Monotonicity and nonexistence results for some fractional elliptic problems in the half space

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Abstract

We study a class of fractional elliptic problems of the form $(-\Delta)^s u = f(u)$ in the half space $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_1 > 0\}$ with the complementary Dirichlet condition $u \equiv 0$ in $\mathbb{R}^N \setminus \mathbb{R}^N_+$. Under mild assumptions on the nonlinearity f, we show that bounded positive solutions are increasing in x_1 . For the special case $f(u) = u^q$, we deduce nonexistence of positive bounded solutions in the case where $q \ge 1$ and $q < \frac{N-1+2s}{N-1-2s}$ if $N \ge 1+2s$. We do not require integrability assumptions on the solutions we study.

1 Introduction

In the present paper, we are concerned with solutions $u \in L^{\infty}(\mathbb{R}^N)$ of the semilinear fractional problem

(1.1)
$$\begin{cases} (-\Delta)^s u = f(u), \quad u \ge 0 \qquad \text{ in } \mathbb{R}^N_+, \\ u = 0 \qquad \text{ in } \mathbb{R}^N \setminus \mathbb{R}^N_+ \end{cases}$$

Here $s \in (0,1)$, $N \in \mathbb{N}$, $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_1 > 0\}$ and $f : [0,\infty) \to [0,\infty)$ is a nonnegative, nondecreasing and locally Lipschitz continuous nonlinearity. Special attention will be given to the case $f(t) = t^q$ with q > 1. Due to applications in physics, biology and finance, linear and nonlinear equations involving the fractional Laplacian $(-\Delta)^s$ have received growing attention in recent years (see e.g. [34, Introduction] for various references), while they are are still much less understood than their non-fractional counterparts.

We briefly explain in which sense we consider problems of type (1.1). For functions $u \in C_c^2(\mathbb{R}^N)$, the fractional Laplacian $(-\Delta)^s$ is defined by

(1.2)
$$(-\Delta)^{s} u(x) = a_{N,s} \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,$$

where $a_{N,s} = s(1-s)\pi^{-N/2}4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(2-s)}$ (see e.g. [14, Remark 3.11]). Let \mathcal{L}_s^1 denote the space of all functions $u: \mathbb{R}^N \to \mathbb{R}$ such that $\int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}} dx < \infty$. If $\Omega \subset \mathbb{R}^N$ is an open subset and $g \in L^1_{loc}(\Omega)$,

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we say that a function $v \in \mathcal{L}^1_s$ solves the equation $(-\Delta)^s v = g$ in Ω in the sense of distributions if

$$\int_{\mathbb{R}^N} v(-\Delta)^s \varphi \, dx = \int_\Omega g \varphi \, dx \qquad \text{for every } \varphi \in C^2_c(\Omega).$$

Note that the integral on the left hand side is well defined since

(1.3)
$$|(-\Delta)^s \varphi(x)| \le \frac{\kappa \|\varphi\|_{C_c^2}}{1+|x|^{N+2s}} \quad \text{for every } \varphi \in C_c^2(\Omega), \, x \in \mathbb{R}^N$$

with a constant $\kappa = \kappa(N, s, \Omega)$ (see for instance [21]). The following is our first main result:

Theorem 1.1 Suppose that $f : [0, \infty) \to [0, \infty)$ is a nonnegative, nondecreasing and locally Lipschitz continuous function satisfying f(t) > 0 for t > 0 and

(1.4)
$$\lim_{\substack{r,t\to 0\\r\neq t}}\frac{f(r)-f(t)}{r-t} = 0.$$

Then every bounded solution u of (1.1) is increasing in x_1 . Moreover, either $u \equiv 0$, or u is strictly increasing in x_1 .

We note that, for C^1 -nonlinearities, condition (1.4) simply amounts to f'(0) = 0. Our second main result is of Liouville type.

Theorem 1.2 If $N \le 1 + 2s$ and q > 1 or N > 1 + 2s and $1 < q < \frac{N-1+2s}{N-1-2s}$, then the problem

(1.5)
$$\begin{cases} (-\Delta)^s u = u^q, \quad u \ge 0 \qquad \text{in } \mathbb{R}^N_+, \\ u = 0 \qquad \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases}$$

only admits the trivial solution $u \equiv 0$.

Our results complement the following recent Liouville type result of Jin, Li and Xiong [27] for the corresponding full space problem

(1.6)
$$(-\Delta)^s u = u^q, \qquad u > 0 \qquad \text{in } \mathbb{R}^N.$$

Theorem 1.3 (see [27])

Suppose that $N \leq 2s$ and q > 0 or N > 2s and $0 < q < \frac{N+2s}{N-2s}$. Then (1.6) has no bounded solution.

