$ is defined by

$$
\begin{equation*}
(-\Delta)^{s} u(x)=a_{N, s} \lim _{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2 s}} d y \tag{1.2}
\end{equation*}
$$

where $a_{N, s}=s(1-s) \pi^{-N / 2} 4^{s} \frac{\Gamma\left(\frac{N}{2}+s\right)}{\Gamma(2-s)}$ (see e.g. [14, Remark 3.11]). Let $\mathcal{L}_{s}^{1}$ denote the space of all functions $u: \mathbb{R}^{N} \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^{N}} \frac{|u(x)|}{1+|x|^{N+2 s}} d x<\infty$. If $\Omega \subset \mathbb{R}^{N}$ is an open subset and $g \in L_{l o c}^{1}(\Omega)$,

[^0]we say that a function $v \in \mathcal{L}_{s}^{1}$ solves the equation $(-\Delta)^{s} v=g$ in $\Omega$ in the sense of distributions if
$$
\int_{\mathbb{R}^{N}} v(-\Delta)^{s} \varphi d x=\int_{\Omega} g \varphi d x \quad \text { for every } \varphi \in C_{c}^{2}(\Omega)
$$

Note that the integral on the left hand side is well defined since

$$
\begin{equation*}
\left|(-\Delta)^{s} \varphi(x)\right| \leq \frac{\kappa\|\varphi\|_{C_{c}^{2}}}{1+|x|^{N+2 s}} \quad \text { for every } \varphi \in C_{c}^{2}(\Omega), x \in \mathbb{R}^{N} \tag{1.3}
\end{equation*}
$$

with a constant $\kappa=\kappa(N, s, \Omega)$ (see for instance [21]). The following is our first main result:
Theorem 1.1 Suppose that $f:[0, \infty) \rightarrow[0, \infty)$ is a nonnegative, nondecreasing and locally Lipschitz continuous function satisfying $f(t)>0$ for $t>0$ and

$$
\begin{equation*}
\lim _{\substack{r, t \rightarrow 0 \\ r \neq t}} \frac{f(r)-f(t)}{r-t}=0 \tag{1.4}
\end{equation*}
$$

Then every bounded solution $u$ of (1.1) is increasing in $x_{1}$. Moreover, either $u \equiv 0$, or $u$ is strictly increasing in $x_{1}$.

We note that, for $C^{1}$-nonlinearities, condition (1.4) simply amounts to $f^{\prime}(0)=0$. Our second main result is of Liouville type.

Theorem 1.2 If $N \leq 1+2 s$ and $q>1$ or $N>1+2 s$ and $1<q<\frac{N-1+2 s}{N-1-2 s}$, then the problem

$$
\begin{cases}(-\Delta)^{s} u=u^{q}, \quad u \geq 0 & \text { in } \mathbb{R}_{+}^{N},  \tag{1.5}\\ u=0 & \text { in } \mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}\end{cases}
$$

only admits the trivial solution $u \equiv 0$.
Our results complement the following recent Liouville type result of Jin, Li and Xiong [27] for the corresponding full space problem

$$
\begin{equation*}
(-\Delta)^{s} u=u^{q}, \quad u>0 \quad \text { in } \mathbb{R}^{N} \tag{1.6}
\end{equation*}
$$

Theorem 1.3 (see [27])
Suppose that $N \leq 2 s$ and $q>0$ or $N>2 s$ and $0<q<\frac{N+2 s}{N-2 s}$. Then (1.6) has no bounded solution.
We note that this result has also been obtained independently in 11 in the case $s \geq \frac{1}{2}$. Before that, the special case $s=\frac{1}{2}$ had been considered in [35], whereas in [16] the result was proved for a restricted class of solutions.

To put our results into perspective, some remarks are in order. Theorem 1.2 is an improvement of [21, Corollary 1.6], where the authors established nonexistence of a restricted class of solutions $u$ of (1.5) in the subcritical case $N>2 s$ and $1<q \leq \frac{N+2 s}{N-2 s}$. More precisely, in [21] we assumed that
$u$ is contained in the Sobolev space $\mathcal{D}^{s, 2}\left(\mathbb{R}_{+}^{N}\right)$ defined as the completion of $C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ with respect to the norm given by

$$
\|u\|^{2}=\int_{\mathbb{R}^{2} N} \frac{(u(x)-u(y))^{2}}{|x-y|^{N+2 s}} d x d y
$$

The argument of [21], relying on the method of moving spheres, does not apply under the assumptions of Theorem 1.2.
Theorem 1.1 is proved by a variant of the moving plane method based on extensions and modifications of techniques in the papers [3] resp. [39], which were devoted to second order and polyharmonic boundary value problems, respectively. The first key step in the argument is to show, without a priori integrability assumptions, that bounded solutions of (1.1) admit a Green function representation. This representation is obtained, via an approximation argument, from Green-Poisson type formulas in balls. Once the Green function representation is obtained, we carry out a moving plane argument for integral equations. We note that moving plane arguments for integral equations have been applied very successfully in recent years, see e.g. [4, 6, 15, 17, 39]. In the present situation, the lack of integrability assumptions creates additional difficulties which require to argue somewhat differently than in earlier papers.
Theorem 1.2 is deduced from Theorems 1.1 and 1.3 by considering the limits of solutions of (1.5) as $x_{1} \rightarrow \infty$, which, considered as functions of $\left(x_{2}, \ldots, x_{N}\right)$, solve (1.6) in $\mathbb{R}^{N-1}$.
The boundedness assumption in Theorem 1.2 can be replaced by only assuming boundedness in compact subsets of $\mathbb{R}_{+}^{N}$ if the assumption on $q$ is strengthened to $1<q<\frac{N+2 s}{N-2 s}$ in case $N>2 s$. This can be deduced from Theorems 1.2 and 1.3 by the argument used in [40, Section 4] for the polyharmonic version of (1.5). The argument is based on the doubling-lemma (see [36]).
The combination of the Liouville type results Theorem 1.2 and 1.3 are expected to give rise, via a Gidas-Spruck type rescaling argument (see [24]), to a priori bounds for solutions to more general integral equations in bounded domains and also to elliptic boundary value problems of second order with mixed nonlinear boundary conditions. For applications of this type, it is essential that Theorems 1.2 and 1.5 do not contain a priori integrability assumptions. This topic will be considered by the authors in a future work.
The combination of Liouville type results with rescaling arguments has already been applied successfully by Cabré and Tan [13] for nonlinear boundary value problems involving the spectral theoretic square root of the Dirichlet Laplacian, denoted by $\mathcal{A}_{1 / 2}$ in [13]. In particular, the analogue of Theorem 1.2 with $(-\Delta)^{s}$ replaced by $\mathcal{A}_{1 / 2}$ has been proved in [13, Theorem 1.5]. The subtle differences between $(-\Delta)^{s}$ and spectral theoretic powers of the Dirichlet Laplacian are discussed in [20, Remark 0.4] from a PDE point of view and in [44] in terms of stochastic processes. Because of these differences, it remains unclear whether Theorem 1.2 can also be obtained via similar methods as in [13]. The approach of the present paper is completely different. Moreover, the monotonicity result given by Theorem 1.1 is not available yet for the corresponding problem with spectral theoretic powers.

The paper is organized as follows. In Section 2 we collect preliminary results on (distributional) solutions of $(-\Delta)^{s} u=f$ on some open subset of $\mathbb{R}^{N}$ with bounded $f$. In Section 3 we show that bounded solutions $u$ of the problem $(-\Delta)^{s} u=f$ in $\mathbb{R}_{+}^{N}$ with $u \equiv 0$ in $\mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}$ admit a Green function representation whenever $f$ is bounded and nonnegative. In Section 4 we complete the proof of Theorem [1.1, and in Section 5 we complete the proof of Theorem [1.2. The appendix
contains a regularity result needed in the proof of Theorem 1.1.
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## 2 Preliminaries

Here and in the following, we consider $N \geq 1$ and $s \in(0,1)$. We write $\mathbf{B}=\left\{x \in \mathbb{R}^{N}:|x|<1\right\}$ for the open unit ball in $\mathbb{R}^{N}$ and set $B_{R}:=\left\{x \in \mathbb{R}^{N}:|x|<R\right\}$ for $R>0$. We start by recalling the following estimate, see [21.
Lemma 2.1 Let $\Omega \subset \mathbb{R}^{N}$ be a bounded open set. Then there exists a constant $C=C(N, s, \Omega)>0$ such that for all $\varphi \in C_{c}^{2}(\Omega)$ and for all $x \in \mathbb{R}^{N}$

$$
\begin{equation*}
\left|\int_{|x-y|>\varepsilon} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2 s}} d y\right| \leq \frac{C\|\varphi\|_{C^{2}(\Omega)}}{1+|x|^{N+2 s}} \quad \text { for all } \varepsilon \in(0,1) \tag{2.1}
\end{equation*}
$$

As a consequence, $(-\Delta)^{s} u$ can be defined for functions $u \in \mathcal{L}_{s}^{1}$ in the following way in distributional sense. Here we recall that $\mathcal{L}_{s}^{1}$ is the space of all functions $u: \mathbb{R}^{N} \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^{N}} \frac{|u(x)|}{1+|x|^{N+2 s}} d x<\infty$.

Definition 2.2 Let $\Omega$ be an open subset of $\mathbb{R}^{N}$. Given $u \in \mathcal{L}_{s}^{1}$, the distribution $(-\Delta)^{s} u \in \mathcal{D}^{\prime}(\Omega)$ is defined as

$$
\left\langle(-\Delta)^{s} u, \varphi\right\rangle=\int_{\mathbb{R}^{N}} u(-\Delta)^{s} \varphi d x \quad \text { for all } \varphi \in C_{c}^{\infty}(\Omega)
$$

The following result is contained in [8, Lemma 3.8].
Lemma 2.3 Let $\Omega \subset \mathbb{R}^{N}$ be open. If $u \in \mathcal{L}_{s}^{1} \cap C^{2}(\Omega)$, then the limit $\lim _{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2 s}} d y$ exists for every $x \in \Omega$. Moreover, $(-\Delta)^{s} u$ is a regular distribution given by

$$
\begin{equation*}
(-\Delta)^{s} u(x)=a_{N, s} \lim _{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2 s}} d y \quad \text { for } x \in \Omega . \tag{2.2}
\end{equation*}
$$

In the situation of this lemma, $(-\Delta)^{s} u$ is a priori only well defined a.e. in $\Omega$ as a regular distribution. Nevertheless, under these hypotheses, we may assume in the following that $(-\Delta)^{s} u$ is defined pointwise by (2.2) in all of $\Omega$.

Lemma 2.4 Let $\Omega \subset \mathbb{R}^{N}$ be open, and suppose that $u \in \mathcal{L}_{s}^{1} \cap C^{2}(\Omega)$ satisfies $(-\Delta)^{s} u \leq 0$ in $\Omega$. Suppose furthermore that there exists $x_{0} \in \Omega$ such that $u\left(x_{0}\right) \geq u(y)$ for a.e. $y \in \mathbb{R}^{N}$. Then $u(y)=u\left(x_{0}\right)$ for every $y \in \Omega$ and a.e. $y \in \mathbb{R}^{N} \backslash \Omega$.

Proof. Let $x_{0} \in \Omega$ satisfy $u\left(x_{0}\right) \geq u(y)$ for a.e. $y \in \mathbb{R}^{N}$. Then the function

$$
\varepsilon \rightarrow h(\varepsilon):=\int_{|x-y|>\varepsilon} \frac{u\left(x_{0}\right)-u(y)}{\left|x_{0}-y\right|^{N+2 s}} d y
$$

is nonnegative and nonincreasing in $(0, \infty)$, whereas $\lim _{\varepsilon \rightarrow 0} h(\varepsilon) \leq 0$ by assumption and (2.2). Hence $h \equiv 0$ in $(0, \infty)$, which, since $u \in C^{2}(\Omega)$, shows that $u(y)=u\left(x_{0}\right)$ for every $y \in \Omega$ and a.e. $y \in \mathbb{R}^{N} \backslash \Omega$.

Next we consider the Poisson kernel of $B_{R}$ (see [5]) which is given by

$$
\begin{equation*}
\Gamma_{R}(x, y)=C_{N, s}\left(\frac{R^{2}-|x|^{2}}{|y|^{2}-R^{2}}\right)^{s}|x-y|^{-N}, \quad|y|>R,|x|<R \tag{2.3}
\end{equation*}
$$

and $\Gamma_{R}(x, y)=0$ elsewhere. The constant $C_{N, s}$ is chosen such that

$$
\int_{\mathbb{R}^{N}} \Gamma_{R}(x, y) d y=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) d y=1 \quad \text { for every } x \in B_{R} .
$$

Lemma 2.5 Let $R>0, g \in \mathcal{L}_{s}^{1}$, and suppose that $g$ is bounded in a neighborhood of $B_{R}$. Then the problem

$$
\left\{\begin{align*}
(-\Delta)^{s} u=0 & \text { in } B_{R}  \tag{2.4}\\
u=g & \text { in } \mathbb{R}^{N} \backslash B_{R}
\end{align*}\right.
$$

has a unique solution $u \in \mathcal{L}_{s}^{1} \cap C^{2}\left(B_{R}\right) \cap L^{\infty}\left(B_{R}\right)$ given by

$$
\begin{equation*}
u(x)=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) g(y) d y \quad \text { for } x \in B_{R} \tag{2.5}
\end{equation*}
$$

Moreover, $u \in C^{\infty}\left(B_{R}\right)$.
Proof. Let $u: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be defined by $u \equiv g$ in $\mathbb{R}^{N} \backslash B_{R}$ and by (2.5) in $B_{R}$. Using the explicit representation (2.3) and the assumption that $g \in \mathcal{L}_{s}^{1}$ is bounded in a neighborhood of $B_{R}$, it is easy to see that $u \in \mathcal{L}_{s}^{1} \cap C^{\infty}\left(B_{R}\right) \cap L^{\infty}\left(B_{R}\right)$, and that

$$
(-\Delta)^{s} u(x)=a_{N, s} \lim _{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{N+2 s}} d y=0 \quad \text { for all } x \in B_{R} .
$$

Conversely, let $u \in \mathcal{L}_{s}^{1} \cap C^{2}\left(B_{R}\right) \cap L^{\infty}\left(B_{R}\right)$ satisfy (2.4). Let $r \in(0, R)$ and define $v_{r} \in C^{2}\left(B_{r}\right) \cap$ $L^{\infty}\left(\mathbb{R}^{N}\right)$ by $v_{r} \equiv u$ in $\mathbb{R}^{N} \backslash B_{r}$ and

$$
\begin{equation*}
v_{r}(x)=\int_{\mathbb{R}^{N} \backslash B_{r}} \Gamma_{r}(x, y) u(y) d y \quad \text { for } x \in B_{r} . \tag{2.6}
\end{equation*}
$$

Since $u$ is continuous in $B_{R}$, it is not difficult to see from (2.6) that $v_{r} \in C\left(B_{R}\right)$. Applying Lemma 2.4 to $\Omega=B_{r}$ and the functions $u-v_{r}, v_{r}-u$ which are continuous on $\mathbb{R}^{N}$ and vanish on $\mathbb{R}^{N} \backslash B_{r}$, we infer that $u \equiv v_{r}$. Passing to the limit $r \mapsto R^{-}$and using the fact that $u$ is bounded in a neighborhood of $B_{R}$, we conclude that

$$
u(x)=\lim _{r \mapsto R^{-}} \int_{\mathbb{R}^{N} \backslash B_{r}} \Gamma_{r}(x, y) u(y) d y=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) u(y) d y=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) g(y) d y,
$$

as claimed.
Next, we wish to remove the $C^{2}$-assumption in Lemma 2.4. For this we consider the regularization of $\Gamma_{R}$ as defined in [8]. Let $\chi \in C_{c}^{\infty}(1 / 2,1)$ such that $\int_{1 / 2}^{1} \chi(r) d r=1$ and define

$$
\tilde{\Gamma}: \mathbb{R}^{N} \rightarrow \mathbb{R}, \quad \tilde{\Gamma}(y)=\int_{1 / 2}^{1} \chi(r) \Gamma_{r}(0, y) d r .
$$

We have $\tilde{\Gamma} \in C^{\infty}\left(\mathbb{R}^{N}\right)$ and

$$
\begin{equation*}
|\tilde{\Gamma}(y)| \leq \frac{C_{N, s}}{1+|y|^{N+2+2 s}} \quad \text { for } y \in \mathbb{R}^{N} \tag{2.7}
\end{equation*}
$$

with a constant $C_{N, s}$, see for instance [8, Lemma 3.11]. For $\varepsilon>0$, we define

$$
\tilde{\Gamma}_{\varepsilon}: \mathbb{R}^{N} \rightarrow \mathbb{R}, \quad \tilde{\Gamma}_{\varepsilon}(x)=\varepsilon^{-N} \tilde{\Gamma}(x / \varepsilon) .
$$

The following result is given in [8] Theorem 3.12] for the case $N \geq 2$, but the same proof also gives the result for $N=1$. For the convenience of the reader, we include the proof here.

Theorem 2.6 Let $\Omega \subset \mathbb{R}^{N}$ be open, and let $u \in \mathcal{L}_{s}^{1}$ satisfy $(-\Delta)^{s} u=0$ in $\Omega$, i.e.,

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} u(-\Delta)^{s} \psi d x=0 \quad \text { for all } \psi \in C_{c}^{\infty}(\Omega) \tag{2.8}
\end{equation*}
$$

Then for every $\varepsilon>0$ we have

$$
\begin{equation*}
u \equiv \tilde{\Gamma}_{\varepsilon} * u \quad \text { a.e. in } \Omega_{\varepsilon} \tag{2.9}
\end{equation*}
$$

where $\Omega_{\varepsilon}:=\{x \in \Omega: \operatorname{dist}(x, \partial \Omega)>\varepsilon\}$. In particular, $u$ is equivalent to a $C^{\infty}$-function in $\Omega$.
Proof. We first remark that, by Lemma (2.5) and Fubini's theorem, the equality (2.9) holds under the additional assumption $u \in C^{2}(\Omega)$. Let $\rho_{n} \in C_{c}^{\infty}\left(B_{\frac{1}{n}}\right)$ denote the standard radially symmetric mollifier for $n \in \mathbb{N}$. For every $\psi \in C_{c}^{\infty}\left(\Omega_{\frac{1}{n}}\right)$, we then have $\rho_{n} *(-\Delta)^{s} \psi=(-\Delta)^{s}\left[\rho_{n} * \psi\right]$ in $\mathbb{R}^{N}$ and therefore, by Fubini's theorem and (2.8),

$$
\int_{\mathbb{R}^{N}}\left[\rho_{n} * u\right](-\Delta)^{s} \psi d x=\int_{\mathbb{R}^{N}} u\left[\rho_{n} *(-\Delta)^{s} \psi\right] d y=\int_{\mathbb{R}^{N}} u(-\Delta)^{s}\left[\rho_{n} * \psi\right] d y=0 .
$$

Letting $u_{n}=\rho_{n} * u \in C^{\infty}\left(\mathbb{R}^{N}\right)$ for $n \in \mathbb{N}$, we deduce that $(-\Delta)^{s} u_{n} \equiv 0$ in $\Omega_{\frac{1}{n}}$. By the remark above, we thus have $u_{n} \equiv \tilde{\Gamma}_{\varepsilon} * u_{n}$ in $\Omega_{\varepsilon+\frac{1}{n}}$. By (2.7), we now may pass to the limit $n \rightarrow \infty$ to get $u \equiv \tilde{\Gamma}_{\varepsilon} * u$ a.e. in $\Omega_{\varepsilon}$, as claimed.

Corollary 2.7 Let $\Omega \subset \mathbb{R}^{N}$ be open and $u \in \mathcal{L}_{s}^{1} \cap C(\Omega)$ such that $(-\Delta)^{s} u=0$ in $\Omega$. Then if $u$ attains it's maximum or it's minimum in $\Omega$ it is constant.

Proof. By Theorem 2.6 and the continuity of $u$, we have $u \in C^{\infty}(\Omega)$, so the result follows from Lemma 2.4.

We finally obtain the following result which improves Lemma 2.5,

Lemma 2.8 Let $R, \delta>0$, and let $u \in \mathcal{L}_{s}^{1} \cap L^{\infty}\left(B_{R+\delta}\right)$ satisfy $(-\Delta)^{s} u=0$ in $B_{R}$. Then

$$
u(x)=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) u(y) d y \quad \text { for a.e. } x \in B_{R}
$$

Moreover, $u$ is equivalent to a $C^{\infty}$-function in $B_{R}$.
Proof. By Theorem [2.6 we may assume that $u \in C^{\infty}\left(B_{R}\right)$. Hence the result follows from Lemma 2.5,

Next we consider the Green function associated with $(-\Delta)^{s}$ and the unit ball B, which was computed by Blumenthal, Getoor and Ray in [5]. It is given by

$$
\begin{aligned}
G_{1}(x, y) & =k_{N}^{s}|x-y|^{2 s-N} \int_{1}^{(\psi(x, y)+1)^{1 / 2}} \frac{\left(z^{2}-1\right)^{s-1}}{z^{N-1}} d z \\
& =\frac{k_{N}^{s}}{2}|x-y|^{2 s-N} \int_{0}^{\psi(x, y)} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \text { with } \psi(x, y)=\frac{\left(1-|x|^{2}\right)\left(1-|y|^{2}\right)}{|x-y|^{2}}
\end{aligned}
$$

for $x, y \in \mathbf{B}$ and $G(x, y)=0$ if $x \notin \mathbf{B}$ or $y \notin B$. Here the normalization constant is given by $k_{N}^{s}=\pi^{-(N / 2+1)} \Gamma(N / 2) \sin (\pi s)$, see [5]. If $N=1=2 s$, then direct computations give

$$
\int_{0}^{\psi(x, y)} \frac{z^{-1 / 2}}{(z+1)^{1 / 2}} d z=2 \log \frac{1-x y+\left(1-x^{2}\right)^{1 / 2}\left(1-y^{2}\right)^{1 / 2}}{|x-y|}
$$

and yet, see (5), in this case

$$
\begin{equation*}
G_{1}(x, y)=\frac{1}{\pi} \log \frac{1-x y+\left(1-x^{2}\right)^{1 / 2}\left(1-y^{2}\right)^{1 / 2}}{|x-y|} . \tag{2.10}
\end{equation*}
$$