We note that this result has also been obtained independently in [11] in the case $s \ge \frac{1}{2}$. Before that, the special case $s = \frac{1}{2}$ had been considered in [35], whereas in [16] the result was proved for a restricted class of solutions.

To put our results into perspective, some remarks are in order. Theorem 1.2 is an improvement of [21, Corollary 1.6], where the authors established nonexistence of a restricted class of solutions u of (1.5) in the subcritical case N > 2s and $1 < q \leq \frac{N+2s}{N-2s}$. More precisely, in [21] we assumed that

u is contained in the Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N_+)$ defined as the completion of $C_c^{\infty}(\mathbb{R}^N)$ with respect to the norm given by

$$||u||^{2} = \int_{\mathbb{R}^{2}N} \frac{(u(x) - u(y))^{2}}{|x - y|^{N + 2s}} \, dx \, dy.$$

The argument of [21], relying on the method of moving spheres, does not apply under the assumptions of Theorem 1.2.

Theorem 1.1 is proved by a variant of the moving plane method based on extensions and modifications of techniques in the papers [3] resp. [39], which were devoted to second order and polyharmonic boundary value problems, respectively. The first key step in the argument is to show, without a priori integrability assumptions, that bounded solutions of (1.1) admit a Green function representation. This representation is obtained, via an approximation argument, from Green-Poisson type formulas in balls. Once the Green function representation is obtained, we carry out a moving plane argument for integral equations. We note that moving plane arguments for integral equations have been applied very successfully in recent years, see e.g. [4, 6, 15–17, 39]. In the present situation, the lack of integrability assumptions creates additional difficulties which require to argue somewhat differently than in earlier papers.

Theorem 1.2 is deduced from Theorems 1.1 and 1.3 by considering the limits of solutions of (1.5) as $x_1 \to \infty$, which, considered as functions of (x_2, \ldots, x_N) , solve (1.6) in \mathbb{R}^{N-1} .

The boundedness assumption in Theorem 1.2 can be replaced by only assuming boundedness in compact subsets of \mathbb{R}^N_+ if the assumption on q is strengthened to $1 < q < \frac{N+2s}{N-2s}$ in case N > 2s. This can be deduced from Theorems 1.2 and 1.3 by the argument used in [40, Section 4] for the polyharmonic version of (1.5). The argument is based on the doubling-lemma (see [36]).

The combination of the Liouville type results Theorem 1.2 and 1.3 are expected to give rise, via a Gidas-Spruck type rescaling argument (see [24]), to a priori bounds for solutions to more general integral equations in bounded domains and also to elliptic boundary value problems of second order with mixed nonlinear boundary conditions. For applications of this type, it is essential that Theorems 1.2 and 1.5 do not contain a priori integrability assumptions. This topic will be considered by the authors in a future work.

The combination of Liouville type results with rescaling arguments has already been applied successfully by Cabré and Tan [13] for nonlinear boundary value problems involving the spectral theoretic square root of the Dirichlet Laplacian, denoted by $\mathcal{A}_{1/2}$ in [13]. In particular, the analogue of Theorem 1.2 with $(-\Delta)^s$ replaced by $\mathcal{A}_{1/2}$ has been proved in [13, Theorem 1.5]. The subtle differences between $(-\Delta)^s$ and spectral theoretic powers of the Dirichlet Laplacian are discussed in [20, Remark 0.4] from a PDE point of view and in [44] in terms of stochastic processes. Because of these differences, it remains unclear whether Theorem 1.2 can also be obtained via similar methods as in [13]. The approach of the present paper is completely different. Moreover, the monotonicity result given by Theorem 1.1 is not available yet for the corresponding problem with spectral theoretic powers.

The paper is organized as follows. In Section 2 we collect preliminary results on (distributional) solutions of $(-\Delta)^s u = f$ on some open subset of \mathbb{R}^N with bounded f. In Section 3 we show that bounded solutions u of the problem $(-\Delta)^s u = f$ in \mathbb{R}^N_+ with $u \equiv 0$ in $\mathbb{R}^N \setminus \mathbb{R}^N_+$ admit a Green function representation whenever f is bounded and nonnegative. In Section 4 we complete the proof of Theorem 1.1, and in Section 5 we complete the proof of Theorem 1.2. The appendix

contains a regularity result needed in the proof of Theorem 1.1.

Acknowledgment: The second author wishes to thank Enrico Valdinoci und Eduardo Colorado for helpful discussions. The first author is funded by the Alexander von Humboldt foundation and would like to thank Krzysztof Bogdan for useful discussions.