The explicit form of $G_{1}$ gives rise to the following estimates for $x, y \in B$ :

$$
G_{\Omega}(x, y) \leq \begin{cases}C|x-y|^{2 s-N} \min \left(\frac{d^{s}(x) d^{s}(y)}{|x-y|^{2 s}}, 1\right) & \text { if } N>2 s ;  \tag{2.11}\\ C \min \left(\frac{d^{1 / 2}(x) d^{1 / 2}(y)}{|x-y|}, \log \frac{3}{|x-y|}\right) \quad \text { if } N=1=2 s ; \\ C|x-y|^{2 s-1} \min \left(\frac{(d(x) d(y))^{(2 s-1) / 2}}{|x-y|^{2 s-1}}, \frac{d^{s}(x) d^{s}(y)}{|x-y|^{2 s}}\right) \quad \text { if } N=1<2 s\end{cases}
$$

Here $z \mapsto d(z)=1-|z|$ is the distance function to $\mathbb{R}^{N} \backslash B$, and $C$ is a constant depending on $N$ and $s$. Similar estimates are available for Greens functions in general $C^{1,1}$-domains, see e.g. [18, 28]. By dilation, the Green function for the ball $B_{R}=\left\{x \in \mathbb{R}^{N}:|x|<R\right\}, R>0$ is given by

$$
\begin{equation*}
G_{R}(x, y)=R^{2 s-N} G_{1}\left(\frac{x}{R}, \frac{y}{R}\right)=\frac{k_{N}^{s}}{2}|x-y|^{2 s-N} \int_{0}^{\psi_{R}(x, y)} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \tag{2.12}
\end{equation*}
$$

with $\psi_{R}(x, y)=\frac{\left(R^{2}-|x|^{2}\right)\left(R^{2}-|y|^{2}\right)}{R^{2}|x-y|^{2}}$. In the next section, we will need the following general GreenPoisson representation formula.

Corollary 2.9 Let $R, \delta>0$ and $f \in L^{\infty}\left(B_{R}\right)$. Moreover, let $v \in \mathcal{L}_{s}^{1} \cap L^{\infty}\left(B_{R+\delta}\right)$ satisfy $(-\Delta)^{s} v=$ $f$ in $B_{R}$. Then

$$
\begin{equation*}
v(x)=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) v(y) d y+\int_{B_{R}} G_{R}(x, y) f(y) d y \quad \text { for a.e. } x \in B_{R} . \tag{2.13}
\end{equation*}
$$

Proof. Without loss of generality, we may assume that $R=1$. Consider

$$
w_{0}: \mathbb{R}^{N} \rightarrow \mathbb{R}, \quad w_{0}(x)=\int_{\mathbf{B}} G_{1}(x, y) f(y) d y
$$

The estimates (2.11) imply that $w_{0} \in L^{\infty}\left(\mathbb{R}^{N}\right)$. Moreover, $(-\Delta)^{s} w_{0}=f$ in $\mathbf{B}$ in distributional sense, since for any $\varphi \in C_{c}^{\infty}(\mathbf{B})$ we have

$$
\varphi(x)=\int_{\mathbf{B}} G_{1}(x, y)\left[(-\Delta)^{s} \varphi\right](y) d y \quad \text { for } x \in \mathbf{B}
$$

We now consider $w:=v-w_{0} \in \mathcal{L}_{s}^{1} \cap L^{\infty}\left(B_{R+\delta}\right)$. Clearly $(-\Delta)^{s} w=0$ in $B_{R}$, and $w \equiv v$ on $\mathbb{R}^{N} \backslash B_{R}$. By Lemma 2.8, we have $w(x)=\int_{\mathbb{R}^{N} \backslash B_{R}} \Gamma_{R}(x, y) v(y) d y$ for a.e. $x \in B_{R}$, and thus (2.13) follows.
We finally add the following boundary estimate.
Lemma 2.10 Let $f \in L^{\infty}(\mathbf{B})$ and consider

$$
v: \mathbb{R}^{N} \rightarrow \mathbb{R}, \quad v(x)=\int_{\mathbf{B}} G_{1}(x, y) f(y) d y .
$$

Then there exists a constant $C=C(N, s)>0$ such that for $x \in \mathbf{B}$ we have

$$
v(x) \leq C \begin{cases}(1-|x|)^{s}\|f\|_{L^{\infty}} & \text { if } 2 s \leq N  \tag{2.14}\\ (1-|x|)^{s-\frac{1}{2}}\|f\|_{L^{\infty}} & \text { if } N=1<2 s\end{cases}
$$

Proof. The second inequality in (2.14) is an immediate consequence of the third inequality in (2.11). To prove the first inequality in (2.14), we let $d(x)=1-|x|$ and $B^{x}:=\left\{y \in \mathbb{R}^{N}:|y-x|<\frac{d(x)}{2}\right\} \subset \mathbf{B}$ for $x \in \mathbf{B}$. We then have

$$
\begin{equation*}
v(x)=\int_{B^{x}} G_{1}(x, y) f(y) d y+\int_{\mathbf{B} \backslash B^{x}} G(x, y) f(y) d y \quad \text { for } x \in \mathbf{B} \tag{2.15}
\end{equation*}
$$

and

$$
\begin{equation*}
d(y) \leq|x-y|+d(x) \leq 3|x-y| \quad \text { for } y \in \mathbb{R}^{N} \backslash B^{x} \tag{2.16}
\end{equation*}
$$

In the following, the letter $C$ stands for positive constants depending only on $N$ and $s$. We first consider the case $N>2 s$. Then the first inequality in (2.11) implies that

$$
\left|\int_{B^{x}} G_{1}(x, y) f(y) d y\right| \leq C\|f\|_{L^{\infty}} \int_{B^{x}}|x-y|^{2 s-N} d y \leq C d^{2 s}(x)\|f\|_{L^{\infty}}
$$

and together with (2.16) it also yields

$$
\begin{array}{r}
\left|\int_{\mathbf{B} \backslash B^{x}} G(x, y) f(y) d y\right| \leq C d^{s}(x)\|f\|_{L^{\infty}} \int_{\mathbf{B} \backslash B^{x}} \frac{d^{s}(y)}{|x-y|^{N}} d y \leq C d^{s}(x)\|f\|_{L^{\infty}} \int_{\mathbf{B}}|x-y|^{s-N} d y \\
\leq C d^{s}(x)\|f\|_{L^{\infty}} \int_{\mathbf{B}}|y|^{s-N} d y=C d^{s}(x)\|f\|_{L^{\infty}} .
\end{array}
$$

Combining these two inequalities, we obtain the assertion in the case $N>2 s$.
Next we consider the case $N=1=2 s$. Applying the second estimate in (2.11) yields

$$
\left|\int_{B^{x}} G_{1}(x, y) f(y) d y\right| \leq C\|f\|_{L^{\infty}} \int_{B^{x}} \log \frac{3}{|x-y|} d y \leq C\|f\|_{L^{\infty}} \int_{0}^{\frac{d(x)}{2}} \log \frac{3}{t} d t \leq C d^{1 / 2}(x)\|f\|_{L^{\infty}}
$$

and, as before, using (2.16)

$$
\begin{aligned}
\left|\int_{\mathbf{B} \backslash B^{x}} G_{1}(x, y) f(y) d y\right| \leq C d^{1 / 2}(x)\|f\|_{L^{\infty}} & \int_{\mathbf{B} \backslash B^{x}} \frac{d^{1 / 2}(y)}{|x-y|} d y \leq C d^{1 / 2}(x)\|f\|_{L^{\infty}} \int_{\mathbf{B} \backslash B^{x}}|x-y|^{-\frac{1}{2}} d y \\
& \leq C d^{1 / 2}(x)\|f\|_{L^{\infty}} \int_{0}^{1} t^{-\frac{1}{2}} d t=C d^{1 / 2}(x)\|f\|_{L^{\infty}} .
\end{aligned}
$$

Combining these two inequalities, we obtain the assertion in the case $N=1=2 s$.

## 3 Green representation on the half-space

The purpose of this section is to state conditions on a function $u$ on the half-space $\mathbb{R}_{+}^{N}$ under which the Green representation formula

$$
u(x)=\int_{\mathbb{R}_{+}^{N}} G_{\infty}^{+}(x, y)(-\Delta)^{s} u(y) d y, \quad x \in \mathbb{R}_{+}^{N}
$$

holds, where $G_{\infty}^{+}$is the half space Green function given by

$$
\begin{equation*}
G_{\infty}^{+}(x, y)=\frac{k_{N}^{s}}{2}|x-y|^{2 s-N} \int_{0}^{\psi_{\infty}(x, y)} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \quad \text { with } \quad \psi_{\infty}(x, y)=\frac{4 x_{1} y_{1}}{|x-y|^{2}} \tag{3.1}
\end{equation*}
$$

for $x, y \in \mathbb{R}_{+}^{N}$. More precisely, we have the following
Theorem 3.1 Let $f \in L^{\infty}\left(\mathbb{R}_{+}^{N}\right)$ be nonnegative, and suppose that $u \in L^{\infty}\left(\mathbb{R}^{N}\right)$ satisfies

$$
(-\Delta)^{s} u=f \quad \text { in } \mathbb{R}_{+}^{N}, \quad u=0 \quad \text { in } \mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}
$$

Then $u$ is continuous, and

$$
\begin{equation*}
u(x)=\int_{\mathbb{R}_{+}^{N}} G_{\infty}^{+}(x, y) f(y) d y \quad \text { for every } x \in \mathbb{R}_{+}^{N} \tag{3.2}
\end{equation*}
$$

Moreover, there exist constants $C>0, \alpha \in(0,1)$ depending only on $N, s,\|f\|_{L^{\infty}},\|u\|_{L^{\infty}}$ such that

$$
\begin{equation*}
0 \leq u(x) \leq C x_{1}^{\alpha} \quad \text { for every } x \in \mathbb{R}_{+}^{N} \tag{3.3}
\end{equation*}
$$

The remainder of this section will be devoted to the proof of this Theorem. We first show how $G_{\infty}^{+}$, as defined in (3.1), arises via an approximation with balls. For this we let $P_{R}:=(R, 0, \ldots, 0) \in \mathbb{R}_{+}^{N}$, and we consider the translated ball $B_{R}^{+}:=\left\{x \in \mathbb{R}^{N}:\left|x-P_{R}\right|<R\right\} \subset \mathbb{R}_{+}^{N}$ for $R>0$. By (2.12), its Green function $G_{R}^{+}$is given by

$$
G_{R}^{+}(x, y)=\frac{k_{N}^{s}}{2}|x-y|^{2 s-N} \int_{0}^{\psi_{R}^{+}(x, y)} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \quad \text { for } x, y \in B_{R}^{+}
$$

with

$$
\begin{equation*}
\psi_{R}^{+}(x, y)=\frac{\left(R^{2}-\left|x-P_{R}\right|^{2}\right)\left(R^{2}-\left|y-P_{R}\right|^{2}\right)}{R^{2}|x-y|^{2}}=\frac{\left(2 x_{1}-\frac{|x|^{2}}{R}\right)\left(2 y_{1}-\frac{|y|^{2}}{R}\right)}{|x-y|^{2}}, \tag{3.4}
\end{equation*}
$$

and its Poisson kernel is given by

$$
\begin{equation*}
\Gamma_{R}^{+}(x, y)=C_{N, s}\left(\frac{R^{2}-\left|x-P_{R}\right|^{2}}{\left|y-P_{R}\right|^{2}-R^{2}}\right)^{s}|x-y|^{-N} \quad \text { for }|x|<R<|y| . \tag{3.5}
\end{equation*}
$$

Note that

$$
\begin{equation*}
B_{R}^{+} \subset B_{R^{\prime}}^{+} \quad \text { if } R^{\prime}>R>0 \quad \text { and } \quad \bigcup_{R>0} B_{R}^{+}=\mathbb{R}_{+}^{N} \tag{3.6}
\end{equation*}
$$

Lemma 3.2 Let $x, y \in \mathbb{R}_{+}^{N}$ and $R_{0}>0$ with $x, y \in B_{R_{0}}^{+}$. Then $G_{R^{\prime}}^{+}(x, y) \geq G_{R}^{+}(x, y)$ for $R^{\prime} \geq R \geq$ $R_{0}$ and $G_{R}^{+}(x, y) \rightarrow G_{\infty}^{+}(x, y)$ as $R \rightarrow \infty$.

Proof. From (3.4) we immediately deduce that $\psi_{R^{\prime}}^{+}(x, y) \geq \psi_{R}^{+}(x, y)$ for $R^{\prime} \geq R \geq R_{0}$, and that $\psi_{R}^{+}(x, y) \rightarrow \psi_{\infty}(x, y)$ as $R \rightarrow \infty$. This shows the claim.

We may now complete the
Proof of Theorem 3.1. It follows from Lemma 6.1 below that $u$ is continuous in $\mathbb{R}_{+}^{N}$. Since $G_{\infty}^{+}$ is invariant under translations of the form $(x, y) \mapsto(x+z, y+z)$ with $z \in\{0\} \times \mathbb{R}^{N-1}$, it suffices to show (3.2) for $x=\left(x_{1}, 0, \ldots, 0\right)$ with $x_{1}>0$. We will fix such a point $x$ from now on. By Corollary 2.9 and the fact that $u \equiv 0$ in $\mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}$, we have

$$
u(x)=\int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}} \Gamma_{R}^{+}(x, y) u(y) d y+\int_{B_{R}^{+}} G_{R}^{+}(x, y) f(y) d y \quad \text { for } x \in B_{R}^{+}
$$

Thanks to Lemma 3.2, the nonnegativity of $f$ and monotone convergence, (3.2) follows once we have shown that

$$
\begin{equation*}
\int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}} \Gamma_{R}^{+}(x, y) u(y) d y \rightarrow 0 \quad \text { as } R \rightarrow \infty \tag{3.7}
\end{equation*}
$$

In the following, we assume that $R>2 x_{1}$, and we let $C$ denote (possibly different) constants which may depend on $N, s, u$ and $x_{1}$ but not on $R$. Using the fact that $u$ is bounded, we have

$$
\begin{align*}
& \left|\int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}} \Gamma_{R}^{+}(x, y) u(y) d y\right| \leq C \int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}} \Gamma_{R}^{+}(x, y) d y=C \int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}}\left(\frac{R^{2}-\left|x-P_{R}\right|^{2}}{\left|y-P_{R}\right|^{2}-R^{2}}\right)^{s}|x-y|^{-N} d y \\
& \quad=C\left(2 x_{1} R-x_{1}^{2}\right)^{s} \int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}}|x-y|^{-N}\left(|y|^{2}-2 y_{1} R\right)^{-s} d y \\
& 3.8) \quad \leq C R^{s} \int_{\mathbb{R}_{+}^{N} \backslash B_{R}^{+}}|x-y|^{-N}\left(|y|^{2}-2 y_{1} R\right)^{-s} d y=C R^{-s} \int_{\mathbb{R}_{+}^{N} \backslash B_{1}^{+}}\left|y-\frac{x}{R}\right|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y . \tag{3.8}
\end{align*}
$$

We will now show that, as $R \rightarrow \infty$,

$$
\int_{\mathbb{R}_{+}^{N} \backslash B_{1}^{+}}\left|y-\frac{x}{R}\right|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y= \begin{cases}O(1) & \text { if } 0<s<\frac{1}{2}  \tag{3.9}\\ O\left(R^{2 s-1}\right) & \text { if } \frac{1}{2}<s<1 \\ O\left(\log R^{2}\right) & \text { if } s=\frac{1}{2}\end{cases}
$$

Together with (3.8) this implies (3.7), since $s<1$. To show (3.9), we decompose the domain of integration as

$$
R_{+}^{\mathbb{N}} \backslash B_{1}^{+}=A_{1} \cup A_{2} \quad \text { with } \quad A_{1}:=\left((0,1) \times \mathbb{R}^{N-1}\right) \backslash B_{1}^{+}, \quad A_{2}:=\left([1, \infty) \times \mathbb{R}^{N-1}\right) \backslash B_{1}^{+} .
$$

We then have $\left|y-\frac{x}{R}\right| \geq\left|y-e_{1}\right|$ for every $y \in A_{2}$, where $e_{1}=(1,0, \ldots, 0)$, and therefore

$$
\begin{align*}
& \int_{A_{2}}\left|y-\frac{x}{R}\right|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y \leq \int_{A_{2}}\left|y-e_{1}\right|^{-N}\left(\left|y-e_{1}\right|^{2}-1\right)^{-s} d y \\
& \leq \int_{\mathbb{R}^{N} \backslash B_{1}(0)}|y|^{-N}\left(|y|^{2}-1\right)^{-s} d y=\omega_{N-1} \int_{1}^{\infty} \tau^{-1}\left(\tau^{2}-1\right)^{-s} d \tau<\infty \tag{3.10}
\end{align*}
$$

In case $N=1, A_{1}$ is empty, and thus (3.9) follows. We now assume that $N \geq 2$, and we put $B_{t}:=\left\{\tilde{y} \in \mathbb{R}^{N-1}:|\tilde{y}|^{2} \geq 2 t-t^{2}\right\}$ for $t \in(0,1)$. By Fubini's theorem, we have

$$
\begin{aligned}
\int_{A_{1}} & \left|y-\frac{x}{R}\right|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y=\int_{0}^{1} \int_{B_{t}}\left(|\tilde{y}|^{2}+\left(t-\frac{x_{1}}{R}\right)^{2}\right)^{-\frac{N}{2}}\left(|\tilde{y}|^{2}+t^{2}-2 t\right)^{-s} d \tilde{y} d t \\
& =\omega_{N-2} \int_{0}^{1} \int_{\sqrt{2 t-t^{2}}}^{\infty} \tau^{N-2}\left(\tau^{2}+\left(t-\frac{x_{1}}{R}\right)^{2}\right)^{-\frac{N}{2}}\left(\tau^{2}-\left[2 t-t^{2}\right]\right)^{-s} d \tau d t \\
& =\omega_{N-2} \int_{0}^{1}\left(2 t-t^{2}\right)^{-\frac{1}{2}-s} \int_{1}^{\infty} \sigma^{N-2}\left(\sigma^{2}+\frac{\left(t-\frac{x_{1}}{R}\right)^{2}}{2 t-t^{2}}\right)^{-\frac{N}{2}}\left(\sigma^{2}-1\right)^{-s} d \sigma d t \\
& \leq \omega_{N-2} \int_{0}^{1} t^{-\frac{1}{2}-s} \int_{0}^{\infty}(\rho+1)^{\frac{N-3}{2}}\left(\rho+1+\frac{\left(t-\frac{x_{1}}{R}\right)^{2}}{2 t}\right)^{-\frac{N}{2}} \rho^{-s} d \rho d t \\
& \leq \omega_{N-2} \int_{0}^{1} t^{-\frac{1}{2}-s}\left(1+\frac{\left(t-\frac{x_{1}}{R}\right)^{2}}{2 t}\right)^{-\frac{1}{2}} \int_{0}^{\infty}(\rho+1)^{-1} \rho^{-s} d \rho d t \\
& \leq C \int_{0}^{1} t^{-\frac{1}{2}-s}\left(1+\frac{\left(t-\frac{x_{1}}{R}\right)^{2}}{2 t}\right)^{-\frac{1}{2}} d t,
\end{aligned}
$$

whereas

$$
\left(1+\frac{\left(t-\frac{x_{1}}{R}\right)^{2}}{2 t}\right)^{-\frac{1}{2}} \leq \sqrt{2 t}\left(2 t+\left(t-\frac{x_{1}}{R}\right)^{2}\right)^{-\frac{1}{2}} \leq \min \left\{1, \frac{\sqrt{2 t} R}{x_{1}}\right\} \quad \text { for } t \in(0,1)
$$

since $\frac{x_{1}}{R} \leq 1$ by assumption. Inserting this in (3.11) yields
$\int_{A_{1}}\left|y-\frac{x}{R}\right|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y \leq C\left(\frac{\sqrt{2} R}{x_{1}} \int_{0}^{\frac{x_{1}^{2}}{2 R^{2}}} t^{-s} d s+\int_{\frac{x_{1}^{2}}{2 R^{2}}}^{1} t^{-\frac{1}{2}-s} d t\right) \leq C \begin{cases}1+R^{2 s-1} & \text { if } s \neq \frac{1}{2} ; \\ 1+\log R^{2} & \text { if } s=\frac{1}{2} .\end{cases}$
Combining the last inequality with (3.10), we deduce (3.9), as required. Thus the proof of (3.2) is finished.
To show (3.3), we may assume without loss that $x=\left(x_{1}, 0, \ldots, 0\right)$ with $0 \leq x_{1} \leq 1$. Moreover, we let $C>0$ denote constants depending on $N, s,\|f\|_{L^{\infty}\left(\mathbb{R}_{+}^{N}\right)}$ and $\|u\|_{L^{\infty}\left(\mathbb{R}_{+}^{N}\right)}$ but not on $x$. By Corollary 2.9, we have

$$
u(x)=\int_{\mathbb{R}^{N} \backslash B_{1}^{+}} \Gamma_{1}^{+}(x, y) u(y) d y+v(x) \quad \text { with } \quad v(x)=\int_{B_{1}^{+}} G_{1}^{+}(x, y) f(y) d y
$$

By Lemma 2.10, it suffices to show (3.3) for $v$ in place of $u$. For this we estimate, similarly as in (3.8),

$$
\begin{align*}
0 \leq v(x) \leq C \int_{\mathbb{R}_{+}^{N} \backslash B_{1}^{+}} \Gamma_{1}^{+}(x, y) d y & \leq C\left(2 x_{1}-x_{1}^{2}\right)^{s} \int_{\mathbb{R}_{+}^{N} \backslash B_{1}^{+}}|x-y|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y \\
& \leq C x_{1}^{s} \int_{\mathbb{R}_{+}^{N} \backslash B_{1}^{+}}|y-x|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y . \tag{3.12}
\end{align*}
$$

Replacing $\frac{x}{R}$ with $x=\left(x_{1}, 0, \ldots, 0\right)$ in (3.9), we also have, as $x_{1} \rightarrow 0^{+}$,

$$
\int_{\mathbb{R}_{+}^{N} \backslash B_{1}^{+}}|y-x|^{-N}\left(|y|^{2}-2 y_{1}\right)^{-s} d y= \begin{cases}O(1) & \text { if } 0<s<\frac{1}{2}  \tag{3.13}\\ O\left(x_{1}^{1-2 s}\right) & \text { if } \frac{1}{2}<s<1 \\ O\left(\log \frac{1}{x_{1}^{2}}\right) & \text { if } s=\frac{1}{2}\end{cases}
$$

Combining (3.12) and (3.13) yields constants $C>0, \alpha \in(0,1)$ such that (3.3) holds for $v$ in place of $u$, and this finishes the proof.