2 Preliminaries

Here and in the following, we consider $N \ge 1$ and $s \in (0, 1)$. We write $\mathbf{B} = \{x \in \mathbb{R}^N : |x| < 1\}$ for the open unit ball in \mathbb{R}^N and set $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ for R > 0. We start by recalling the following estimate, see [21].

Lemma 2.1 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then there exists a constant $C = C(N, s, \Omega) > 0$ such that for all $\varphi \in C_c^2(\Omega)$ and for all $x \in \mathbb{R}^N$

(2.1)
$$\left| \int_{|x-y|>\varepsilon} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy \right| \le \frac{C \|\varphi\|_{C^2(\Omega)}}{1 + |x|^{N+2s}} \quad \text{for all } \varepsilon \in (0,1)$$

As a consequence, $(-\Delta)^s u$ can be defined for functions $u \in \mathcal{L}^1_s$ in the following way in distributional sense. Here we recall that \mathcal{L}^1_s is the space of all functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $\int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}} dx < \infty$.

Definition 2.2 Let Ω be an open subset of \mathbb{R}^N . Given $u \in \mathcal{L}^1_s$, the distribution $(-\Delta)^s u \in \mathcal{D}'(\Omega)$ is defined as

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u(-\Delta)^s \varphi dx \quad \text{for all } \varphi \in C^\infty_c(\Omega).$$

The following result is contained in [8, Lemma 3.8].

Lemma 2.3 Let $\Omega \subset \mathbb{R}^N$ be open. If $u \in \mathcal{L}^1_s \cap C^2(\Omega)$, then the limit $\lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy$ exists for every $x \in \Omega$. Moreover, $(-\Delta)^s u$ is a regular distribution given by

(2.2)
$$(-\Delta)^s u(x) = a_{N,s} \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy \quad \text{for } x \in \Omega.$$

In the situation of this lemma, $(-\Delta)^s u$ is a priori only well defined a.e. in Ω as a regular distribution. Nevertheless, under these hypotheses, we may assume in the following that $(-\Delta)^s u$ is defined pointwise by (2.2) in all of Ω .

Lemma 2.4 Let $\Omega \subset \mathbb{R}^N$ be open, and suppose that $u \in \mathcal{L}^1_s \cap C^2(\Omega)$ satisfies $(-\Delta)^s u \leq 0$ in Ω . Suppose furthermore that there exists $x_0 \in \Omega$ such that $u(x_0) \geq u(y)$ for a.e. $y \in \mathbb{R}^N$. Then $u(y) = u(x_0)$ for every $y \in \Omega$ and a.e. $y \in \mathbb{R}^N \setminus \Omega$.

Proof. Let $x_0 \in \Omega$ satisfy $u(x_0) \ge u(y)$ for a.e. $y \in \mathbb{R}^N$. Then the function

$$\varepsilon \to h(\varepsilon) := \int_{|x-y| > \varepsilon} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} \, dy$$

is nonnegative and nonincreasing in $(0, \infty)$, whereas $\lim_{\varepsilon \to 0} h(\varepsilon) \leq 0$ by assumption and (2.2). Hence $h \equiv 0$ in $(0, \infty)$, which, since $u \in C^2(\Omega)$, shows that $u(y) = u(x_0)$ for every $y \in \Omega$ and a.e. $y \in \mathbb{R}^N \setminus \Omega$.

Next we consider the Poisson kernel of B_R (see [5]) which is given by

(2.3)
$$\Gamma_R(x,y) = C_{N,s} \left(\frac{R^2 - |x|^2}{|y|^2 - R^2}\right)^s |x - y|^{-N}, \qquad |y| > R, \ |x| < R$$

and $\Gamma_R(x, y) = 0$ elsewhere. The constant $C_{N,s}$ is chosen such that

$$\int_{\mathbb{R}^N} \Gamma_R(x, y) \, dy = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) \, dy = 1 \quad \text{for every } x \in B_R.$$

Lemma 2.5 Let R > 0, $g \in \mathcal{L}_s^1$, and suppose that g is bounded in a neighborhood of B_R . Then the problem

(2.4)
$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_R, \\ u = g & \text{in } \mathbb{R}^N \setminus B_R \end{cases}$$

has a unique solution $u \in \mathcal{L}^1_s \cap C^2(B_R) \cap L^\infty(B_R)$ given by

(2.5)
$$u(x) = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) g(y) dy \quad \text{for } x \in B_R$$

Moreover, $u \in C^{\infty}(B_R)$.

Proof. Let $u : \mathbb{R}^N \to \mathbb{R}$ be defined by $u \equiv g$ in $\mathbb{R}^N \setminus B_R$ and by (2.5) in B_R . Using the explicit representation (2.3) and the assumption that $g \in \mathcal{L}^1_s$ is bounded in a neighborhood of B_R , it is easy to see that $u \in \mathcal{L}^1_s \cap C^{\infty}(B_R) \cap L^{\infty}(B_R)$, and that

$$(-\Delta)^s u(x) = a_{N,s} \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = 0 \quad \text{for all } x \in B_R.$$

Conversely, let $u \in \mathcal{L}^1_s \cap C^2(B_R) \cap L^\infty(B_R)$ satisfy (2.4). Let $r \in (0, R)$ and define $v_r \in C^2(B_r) \cap L^\infty(\mathbb{R}^N)$ by $v_r \equiv u$ in $\mathbb{R}^N \setminus B_r$ and

(2.6)
$$v_r(x) = \int_{\mathbb{R}^N \setminus B_r} \Gamma_r(x, y) u(y) dy \quad \text{for } x \in B_r$$

Since u is continuous in B_R , it is not difficult to see from (2.6) that $v_r \in C(B_R)$. Applying Lemma 2.4 to $\Omega = B_r$ and the functions $u - v_r$, $v_r - u$ which are continuous on \mathbb{R}^N and vanish on $\mathbb{R}^N \setminus B_r$, we infer that $u \equiv v_r$. Passing to the limit $r \mapsto R^-$ and using the fact that u is bounded in a neighborhood of B_R , we conclude that

$$u(x) = \lim_{r \mapsto R^-} \int_{\mathbb{R}^N \setminus B_r} \Gamma_r(x, y) u(y) dy = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) u(y) dy = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) g(y) dy,$$

as claimed.

Next, we wish to remove the C^2 -assumption in Lemma 2.4. For this we consider the regularization of Γ_R as defined in [8]. Let $\chi \in C_c^{\infty}(1/2, 1)$ such that $\int_{1/2}^1 \chi(r) dr = 1$ and define

$$\tilde{\Gamma}: \mathbb{R}^N \to \mathbb{R}, \qquad \tilde{\Gamma}(y) = \int_{1/2}^1 \chi(r) \Gamma_r(0, y) dr$$

We have $\tilde{\Gamma} \in C^{\infty}(\mathbb{R}^N)$ and

(2.7)
$$|\tilde{\Gamma}(y)| \le \frac{C_{N,s}}{1+|y|^{N+2+2s}} \quad \text{for } y \in \mathbb{R}^N$$

with a constant $C_{N,s}$, see for instance [8, Lemma 3.11]. For $\varepsilon > 0$, we define

$$\widetilde{\Gamma}_{\varepsilon} : \mathbb{R}^N \to \mathbb{R}, \qquad \widetilde{\Gamma}_{\varepsilon}(x) = \varepsilon^{-N} \widetilde{\Gamma}(x/\varepsilon).$$

The following result is given in [8, Theorem 3.12] for the case $N \ge 2$, but the same proof also gives the result for N = 1. For the convenience of the reader, we include the proof here.

Theorem 2.6 Let $\Omega \subset \mathbb{R}^N$ be open, and let $u \in \mathcal{L}^1_s$ satisfy $(-\Delta)^s u = 0$ in Ω , *i.e.*,

(2.8)
$$\int_{\mathbb{R}^N} u(-\Delta)^s \psi \, dx = 0 \quad \text{for all } \psi \in C_c^\infty(\Omega).$$

Then for every $\varepsilon > 0$ we have

(2.9)
$$u \equiv \tilde{\Gamma}_{\varepsilon} * u \quad a.e. \ in \ \Omega_{\varepsilon},$$

where $\Omega_{\varepsilon} := \{x \in \Omega : dist(x, \partial \Omega) > \varepsilon\}$. In particular, u is equivalent to a C^{∞} -function in Ω .

Proof. We first remark that, by Lemma 2.5 and Fubini's theorem, the equality (2.9) holds under the additional assumption $u \in C^2(\Omega)$. Let $\rho_n \in C_c^{\infty}(B_{\frac{1}{n}})$ denote the standard radially symmetric mollifier for $n \in \mathbb{N}$. For every $\psi \in C_c^{\infty}(\Omega_{\frac{1}{n}})$, we then have $\rho_n * (-\Delta)^s \psi = (-\Delta)^s [\rho_n * \psi]$ in \mathbb{R}^N and therefore, by Fubini's theorem and (2.8),

$$\int_{\mathbb{R}^N} [\rho_n * u] (-\Delta)^s \psi \, dx = \int_{\mathbb{R}^N} u \left[\rho_n * (-\Delta)^s \psi \right] dy = \int_{\mathbb{R}^N} u \left(-\Delta \right)^s [\rho_n * \psi] \, dy = 0.$$