## 4 Proof of the monotonicity result

In this section we complete the proof of Theorem 1.1. We know from Theorem 3.1 that every bounded solution $u$ of (1.1) obeys the integral representation

$$
u(x)=\int_{\mathbb{R}_{+}^{N}} G_{\infty}^{+}(x, y) f(u(y)) d y \quad \text { for all } x \in \mathbb{R}_{+}^{N},
$$

where $G_{\infty}^{+}$is the half-space Green function given by (3.1). We note the following simple estimate.

Lemma 4.1 Let $L>0$. Then there exists a constant $C=C(N, s, L)>0$ such that for every $x, y \in \mathbb{R}_{+}^{N}$ with $x_{1}, y_{1} \leq L$ we have

$$
G_{\infty}^{+}(x, y) \leq C \begin{cases}\min \left\{|x-y|^{2 s-N},|x-y|^{-N}\right\} & \text { if } N>2 s  \tag{4.1}\\ 1 & \text { if } N=1<2 s \\ 1+\log \frac{L}{|x-y|} & \text { if } N=1=2 s\end{cases}
$$

Proof. In the case $N>2 s$ we have $\int_{0}^{\infty} \frac{z^{s-1}}{(z+1)^{N / 2}} d z<\infty$, which immediately implies that $G_{\infty}^{+}(x, y) \leq$ $C|x-y|^{2 s-N}$ for all $x, y \in \mathbb{R}_{+}^{N}$ with a constant $C>0$ depending only on $N$ and $s$. Moreover, for $t>0$ we have

$$
\int_{0}^{t} \frac{z^{s-1}}{(z+1)^{N / 2}} d z<\frac{t^{s}}{s}
$$

and

$$
\int_{0}^{t} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \leq \begin{cases}\frac{t^{s-N / 2}}{s-N / 2} & \text { if } N<2 s \\ \frac{1}{s}+\log \max \{1, t\} & \text { if } N=2 s\end{cases}
$$

For $x, y \in \mathbb{R}_{+}^{N}$ with $x_{1}, y_{1} \leq L$ we have also $\psi_{\infty}(x, y) \leq \frac{L^{2}}{|x-y|^{2}}$ and therefore

$$
\int_{0}^{\psi_{\infty}(x, y)} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \leq C|x-y|^{-2 s}
$$

as well as

$$
\int_{0}^{\psi_{\infty}(x, y)} \frac{z^{s-1}}{(z+1)^{N / 2}} d z \leq C \begin{cases}|x-y|^{N-2 s} & \text { if } N<2 s \\ 1+\ln \frac{L}{|x-y|} & \text { if } N<2 s\end{cases}
$$

with a constant $C>0$ depending only on $N, s, L$. This readily implies the assertion.
We need some notation. For $\lambda \geq 0$, we consider the set

$$
\Sigma_{\lambda}=\left\{x \in \mathbb{R}^{N}: 0<x_{1}<\lambda\right\} \subset \mathbb{R}_{+}^{N}
$$

From Lemma 4.1 we easily deduce the following.
Corollary 4.2 As $\lambda \rightarrow 0, \sup _{x \in \Sigma_{\lambda}} \int_{\Sigma_{\lambda}} G_{\infty}^{+}(x, y) d y \rightarrow 0$.
We also consider the reflection $x \mapsto x^{\lambda}:=\left(2 \lambda-x_{1}, x_{2}, \ldots, x_{N}\right)$ at the hyperplane $\left\{x_{1}=\lambda\right\}$. We need the following fact which also follows from Lemma 4.1 in a straightforward way.

Corollary 4.3 If $\left(\lambda_{n}\right)_{n} \subset(0, \infty)$ and $\left(z^{n}\right)_{n} \subset \mathbb{R}_{+}^{N}$ are sequences with

$$
\lambda_{n} \rightarrow \lambda>0 \quad \text { and } \quad z^{n} \rightarrow z \in \mathbb{R}_{+}^{N} \quad \text { as } n \rightarrow \infty
$$

then

$$
\int_{\Sigma_{\lambda_{n}}} G_{\infty}^{+}\left(z^{n}, y^{\lambda_{n}}\right) d y \rightarrow \int_{\Sigma_{\lambda}} G_{\infty}^{+}\left(z, y^{\lambda}\right) d y \quad \text { as } n \rightarrow \infty
$$

In the sequel we also consider $J_{\lambda}:=\left\{x \in \mathbb{R}_{+}^{N}: x_{1} \geq 2 \lambda\right\}$. We need the following "reflection inequalities" for the Green function.

Lemma 4.4 We have

$$
\left.\begin{array}{l}
G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right)>G_{\infty}^{+}\left(x, y^{\lambda}\right) \quad \text { and }  \tag{4.2}\\
G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right)-G_{\infty}^{+}(x, y)>G_{\infty}^{+}\left(x, y^{\lambda}\right)-G_{\infty}^{+}\left(x^{\lambda}, y\right)
\end{array}\right\} \quad \text { for all } x, y \in \Sigma_{\lambda}
$$

and

$$
\begin{equation*}
G_{\infty}^{+}\left(x^{\lambda}, y\right)-G_{\infty}^{+}(x, y)>0 \quad \text { for } x \in \Sigma_{\lambda}, y \in J_{\lambda} . \tag{4.3}
\end{equation*}
$$

Proof. We note that $G(x, y)=H(s(x, y), t(x, y))$, where

$$
H:(0, \infty) \times[0, \infty) \rightarrow \mathbb{R}, \quad H(r, t)=r^{s-\frac{N}{2}} \int_{0}^{\frac{t}{r}} \frac{z^{s-1}}{(z+1)^{N / 2}} d z
$$

and

$$
r(x, y)=|x-y|^{2}, \quad t(x, y)=4 x_{1} y_{1} \quad \text { for } x, y \in \mathbb{R}_{+}^{N} .
$$

By [4, Lemma 2] we have

$$
\begin{equation*}
\partial_{r} H(r, t)<0, \quad \partial_{t} H(r, t)>0 \quad \text { and } \quad \partial_{r} \partial_{t} H(r, t)<0 \quad \text { for } r, t>0 . \tag{4.4}
\end{equation*}
$$

Hence

$$
\begin{equation*}
H\left(r_{1}, t_{1}\right)>H\left(r_{2}, t_{2}\right) \quad \text { if } r_{1}<r_{2}, t_{1}>t_{2}, \tag{4.5}
\end{equation*}
$$

and

$$
\begin{align*}
H\left(r_{1}, t_{4}\right)-H\left(r_{1}, t_{1}\right) & =\int_{t_{1}}^{t_{4}} \partial_{t} H\left(r_{1}, t\right) d t>\int_{t_{1}}^{t_{4}} \partial_{t} H\left(r_{2}, t\right) d t>\int_{\min \left\{t_{2}, t_{3}\right\}}^{\max \left\{t_{2}, t_{3}\right\}} \partial_{t} H\left(r_{2}, t\right) d t \\
& =\left|H\left(r_{2}, t_{2}\right)-H\left(r_{2}, t_{3}\right)\right| \quad \text { if } 0<r_{1}<r_{2}, 0<t_{1}<t_{2}, t_{3}<t_{4} \tag{4.6}
\end{align*}
$$

Now fix $\lambda>0$. Then for $x, y \in \Sigma_{\lambda}$ we have

$$
\begin{equation*}
r(x, y)=r\left(x^{\lambda}, y^{\lambda}\right)<r\left(x, y^{\lambda}\right)=r\left(x^{\lambda}, y\right) \tag{4.7}
\end{equation*}
$$

and

$$
\begin{equation*}
t\left(x^{\lambda}, y^{\lambda}\right)>\max \left\{t\left(x^{\lambda}, y\right), t\left(x, y^{\lambda}\right)\right\} \geq \min \left\{t\left(x^{\lambda}, y\right), t\left(x, y^{\lambda}\right)\right\}>t(x, y) \tag{4.8}
\end{equation*}
$$

The inequalities given in Lemma 4.4 now follow from (4.5), (4.6), (4.7) and (4.8).
We now fix a solution $u$ of (1.1), and we let $C_{u}>0$ be a Lipschitz constant for $f$ on $\left[0,\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right]$, so that

$$
|f(t)-f(r)| \leq C_{u}|t-r| \quad \text { for all } r, t \in\left[0,\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right] .
$$

Inequality (4.3) and the nonnegativity of $f$ imply that

$$
\begin{equation*}
\int_{J_{\lambda}}\left[G_{\infty}^{+}\left(x^{\lambda}, y\right)-G_{\infty}^{+}(x, y)\right] f(u(y)) d y \geq 0 \quad \text { for } \lambda>0 \text { and } x \in \Sigma_{\lambda} . \tag{4.9}
\end{equation*}
$$

We claim that the following reflection inequality holds for every $\lambda>0$ :

$$
u(x) \leq u\left(x^{\lambda}\right) \quad \text { for all } x \in \Sigma_{\lambda}
$$

As a first step, we prove
Lemma 4.5 There exists $\lambda_{0}>0$ such that $\left(\mathcal{C}_{\lambda}\right)$ holds for $\lambda \in\left[0, \lambda_{0}\right]$.
Proof. By Corollary 4.2, we may fix $\lambda_{0}>0$ such that

$$
\begin{equation*}
\int_{\Sigma_{2 \lambda_{0}}} G_{\infty}^{+}(x, y) d y<C_{u}^{-1} \quad \text { for every } x \in \Sigma_{\lambda_{0}} \tag{4.10}
\end{equation*}
$$

For fixed $\lambda \in\left[0, \lambda_{0}\right]$, we consider the difference function

$$
v: \Sigma_{\lambda} \rightarrow \mathbb{R}, \quad v(x)=u\left(x^{\lambda}\right)-u(x)
$$

and the set

$$
W:=\left\{x \in \Sigma_{\lambda}: v(x)<0\right\} .
$$

For $x \in W$ we estimate, using Lemma 4.4 and (4.9),

$$
\begin{aligned}
0 & >v(x)=\int_{\mathbb{R}_{+}^{N}}\left[G_{\infty}^{+}\left(x^{\lambda}, y\right)-G_{\infty}^{+}(x, y)\right] f(u(y)) d y=\int_{\Sigma_{\lambda}} \ldots d y+\int_{\mathbb{R}_{+}^{N} \backslash \Sigma_{\lambda}} \ldots d y \\
& =\int_{\Sigma_{\lambda}}\left(\left[G_{\infty}^{+}\left(x^{\lambda}, y\right)-G_{\infty}^{+}(x, y)\right] f(u(y))+\left[G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right)-G_{\infty}^{+}\left(x, y^{\lambda}\right)\right] f\left(u\left(y^{\lambda}\right)\right)\right) d y \\
& +\int_{J_{\lambda}}\left[G\left(x^{\lambda}, y\right)-G_{\infty}^{+}(x, y)\right] f(u(y)) d y \geq \int_{\Sigma_{\lambda}}\left[G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right)-G_{\infty}^{+}\left(x, y^{\lambda}\right)\right]\left[f\left(u\left(y^{\lambda}\right)\right)-f(u(y))\right] d y \\
& \geq \int_{W}\left[G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right)-G_{\infty}^{+}\left(x, y^{\lambda}\right)\right]\left[f\left(u\left(y^{\lambda}\right)\right)-f(u(y))\right] d y \geq \int_{W} G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right)\left[f\left(u\left(y^{\lambda}\right)\right)-f(u(y))\right] d y \\
& \geq C_{u} \int_{W} G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right) v(y) d y \geq-C_{u}\|v\|_{L^{\infty}(W)} \int_{W} G_{\infty}^{+}\left(x^{\lambda}, y^{\lambda}\right) d y \geq-C_{u}\|v\|_{L^{\infty}(W)} \int_{\Sigma_{2 \lambda_{0}}} G_{\infty}^{+}\left(x^{\lambda}, y\right) d y
\end{aligned}
$$

Combining this with (4.10) we infer $\|v\|_{L^{\infty}(W)} \leq C\|v\|_{L^{\infty}(W)}$ with some constant $C \in(0,1)$, hence $v \equiv 0$ on $W$ and therefore $W=\varnothing$ by definition. We therefore conclude that $\left(\mathcal{C}_{\lambda}\right)$ holds.