Letting $u_n = \rho_n * u \in C^{\infty}(\mathbb{R}^N)$ for $n \in \mathbb{N}$, we deduce that $(-\Delta)^s u_n \equiv 0$ in $\Omega_{\frac{1}{n}}$. By the remark above, we thus have $u_n \equiv \tilde{\Gamma}_{\varepsilon} * u_n$ in $\Omega_{\varepsilon + \frac{1}{n}}$. By (2.7), we now may pass to the limit $n \to \infty$ to get $u \equiv \tilde{\Gamma}_{\varepsilon} * u$ a.e. in Ω_{ε} , as claimed.

Corollary 2.7 Let $\Omega \subset \mathbb{R}^N$ be open and $u \in \mathcal{L}^1_s \cap C(\Omega)$ such that $(-\Delta)^s u = 0$ in Ω . Then if u attains it's maximum or it's minimum in Ω it is constant.

Proof. By Theorem 2.6 and the continuity of u, we have $u \in C^{\infty}(\Omega)$, so the result follows from Lemma 2.4.

We finally obtain the following result which improves Lemma 2.5.

Lemma 2.8 Let $R, \delta > 0$, and let $u \in \mathcal{L}^1_s \cap L^{\infty}(B_{R+\delta})$ satisfy $(-\Delta)^s u = 0$ in B_R . Then

$$u(x) = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) u(y) dy$$
 for a.e. $x \in B_R$.

Moreover, u is equivalent to a C^{∞} -function in B_R .

Proof. By Theorem 2.6 we may assume that $u \in C^{\infty}(B_R)$. Hence the result follows from Lemma 2.5.

Next we consider the Green function associated with $(-\Delta)^s$ and the unit ball **B**, which was computed by Blumenthal, Getoor and Ray in [5]. It is given by

$$G_1(x,y) = k_N^s |x-y|^{2s-N} \int_1^{(\psi(x,y)+1)^{1/2}} \frac{(z^2-1)^{s-1}}{z^{N-1}} dz$$

= $\frac{k_N^s}{2} |x-y|^{2s-N} \int_0^{\psi(x,y)} \frac{z^{s-1}}{(z+1)^{N/2}} dz$ with $\psi(x,y) = \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}$

for $x, y \in \mathbf{B}$ and G(x, y) = 0 if $x \notin \mathbf{B}$ or $y \notin B$. Here the normalization constant is given by $k_N^s = \pi^{-(N/2+1)} \Gamma(N/2) \sin(\pi s)$, see [5]. If N = 1 = 2s, then direct computations give

$$\int_0^{\psi(x,y)} \frac{z^{-1/2}}{(z+1)^{1/2}} \, dz = 2\log\frac{1-xy+(1-x^2)^{1/2}(1-y^2)^{1/2}}{|x-y|}$$

and yet, see [5], in this case

(2.10)
$$G_1(x,y) = \frac{1}{\pi} \log \frac{1 - xy + (1 - x^2)^{1/2} (1 - y^2)^{1/2}}{|x - y|}.$$

The explicit form of G_1 gives rise to the following estimates for $x, y \in B$:

$$(2.11) G_{\Omega}(x,y) \leq \begin{cases} C|x-y|^{2s-N} \min\left(\frac{d^{s}(x)d^{s}(y)}{|x-y|^{2s}},1\right) & \text{if } N > 2s; \\ C\min\left(\frac{d^{1/2}(x)d^{1/2}(y)}{|x-y|},\log\frac{3}{|x-y|}\right) & \text{if } N = 1 = 2s; \\ C|x-y|^{2s-1}\min\left(\frac{(d(x)d(y))^{(2s-1)/2}}{|x-y|^{2s-1}},\frac{d^{s}(x)d^{s}(y)}{|x-y|^{2s}}\right) & \text{if } N = 1 < 2s. \end{cases}$$

Here $z \mapsto d(z) = 1 - |z|$ is the distance function to $\mathbb{R}^N \setminus B$, and C is a constant depending on N and s. Similar estimates are available for Greens functions in general $C^{1,1}$ -domains, see e.g. [18,28]. By dilation, the Green function for the ball $B_R = \{x \in \mathbb{R}^N : |x| < R\}, R > 0$ is given by

(2.12)
$$G_R(x,y) = R^{2s-N} G_1\left(\frac{x}{R}, \frac{y}{R}\right) = \frac{k_N^s}{2} |x-y|^{2s-N} \int_0^{\psi_R(x,y)} \frac{z^{s-1}}{(z+1)^{N/2}} dz$$

with $\psi_R(x,y) = \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2 |x-y|^2}$. In the next section, we will need the following general Green-Poisson representation formula.