Next we put

$$
\lambda_{*}:=\sup \left\{\lambda>0:\left(\mathcal{C}_{\lambda^{\prime}}\right) \text { holds for all } \lambda^{\prime} \leq \lambda\right\} .
$$

Then $\lambda_{*} \geq \lambda_{0}$. Using the continuity of $u$, it is easy to see that $\left(\mathcal{C}_{\lambda_{*}}\right)$ holds. We suppose by contradiction that

$$
\begin{equation*}
\lambda_{*}<\infty \tag{4.11}
\end{equation*}
$$

Then there exists a sequence of numbers $\lambda_{n}>\lambda^{*}, n \in \mathbb{N}$ and points $x^{n} \in \Sigma_{\lambda_{n}}$ such that

$$
\begin{equation*}
u\left(x^{n}\right)>u\left(\left(x^{n}\right)^{\lambda_{n}}\right) \quad \text { for all } n \tag{4.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{n} \rightarrow \lambda^{*} \quad \text { as } n \rightarrow \infty \tag{4.13}
\end{equation*}
$$

We may further assume that

$$
\begin{equation*}
u\left(x^{n}\right)-u\left(\left(x^{n}\right)^{\lambda_{n}}\right)>\frac{1}{2} \sup _{x \in \Sigma_{\lambda_{n}}}\left(u(x)-u\left(x^{\lambda_{n}}\right)\right) . \tag{4.14}
\end{equation*}
$$

In the following we write $x=\left(x_{1}, \hat{x}\right)$ for $x \in \mathbb{R}^{N}$ with $\hat{x} \in \mathbb{R}^{N-1}$. For $n \in \mathbb{N}$ we define the translated functions

$$
u_{n}: \mathbb{R}^{N} \rightarrow \mathbb{R}, \quad u_{n}(y)=u\left(y_{1}, \hat{x}^{n}+\hat{y}\right)
$$

and

$$
v_{n}: \Sigma_{\lambda_{n}} \rightarrow \mathbb{R}, \quad v_{n}(y)=u_{n}\left(y^{\lambda_{n}}\right)-u(y) .
$$

Then (4.12) is rewritten as

$$
\begin{equation*}
v_{n}\left(x_{1}^{n}, 0\right)=u_{n}\left(2 \lambda_{n}-x_{1}^{n}, 0\right)-u_{n}\left(x_{1}^{n}, 0\right)<0 \quad \text { for all } n \tag{4.15}
\end{equation*}
$$

We also consider the sets

$$
W_{n}:=\left\{x \in \Sigma_{\lambda_{n}}: v_{n}(x)<0\right\} .
$$

We let $\Lambda:=2 \max _{n} \lambda_{n}<\infty$. For abbreviation, we also put $z^{n}=\left(x_{1}^{n}, 0\right) \in \Sigma_{\lambda_{n}}$ for $n \in \mathbb{N}$. Using (4.14) and arguing similarly as in the proof of Lemma 4.5, we find

$$
\begin{equation*}
-\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \geq 2 v_{n}\left(z^{n}\right) \geq 2 \int_{W_{n}}\left(G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)-G_{\infty}^{+}\left(z^{n}, y^{\lambda_{n}}\right)\right)\left[f\left(u_{n}\left(y^{\lambda_{n}}\right)\right)-f\left(u_{n}(y)\right)\right] d y \tag{4.16}
\end{equation*}
$$

We now pass to a subsequence such that $x_{1}^{n} \rightarrow t \in\left[0, \lambda_{*}\right]$. Moreover, we note that the sequence $\left(u_{n}\right)_{n}$ is uniformly equicontinuous on compact subsets of $\mathbb{R}^{N}$. Indeed, this follows from Lemma 6.1 below and the boundary estimate (3.3) which holds uniformly for $u_{n}, n \in \mathbb{N}$ in place of $u$. Hence we may pass to a subsequence such that $u_{n} \rightarrow \bar{u}$ in $C_{\text {loc }}^{0}\left(\mathbb{R}^{N}\right) \cap L^{\infty}\left(\mathbb{R}^{N}\right)$ and $\bar{u} \equiv 0$ on $\mathbb{R}^{N} \backslash \mathbb{R}_{+}^{N}$. We distinguish three cases.
Case 1: $t=\lambda_{*}$, i.e. $z^{n} \rightarrow\left(\lambda_{*}, 0\right)$. In this case (4.16) implies that

$$
\begin{align*}
-\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} & \geq 2 C_{u} \int_{W_{n}}\left(G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)-G_{\infty}^{+}\left(z^{n}, y^{\lambda_{n}}\right)\right) v_{n}(y) d y \\
& \geq-2 C_{u}\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \int_{\Sigma_{\lambda_{n}}}\left(G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)-G_{\infty}^{+}\left(z^{n}, y^{\lambda_{n}}\right)\right) d y \\
& =-o(1)\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \tag{4.17}
\end{align*}
$$

where in the last step we used Corollary 4.3 and the fact that also $\left(z^{n}\right)^{\lambda_{n}} \rightarrow\left(\lambda_{*}, 0\right)$. Hence $\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)}=0$ for $n$ large, contrary to (4.15).
Case 2: $t<\lambda^{*}$ and $\bar{u} \not \equiv 0$.
We have

$$
\int_{\mathbb{R}^{N}} u_{n}(x)(-\Delta)^{s} \varphi(x) d x=\int_{\mathbb{R}_{+}^{N}} f\left(u_{n}(x)\right) \varphi(x) d x \quad \text { for all } \varphi \in C_{c}^{2}\left(\mathbb{R}_{+}^{N}\right)
$$

It thus follows from the dominated convergence theorem that

$$
\int_{\mathbb{R}^{N}} \bar{u}(x)(-\Delta)^{s} \varphi(x) d x=\int_{\mathbb{R}_{+}^{N}} f(\bar{u}(x)) \varphi(x) d x \quad \text { for all } \varphi \in C_{c}^{2}\left(\mathbb{R}_{+}^{N}\right)
$$

Hence, by Theorem 3.1, $\bar{u}$ is represented as

$$
\bar{u}(x)=\int_{\mathbb{R}_{+}^{N}} G_{\infty}^{+}(x, y) f(\bar{u}(y)) d y \quad \text { for all } x \in \mathbb{R}_{+}^{N}
$$

Since $u \not \equiv 0$ and $f(t)>0$ for $t>0$, it then follows that $\bar{u}$ and $f \circ \bar{u}$ are strictly positive on $\mathbb{R}_{+}^{N}$. Moreover, by the locally uniform convergence $u_{n} \rightarrow \bar{u}$, we have

$$
\bar{u}(x) \leq \bar{u}\left(x^{\lambda_{*}}\right) \quad \text { for all } x \in \Sigma_{\lambda_{*}} .
$$

Indeed this inequality is strict, since we have a strict inequality in (4.9) for $\bar{u}$ in place of $u$ and $\lambda=\lambda^{*}$, so that

$$
\begin{align*}
& \bar{u}\left(x^{\lambda_{*}}\right)-\bar{u}(x)=\int_{\mathbb{R}_{+}^{N}}\left[G_{\infty}^{+}\left(x^{\lambda_{*}}, y\right)-G_{\infty}^{+}(x, y)\right] f(\bar{u}(y)) d y=\int_{\Sigma_{\lambda_{*}}} \ldots d y+\int_{\mathbb{R}_{+}^{N} \backslash \Sigma_{\lambda_{*}}} \ldots d y \\
& \quad=\int_{\Sigma_{\lambda_{*}}}\left(\left[G_{\infty}^{+}\left(x^{\lambda_{*}}, y\right)-G_{\infty}^{+}(x, y)\right] f(\bar{u}(y))+\left[G_{\infty}^{+}\left(x^{\lambda_{*}}, y^{\lambda_{*}}\right)-G_{\infty}^{+}\left(x, y^{\lambda_{*}}\right)\right] f\left(\bar{u}\left(y^{\lambda_{*}}\right)\right)\right) d y \\
& \quad+\int_{J_{\lambda_{*}}}\left[G_{\infty}^{+}\left(x^{\lambda_{*}}, y\right)-G_{\infty}^{+}(x, y)\right] f(\bar{u}(y)) d y \\
& \left.\quad>\int_{\Sigma_{\lambda^{*}}}\left[G_{\infty}^{+}\left(x^{\lambda_{*}}, y^{\lambda_{*}}\right)-G_{\infty}^{+}\left(x, y^{\lambda_{*}}\right)\right] f\left(\bar{u}\left(y^{\lambda_{*}}\right)\right)-f(\bar{u}(y))\right] d y \geq 0 \quad \text { for } x \in \Sigma_{\lambda_{*}} . \tag{4.18}
\end{align*}
$$

On the other hand, also by the locally uniform convergence and (4.15), we have

$$
\bar{u}\left(2 \lambda_{*}-t, 0\right)-\bar{u}(t, 0)=\lim _{n \rightarrow \infty}\left(u_{n}\left(\left(z^{n}\right)^{\lambda_{n}}\right)-u\left(z_{n}\right)\right) \leq 0 .
$$

This is a contradiction.
Case 3: $\bar{u} \equiv 0$. We use (4.16) and Lemma 4.4 to estimate, for $r>0$,

$$
\begin{align*}
-\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \geq & 2 \int_{W_{n}} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)\left[f\left(u_{n}\left(y^{\lambda_{n}}\right)\right)-f\left(u_{n}(y)\right)\right] d y \\
= & 2 \int_{W_{n} \cap B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)\left[f\left(u_{n}\left(y^{\lambda_{n}}\right)\right)-f\left(u_{n}(y)\right)\right] d y \\
& +2 \int_{W_{n} \backslash B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)\left[f\left(u_{n}\left(y^{\lambda_{n}}\right)\right)-f\left(u_{n}(y)\right)\right] d y \tag{4.19}
\end{align*}
$$

where

$$
\begin{align*}
& \int_{W_{n} \backslash B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)\left[f\left(u\left(y^{\lambda_{n}}\right)\right)-f(u(y))\right] d y \geq C_{u} \int_{W_{n} \backslash B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right) v_{n}(u(y)) d y \\
& \quad 20) \quad \geq-C_{u}\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \int_{\Sigma_{\Lambda} \backslash B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y\right) d y \tag{4.20}
\end{align*}
$$