Corollary 2.9 Let $R, \delta > 0$ and $f \in L^{\infty}(B_R)$. Moreover, let $v \in \mathcal{L}^1_s \cap L^{\infty}(B_{R+\delta})$ satisfy $(-\Delta)^s v = f$ in B_R . Then

(2.13)
$$v(x) = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) v(y) dy + \int_{B_R} G_R(x, y) f(y) dy \quad \text{for a.e. } x \in B_R.$$

Proof. Without loss of generality, we may assume that R = 1. Consider

$$w_0 : \mathbb{R}^N \to \mathbb{R}, \qquad w_0(x) = \int_{\mathbf{B}} G_1(x, y) f(y) dy.$$

The estimates (2.11) imply that $w_0 \in L^{\infty}(\mathbb{R}^N)$. Moreover, $(-\Delta)^s w_0 = f$ in **B** in distributional sense, since for any $\varphi \in C_c^{\infty}(\mathbf{B})$ we have

$$\varphi(x) = \int_{\mathbf{B}} G_1(x, y) [(-\Delta)^s \varphi](y) dy \quad \text{for } x \in \mathbf{B}.$$

We now consider $w := v - w_0 \in \mathcal{L}^1_s \cap L^\infty(B_{R+\delta})$. Clearly $(-\Delta)^s w = 0$ in B_R , and $w \equiv v$ on $\mathbb{R}^N \setminus B_R$. By Lemma 2.8, we have $w(x) = \int_{\mathbb{R}^N \setminus B_R} \Gamma_R(x, y) v(y) dy$ for a.e. $x \in B_R$, and thus (2.13) follows. \Box

We finally add the following boundary estimate.

Lemma 2.10 Let $f \in L^{\infty}(\mathbf{B})$ and consider

$$v : \mathbb{R}^N \to \mathbb{R}, \qquad v(x) = \int_{\mathbf{B}} G_1(x, y) f(y) dy.$$

Then there exists a constant C = C(N, s) > 0 such that for $x \in \mathbf{B}$ we have

(2.14)
$$v(x) \le C \begin{cases} (1-|x|)^s ||f||_{L^{\infty}} & \text{if } 2s \le N; \\ (1-|x|)^{s-\frac{1}{2}} ||f||_{L^{\infty}} & \text{if } N = 1 < 2s \end{cases}$$

Proof. The second inequality in (2.14) is an immediate consequence of the third inequality in (2.11). To prove the first inequality in (2.14), we let d(x) = 1 - |x| and $B^x := \{y \in \mathbb{R}^N : |y-x| < \frac{d(x)}{2}\} \subset \mathbf{B}$ for $x \in \mathbf{B}$. We then have

(2.15)
$$v(x) = \int_{B^x} G_1(x,y)f(y)dy + \int_{\mathbf{B}\setminus B^x} G(x,y)f(y)dy \quad \text{for } x \in \mathbf{B}$$

and

(2.16)
$$d(y) \le |x - y| + d(x) \le 3|x - y| \quad \text{for } y \in \mathbb{R}^N \setminus B^x.$$

In the following, the letter C stands for positive constants depending only on N and s. We first consider the case N > 2s. Then the first inequality in (2.11) implies that

$$\left| \int_{B^x} G_1(x,y) f(y) dy \right| \le C ||f||_{L^{\infty}} \int_{B^x} |x-y|^{2s-N} dy \le C d^{2s}(x) ||f||_{L^{\infty}},$$

and together with (2.16) it also yields

$$\begin{split} \left| \int_{\mathbf{B}\setminus B^x} G(x,y)f(y)dy \right| &\leq Cd^s(x) \|f\|_{L^{\infty}} \int_{\mathbf{B}\setminus B^x} \frac{d^s(y)}{|x-y|^N} dy \leq Cd^s(x) \|f\|_{L^{\infty}} \int_{\mathbf{B}} |x-y|^{s-N} dy \\ &\leq Cd^s(x) \|f\|_{L^{\infty}} \int_{\mathbf{B}} |y|^{s-N} dy = Cd^s(x) \|f\|_{L^{\infty}}. \end{split}$$