Since $\left(\left(z^{n}\right)^{\lambda_{n}}\right)_{n} \subset \Sigma_{\Lambda}$ is a bounded sequence, Lemma 4.1 implies that

$$
\sup _{n \in \mathbb{N}} \int_{\Sigma_{\Lambda} \backslash B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y\right) d y \rightarrow 0 \quad \text { as } r \rightarrow \infty
$$

Hence we may fix $r>0$ such that

$$
\begin{equation*}
\int_{\Sigma_{\Lambda} \backslash B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y\right) d y<\frac{1}{4 C_{u}} \quad \text { for all } n \in \mathbb{N} . \tag{4.21}
\end{equation*}
$$

Moreover, assumption (1.4) and the locally uniform convergence $u_{n} \rightarrow 0$ imply that there exists a sequence of numbers $\varepsilon_{n}>0, \varepsilon_{n} \rightarrow 0$ such that

$$
f\left(u_{n}\left(y^{\lambda_{n}}\right)\right)-f\left(u_{n}(y)\right) \geq \varepsilon_{n} v_{n}(y) \quad \text { for } y \in W_{n} \cap B_{r}(0)
$$

so that

$$
\begin{align*}
\int_{W_{n} \cap B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right)\left[f\left(u_{n}\left(y^{\lambda_{n}}\right)\right)-f\left(u_{n}(y)\right)\right] d y & \geq-\varepsilon_{n}\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \int_{B_{r}(0)} G_{\infty}^{+}\left(\left(z^{n}\right)^{\lambda_{n}}, y^{\lambda_{n}}\right) d y \\
& \geq-C \varepsilon_{n}\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \tag{4.22}
\end{align*}
$$

with some constant $C=C(r)>0$. Combining (4.19), (4.20), (4.21) and (4.22), we get

$$
\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)} \leq\left(\frac{1}{2}+2 C \varepsilon_{n}\right)\left\|v_{n}\right\|_{L^{\infty}\left(W_{n}\right)}
$$

so we conclude that $\left\|v_{n}\right\|_{L^{\infty}}\left(W_{n}\right)=0$ for large $n$, contradicting again (4.15). We have thus proved that property $\left(\mathcal{C}_{\lambda}\right)$ holds for all $\lambda>0$, which implies that $u$ is increasing in $x_{1}$. It thus remains to show that $u$ is strictly increasing in $x_{1}$ if $u \not \equiv 0$. This however follows since we can now derive inequality (4.18) for $u$ in place of $\bar{u}$ and all $\lambda>0$ in place of $\lambda^{*}$. The proof of Theorem 1.1 is thus finished.

## 5 Proof of the Liouville Theorem in the half-space

This section is devoted to the proof of Theorem [1.2, Let $s \in(0,1)$ and assume that $q>1$ if $N \leq 1+2 s$ and $1<q<\frac{N-1+2 s}{N-1-2 s}$ if $N>1+2 s$. Suppose by contradiction that there exists a bounded solution $u \not \equiv 0$ of (1.5). Theorem 1.1 implies that $u$ is strictly increasing in $x_{1}$. In particular, we may define

$$
\tilde{u} \in L^{\infty}\left(\mathbb{R}^{N-1}\right), \quad \tilde{u}\left(x^{\prime}\right)=\lim _{x_{1} \rightarrow \infty} \tilde{u}\left(x_{1}, x^{\prime}\right) \quad \text { for } x^{\prime} \in \mathbb{R}^{N-1}
$$

Here and in the following, we write $x=\left(x_{1}, x^{\prime}\right) \in \mathbb{R}^{N}$ with $x^{\prime} \in \mathbb{R}^{N-1}$. Note that $\tilde{u}>0$ on $\mathbb{R}^{N-1}$. For $\tau>0$, define $u_{\tau} \in L^{\infty}\left(\mathbb{R}^{N}\right) \cap C\left(\mathbb{R}^{N}\right)$ by $u_{\tau}(x)=u\left(x+\tau e_{1}\right)$. Then

$$
\begin{equation*}
u_{\tau} \rightarrow u_{\infty} \quad \text { pointwise on } \mathbb{R}^{N} \text { with } u_{\infty} \in L^{\infty}\left(\mathbb{R}^{N}\right) \text { defined by } u_{\infty}(x)=\tilde{u}\left(x^{\prime}\right) . \tag{5.1}
\end{equation*}
$$

Moreover, we have $(-\Delta)^{s} u_{\tau}=u_{\tau}^{q}$ in $H_{\tau}:=\left\{x \in \mathbb{R}^{N}: x_{1}>-\tau\right\}$. Hence (5.1) implies that $(-\Delta)^{s} u_{\infty}=u_{\infty}^{q}$ in $\mathbb{R}^{N}$ in distributional sense. Indeed, let $\varphi \in C_{c}^{2}\left(\mathbb{R}^{N}\right)$. Then

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} u_{\infty}^{q}(x) \varphi(x) d x=\lim _{\tau \rightarrow \infty} \int_{H_{\tau}} u_{\tau}^{q}(x) \varphi(x) d x & =\lim _{\tau \rightarrow \infty} \int_{\mathbb{R}^{N}} u_{\tau}(x)\left[(-\Delta)^{s} \varphi\right](x) d x \\
& =\int_{\mathbb{R}^{N}} u_{\infty}(x)\left[(-\Delta)^{s} \varphi\right](x) d x
\end{aligned}
$$

by Lebesgue's theorem and the estimate (1.3). It then follows from the regularity results in 42 and [14] that also $(-\Delta)^{s} u_{\infty}=u_{\infty}^{q}$ in classical sense in $\mathbb{R}^{N}$. Moreover, $u_{\infty}>0$ in $\mathbb{R}^{N}$ since $\tilde{u}>0$ in $\mathbb{R}^{N-1}$. In case $N=1, u_{\infty}$ is a positive constant on $\mathbb{R}$ which contradicts the equation $(-\Delta)^{s} u_{\infty}=u_{\infty}^{q}$. In case $N \geq 2$, we deduce that

$$
\begin{equation*}
(-\Delta)^{s} \tilde{u}=\tilde{u}^{q} \quad \text { in } \mathbb{R}^{N-1} \tag{5.2}
\end{equation*}
$$

Indeed,

$$
\begin{aligned}
\tilde{u}^{q}\left(x^{\prime}\right) & =u_{\infty}{ }^{q}(x)=(-\Delta)^{s} u_{\infty}(x)=a_{N, s} \int_{\mathbb{R}^{N}} \frac{u_{\infty}(x+z)+u_{\infty}(x-z)-2 u_{\infty}(x)}{|z|^{N+2 s}} d z \\
& =a_{N, s} \int_{\mathbb{R}^{N}} \frac{\tilde{u}\left(x^{\prime}+z^{\prime}\right)+\tilde{u}\left(x^{\prime}-z^{\prime}\right)-2 \tilde{u}\left(x^{\prime}\right)}{\left(\left|z^{\prime}\right|^{2}+z_{1}^{2}\right)^{\frac{N}{2}+s}} d z \\
& =a_{N, s} \int_{\mathbb{R}^{N-1}} \frac{\tilde{u}\left(x^{\prime}+z^{\prime}\right)+\tilde{u}\left(x^{\prime}-z^{\prime}\right)-2 \tilde{u}\left(x^{\prime}\right)}{\left|z^{\prime}\right|^{N-1+2 s}}\left[\frac{1}{\left|z^{\prime}\right|} \int_{\mathbb{R}}\left[1+\left(\frac{z_{1}}{\left|z^{\prime}\right|}\right)^{2}\right]^{-\frac{N}{2}-s} d z_{1}\right] d z^{\prime} \\
& =\frac{a_{N, s}}{a_{N-1, s}}\left[(-\Delta)^{s} \tilde{u}\right]\left(x^{\prime}\right) \int_{\mathbb{R}}\left(1+\lambda^{2}\right)^{-\frac{N}{2}-s} d \lambda=\left[(-\Delta)^{s} \tilde{u}\right]\left(x^{\prime}\right) \quad \text { for } x^{\prime} \in \mathbb{R}^{N-1}
\end{aligned}
$$

since

$$
\int_{\mathbb{R}}\left(1+\lambda^{2}\right)^{-\frac{N}{2}-s} d \lambda=B\left(\frac{1}{2}, \frac{N-1}{2}+s\right)=\frac{\sqrt{\pi} \Gamma\left(\frac{N-1}{2}+s\right)}{\Gamma\left(\frac{N}{2}+s\right)}=\frac{a_{N-1, s}}{a_{N, s}} .
$$

However, since $\tilde{u}>0$, this is impossible by Theorem 1.3 applied to $N-1 \geq 1$ in place of $N$. The proof is finished.

## 6 Appendix: Interior Hölder estimates for distributional solutions to the equation $(-\Delta)^{s} u=f$

In the proof of Theorem 1.1 in Section 4 we used the following lemma on local Hölder regularity. Since the proofs available in the literature are only concerned with the case $N>2 s$ and additional assumptions (see for instance [42, Proposition 2.1.9]), we will give a proof for the convenience of the reader. The proof is similar to arguments in [31, page 263].

Lemma 6.1 Let $f \in L^{\infty}\left(B_{1}\right)$, and let $u \in \mathcal{L}_{s}^{1} \cap L_{\text {loc }}^{\infty}\left(B_{1}\right)$ such that $(-\Delta)^{s} u=f$ in $B_{1}$. Then for $r \in(0,1)$ and every $\alpha \in(0, \min (1,2 s))$ there exists a constant $C_{s, N, r, \alpha}>0$ such that

$$
\begin{equation*}
\|u\|_{C^{0, \alpha}\left(B_{r}\right)} \leq C_{s, N, r, \alpha}\left(\|u\|_{L^{\infty}\left(B_{r}\right)}+\|f\|_{L^{\infty}\left(B_{1}\right)}\right) . \tag{6.1}
\end{equation*}
$$

Proof. Let $\eta \in C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$ be such that $\eta=1$ on $B_{r}, \eta=0$ on $\mathbb{R}^{N} \backslash B_{1}$ and $0 \leq \eta \leq 1$ on $\mathbb{R}^{N}$. Consider the Riesz potential, see [5] and [7],

$$
\Phi(x, y)=\left\{\begin{array}{l}
C_{N, s}|x-y|^{2 s-N} \quad \text { for } N>2 s \\
\frac{1}{\pi} \log \frac{1}{|x|} \quad \text { for } 1=N=2 s \\
-C_{1, s}|x-y|^{2 s-N} \quad \text { for } N=1<2 s
\end{array}\right.
$$

with a positive normalization constant $C_{N, s}$. Then the function $v(x)=\int_{\mathbb{R}^{N}} \Phi(x, y)(\eta f)(y) d y$ satisfies

$$
\begin{equation*}
(-\Delta)^{s} v(x)=\eta(x) f(x) \quad \text { for all } x \in \mathbb{R}^{N} . \tag{6.2}
\end{equation*}
$$