Combining these two inequalities, we obtain the assertion in the case N > 2s. Next we consider the case N = 1 = 2s. Applying the second estimate in (2.11) yields

$$\left| \int_{B^x} G_1(x,y) f(y) dy \right| \le C \|f\|_{L^{\infty}} \int_{B^x} \log \frac{3}{|x-y|} \, dy \le C \|f\|_{L^{\infty}} \int_0^{\frac{d(x)}{2}} \log \frac{3}{t} \, dt \le C d^{1/2}(x) \|f\|_{L^{\infty}}$$

and, as before, using (2.16)

$$\begin{split} \left| \int_{\mathbf{B}\setminus B^x} G_1(x,y) f(y) dy \right| &\leq C d^{1/2}(x) \|f\|_{L^{\infty}} \int_{\mathbf{B}\setminus B^x} \frac{d^{1/2}(y)}{|x-y|} \, dy \leq C d^{1/2}(x) \|f\|_{L^{\infty}} \int_{\mathbf{B}\setminus B^x} |x-y|^{-\frac{1}{2}} \, dy \\ &\leq C d^{1/2}(x) \|f\|_{L^{\infty}} \int_0^1 t^{-\frac{1}{2}} \, dt = C d^{1/2}(x) \|f\|_{L^{\infty}}. \end{split}$$

Combining these two inequalities, we obtain the assertion in the case N = 1 = 2s.

3 Green representation on the half-space

The purpose of this section is to state conditions on a function u on the half-space \mathbb{R}^N_+ under which the Green representation formula

$$u(x) = \int_{\mathbb{R}^N_+} G^+_{\infty}(x, y) (-\Delta)^s u(y) \, dy, \qquad x \in \mathbb{R}^N_+$$

holds, where G_{∞}^+ is the half space Green function given by

(3.1)
$$G_{\infty}^{+}(x,y) = \frac{k_{N}^{s}}{2}|x-y|^{2s-N} \int_{0}^{\psi_{\infty}(x,y)} \frac{z^{s-1}}{(z+1)^{N/2}} dz \quad \text{with} \quad \psi_{\infty}(x,y) = \frac{4x_{1}y_{1}}{|x-y|^{2s-N}} dz$$

for $x, y \in \mathbb{R}^N_+$. More precisely, we have the following

Theorem 3.1 Let $f \in L^{\infty}(\mathbb{R}^N_+)$ be nonnegative, and suppose that $u \in L^{\infty}(\mathbb{R}^N)$ satisfies

 $(-\Delta)^s u = f$ in \mathbb{R}^N_+ , u = 0 in $\mathbb{R}^N \setminus \mathbb{R}^N_+$.

Then u is continuous, and

(3.2)
$$u(x) = \int_{\mathbb{R}^N_+} G^+_{\infty}(x, y) f(y) \, dy \qquad \text{for every } x \in \mathbb{R}^N_+.$$

Moreover, there exist constants C > 0, $\alpha \in (0,1)$ depending only on N, s, $||f||_{L^{\infty}}$, $||u||_{L^{\infty}}$ such that (3.3) $0 \le u(x) \le Cx_1^{\alpha}$ for every $x \in \mathbb{R}^N_+$. The remainder of this section will be devoted to the proof of this Theorem. We first show how G_{∞}^+ , as defined in (3.1), arises via an approximation with balls. For this we let $P_R := (R, 0, \ldots, 0) \in \mathbb{R}_+^N$, and we consider the translated ball $B_R^+ := \{x \in \mathbb{R}^N : |x - P_R| < R\} \subset \mathbb{R}_+^N$ for R > 0. By (2.12), its Green function G_R^+ is given by

$$G_R^+(x,y) = \frac{k_N^s}{2} |x-y|^{2s-N} \int_0^{\psi_R^+(x,y)} \frac{z^{s-1}}{(z+1)^{N/2}} dz \quad \text{for } x, y \in B_R^+$$

with

(3.4)
$$\psi_R^+(x,y) = \frac{(R^2 - |x - P_R|^2)(R^2 - |y - P_R|^2)}{R^2 |x - y|^2} = \frac{(2x_1 - \frac{|x|^2}{R})(2y_1 - \frac{|y|^2}{R})}{|x - y|^2},$$

and its Poisson kernel is given by

(3.5)
$$\Gamma_R^+(x,y) = C_{N,s} \left(\frac{R^2 - |x - P_R|^2}{|y - P_R|^2 - R^2}\right)^s |x - y|^{-N} \quad \text{for } |x| < R < |y|$$

Note that

(3.6)
$$B_R^+ \subset B_{R'}^+$$
 if $R' > R > 0$ and $\bigcup_{R>0} B_R^+ = \mathbb{R}_+^N$.

Lemma 3.2 Let $x, y \in \mathbb{R}^N_+$ and $R_0 > 0$ with $x, y \in B^+_{R_0}$. Then $G^+_{R'}(x, y) \ge G^+_R(x, y)$ for $R' \ge R \ge R_0$ and $G^+_R(x, y) \to G^+_\infty(x, y)$ as $R \to \infty$.