The following inequalities hold for all $a, b>0, \alpha \in(0,1)$ and $m \in \mathbb{R}$ with $m+\alpha>0$ :

$$
\begin{gather*}
\left|b^{-m}-a^{-m}\right| \leq \frac{|m|}{m+\alpha}|b-a|^{\alpha} \max \left(a^{-(m+\alpha)}, b^{-(m+\alpha)}\right)  \tag{6.3}\\
|\log b-\log a| \leq \frac{1}{\alpha}|b-a|^{\alpha} \max \left(a^{-\alpha}, b^{-\alpha}\right) \tag{6.4}
\end{gather*}
$$

These estimates were proved in [31, page 263] in the case $m \geq 1$, and a slight change of their argument yields the more general case considered here. We thus get, for every $x, y, z \in \mathbb{R}^{N}$,

$$
\left||x-z|^{-m}-|y-z|^{-m}\right| \leq \frac{|m|}{m+\alpha}|x-y|^{\alpha}\left(|x-z|^{-(m+\alpha)}+|y-z|^{-(m+\alpha)}\right)
$$

and

$$
|\log | x-z|-\log | y-z| | \leq \frac{1}{\alpha}|x-y|^{\alpha}\left(|x-z|^{-\alpha}+|y-z|^{-\alpha}\right) .
$$

In the case $N \geq 2 s$ we therefore have, for $\alpha \in(0, \min (2 s, 1))$ and $x, y \in B_{r}$,

$$
\begin{align*}
|v(x)-v(y)| & \leq C_{s, \alpha}|x-y|^{\alpha} \int_{\mathbb{R}^{N}}|x-y|^{2 s-N-\alpha} \eta(y)|f(y)| d y \\
& \leq C_{s, \alpha}|x-y|^{\alpha}\|f\|_{L^{\infty}\left(B_{1}\right)} \int_{B(x, 2)}|x-y|^{2 s-N-\alpha} d y \\
& \leq C_{s, \alpha}\|f\|_{L^{\infty}\left(B_{1}\right)}|x-y|^{\alpha} . \tag{6.5}
\end{align*}
$$

Moreover, in the case $N=1<2 s$ we have, for $\alpha \in(0, \min (2 s, 1))$ and $x, y \in B_{r}$,

$$
\begin{equation*}
|v(x)-v(y)| \leq C_{s, \alpha}|x-y|^{\alpha}\|f\|_{L^{\infty}\left(B_{1}\right)} \int_{B_{1}}|x-y|^{2 s-1-\alpha} d y \leq C_{s, \alpha}\|f\|_{L^{\infty}(\Omega)}|x-y|^{\alpha} . \tag{6.6}
\end{equation*}
$$

Hence combing (6.5) and (6.6), we conclude that

$$
\begin{equation*}
\|v\|_{C^{0, \alpha}\left(B_{r}\right)} \leq C_{s, N}\|f\|_{L^{\infty}\left(B_{1}\right)} \tag{6.7}
\end{equation*}
$$

for every $\alpha \in(0, \min (1,2 s))$.
Next we note that the function $w:=u-v$ satisfies $(-\Delta)^{s} w=0$ in $B_{r}$ by (6.2). Therefore, thanks to [7. Lemma 3.2], we get, for every $r^{\prime} \in(0, r)$,

$$
\|\nabla w\|_{L^{\infty}\left(B_{r}^{\prime}\right)} \leq C_{N, s, r^{\prime}}\|w\|_{L^{\infty}\left(B_{1}\right)} \leq C_{N, s, r^{\prime}}\left(\|u\|_{L^{\infty}\left(B_{r}\right)}+\|v\|_{L^{\infty}\left(B_{r}\right)}\right) .
$$

From this, together with (6.7), we conclude that

$$
\|u\|_{C^{0, \alpha}\left(B_{r^{\prime}}\right)}=\|w+v\|_{C^{0, \alpha}\left(B_{r^{\prime}}\right)} \leq C_{s, N, r^{\prime}}\left(\|u\|_{L^{\infty}\left(B_{r}\right)}+\|f\|_{L^{\infty}\left(B_{1}\right)}\right),
$$

for every $\alpha \in(0, \min (1,2 s))$.

## References

[1] R.A. Adams and J.J.F. Fournier: Sobolev spaces. Pure and Applied Mathematics, Academic Press, 2nd edition, 2003. xiii+305 pp.
[2] A.D. Alexandrov: A characteristic property of the spheres. Ann. Mat. Pura Appl. 58 (1962), 303-354.
[3] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, Inequalities for second-order elliptic equations with applications to unbounded domains I. Duke Math. J. Volume 81, Number 2 (1996), 467494.
[4] E. Berchio, F. Gazzola and T. Weth, Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems. J. Reine Angew. Math. 620 (2008), 165-183.
[5] R. M. Blumenthal, R. K. Getoor and D. B. Ray, On the distribution of first hits for the symmetric stable processes, Trans. Amer. Math. Soc. 99 (1961), pp. 54554.
[6] M. Birkner, J.A. López-Mimbela, A. Wakolbinger: Comparison results and steady states for the Fujita equation with fractional Laplacian. Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), 83-97.
[7] K. Bogdan, T. Kulczycki, A. Nowak, Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes. Illinois J. Math. 46 (2002), no. 2, 541-556.
[8] K. Bogdan, T. Byczkowski, Potential theory for the $\alpha$-stable Schrödinger operator on bounded Lipschitz domains. Studia Math. 133 (1999), no. 1, 53-92.
[9] K. Bogdan, T. Byczkowski, Potential theory of Schrödinger operator based on fractional Laplacian. Probab. Math. Statist. 20 (2000), no. 2, Acta Univ. Wratislav. No. 2256, 293-335.
[10] K. Bogdan, The boundary Harnack principle for the fractional Laplacian. Studia Math. 123 (1997), no. 1, 43-80.
[11] C. Brändle, E. Colorado, A. de Pablo, U. Sánchez: A concave-convex elliptic problem involving the fractional Laplacian. Proc. Roy. Soc. Edinburgh, 143 (2013), 39-71.
[12] H. Brézis, R.E.L. Turner: On a class of superlinear elliptic problems. Comm. Partial Differential Equations 2 (1977), 601-614.
[13] X. Cabré, J.G. Tan: Positive solutions of nonlinear problems involving the square root of the Laplacian. Adv. Math. 224 (2010), 2052-2093.
[14] X. Cabré, Y. Sire: Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates. Available online at http://arxiv.org/abs/1012.0867.
[15] A. S-Y Chang, P. C. Yang: On uniqueness of solutions of $n$th order differential equations in conformal geometry. Math. Res. Lett. 4 (1997), 91-102.
[16] W. Chen, C. Li and Biao Ou: Qualitative properties of solutions for an integral equation. Discrete Contin. Dyn. Syst. 12 (2005), 347-354.
[17] W. Chen, C. Li and Biao Ou: Classification of solutions for an integral equation. Comm. Pure Appl. Math. 59 (2006), 330-343.
[18] Z.-Q. Chen and R. Song: Estimates on Green functions and Poisson kernels for symmetric stable processes. Math. Ann., 312(3):465-501, 1998.
[19] D.G. de Figueiredo, P.-L. Lions, R.D. Nussbaum: A priori estimates and existence of positive solutions of semilinear elliptic equations. J. Math. Pures Appl. 61 (1982), no. 1, 41-63.
[20] M. M. Fall, Semilinear elliptic equations for the fractional Laplacian with Hardy potential. Preprint, see arXiv:1109.5530.
[21] M. M. Fall and T. Weth: Nonexistence results for a class of fractional elliptic boundary value problems. J. Funct. Anal. 263 (2012), no. 8, 2205-2227.
[22] A. Ferrero, F. Gazzola and T. Weth: Positivity, symmetry and uniqueness for minimizers of second order Sobolev inequalities. Ann. Mat. Pura Appl. 186 (2007), 565-578.
[23] B. Gidas, Wei-Ming Ni and L. Nirenberg: Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (1979), 209-243.
[24] B. Gidas and J. Spruck: A priori bounds for positive solutions of nonlinear elliptic equations. Comm. Partial Differential Equations 6 (1981), 883-901.
[25] B. Gidas and J. Spruck: Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math. 34 (1981), 525-598.
[26] H.-Ch. Grunau and G. Sweers: Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions. Math. Ann. 307 (1997), 589-626.
[27] T. Jin, Y.Y. Li, J. Xiong: On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, J. Eur. Math. Soc. (JEMS), to appear, arXiv:1111.1332v1.
[28] T. Kulczycki, Properties of Green function of symmetric stable processes, Probab. Math. Statist. 17 (2) (1997), pp. 339-364.
[29] N. S. Landkof. Foundations of modern potential theory. Springer-Verlag, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.
[30] Yan Yan Li: Remark on some conformally invariant integral equations: the method of moving spheres. J. Eur. Math. Soc. 6 (2004), 153-180.
[31] E.H. Lieb and M. Loss: Analysis. Second Edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 1997.
[32] C.-S. Lin: A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^{n}$. Comment. Math. Helv. 73 (1998), 206-231.
[33] P.J. McKenna and W. Reichel: A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains. J. Funct. Anal. 244 (2007), 220-246.
[34] E. di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's Guide to the Fractional Sobolev Spaces. Bulletin des Sciences Mathématiques 136 (2012), 521-573.
[35] B. Ou: Positive harmonic functions on the upper half space satisfying a nonlinear boundary condition. Differential Integral Equations 9 (1996), 1157-1164.
[36] P. Poláčik, P. Quittner and Ph. Souplet: Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems. Duke Math. J. 139 (2007), 555-579.
[37] P. Quittner and Ph. Souplet: A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces. Arch. Ration. Mech. Anal. 174 (2004), 49-81.
[38] X. Ros-Oton and J. Serra: The Dirichlet Problem for the fractional Laplacian: Regularity up to the boundary, submitted, available online at http://arxiv.org/abs/1207.5985.
[39] W. Reichel and T. Weth: A priori bounds and a Liouville theorem on a half-space for higherorder elliptic Dirichlet problems Math. Z. 261 (2009), no. 4, 805-827.
[40] W. Reichel and T. Weth: Existence of solutions to nonlinear, subcritical higher-order elliptic Dirichlet problems J. Differential Equations 248 (2010) 1866-1878.
[41] J. Serrin: A symmetry problem in potential theory. Arch. Ration. Mech. Anal. 43 (1971), 304-318.
[42] L. Silvestre: Regularity of the obstacle problem for a fractional power of the Laplace operator. Comm. Pure Appl. Math., 60(1):67-112, 2007.
[43] B. Sirakov: Existence results and a priori bounds for higher order elliptic equations, preprint.
[44] R. Song and Z. Vondraček: Potential theory of subordinate killed Brownian motion in a domain. Probability Theory and Related Fields 123 (2003), 578-592.
[45] R. Soranzo: A priori estimates and existence of positive solutions of a superlinear polyharmonic equation. Dynam. Systems Appl. 3 (1994), 465-487.
[46] Juncheng Wei and Xingwang Xu: Classification of solutions of higher order conformally invariant equations. Math. Ann. 313 (1999), 207-228.


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