Proof. From (3.4) we immediately deduce that $\psi_{R'}^+(x,y) \ge \psi_R^+(x,y)$ for $R' \ge R \ge R_0$, and that $\psi_R^+(x,y) \to \psi_\infty(x,y)$ as $R \to \infty$. This shows the claim.

We may now complete the

Proof of Theorem 3.1. It follows from Lemma 6.1 below that u is continuous in \mathbb{R}^N_+ . Since G^+_{∞} is invariant under translations of the form $(x, y) \mapsto (x + z, y + z)$ with $z \in \{0\} \times \mathbb{R}^{N-1}$, it suffices to show (3.2) for $x = (x_1, 0, \ldots, 0)$ with $x_1 > 0$. We will fix such a point x from now on. By Corollary 2.9 and the fact that $u \equiv 0$ in $\mathbb{R}^N \setminus \mathbb{R}^N_+$, we have

$$u(x) = \int_{\mathbb{R}^N_+ \setminus B^+_R} \Gamma^+_R(x, y) u(y) dy + \int_{B^+_R} G^+_R(x, y) f(y) \, dy \quad \text{ for } x \in B^+_R.$$

Thanks to Lemma 3.2, the nonnegativity of f and monotone convergence, (3.2) follows once we have shown that

(3.7)
$$\int_{\mathbb{R}^N_+ \setminus B^+_R} \Gamma^+_R(x, y) u(y) dy \to 0 \quad \text{as } R \to \infty.$$

In the following, we assume that $R > 2x_1$, and we let C denote (possibly different) constants which may depend on N, s, u and x_1 but not on R. Using the fact that u is bounded, we have

$$\begin{split} \left| \int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{R}} \Gamma^{+}_{R}(x,y) u(y) dy \right| &\leq C \int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{R}} \Gamma^{+}_{R}(x,y) dy = C \int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{R}} \left(\frac{R^{2} - |x - P_{R}|^{2}}{|y - P_{R}|^{2} - R^{2}} \right)^{s} |x - y|^{-N} dy \\ &= C (2x_{1}R - x_{1}^{2})^{s} \int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{R}} |x - y|^{-N} (|y|^{2} - 2y_{1}R)^{-s} dy \\ (3.8) \qquad \leq C R^{s} \int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{R}} |x - y|^{-N} (|y|^{2} - 2y_{1}R)^{-s} dy = C R^{-s} \int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{1}} |y - \frac{x}{R}|^{-N} (|y|^{2} - 2y_{1})^{-s} dy. \end{split}$$

We will now show that, as $R \to \infty$,

(3.9)
$$\int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{1}} |y - \frac{x}{R}|^{-N} (|y|^{2} - 2y_{1})^{-s} dy = \begin{cases} O(1) & \text{if } 0 < s < \frac{1}{2}; \\ O(R^{2s-1}) & \text{if } \frac{1}{2} < s < 1; \\ O(\log R^{2}) & \text{if } s = \frac{1}{2}. \end{cases}$$

Together with (3.8) this implies (3.7), since s < 1. To show (3.9), we decompose the domain of integration as

$$R^{\mathbb{N}}_+ \setminus B^+_1 = A_1 \cup A_2 \qquad \text{with} \quad A_1 := ((0,1) \times \mathbb{R}^{N-1}) \setminus B^+_1, \quad A_2 := ([1,\infty) \times \mathbb{R}^{N-1}) \setminus B^+_1.$$

We then have $|y - \frac{x}{R}| \ge |y - e_1|$ for every $y \in A_2$, where $e_1 = (1, 0, \dots, 0)$, and therefore

(3.10)
$$\int_{A_2} |y - \frac{x}{R}|^{-N} (|y|^2 - 2y_1)^{-s} dy \le \int_{A_2} |y - e_1|^{-N} (|y - e_1|^2 - 1)^{-s} dy \le \int_{\mathbb{R}^N \setminus B_1(0)} |y|^{-N} (|y|^2 - 1)^{-s} dy = \omega_{N-1} \int_1^\infty \tau^{-1} (\tau^2 - 1)^{-s} d\tau < \infty.$$

In case N = 1, A_1 is empty, and thus (3.9) follows. We now assume that $N \ge 2$, and we put $B_t := \{\tilde{y} \in \mathbb{R}^{N-1} : |\tilde{y}|^2 \ge 2t - t^2\}$ for $t \in (0, 1)$. By Fubini's theorem, we have

$$\begin{aligned} \int_{A_1} |y - \frac{x}{R}|^{-N} (|y|^2 - 2y_1)^{-s} dy &= \int_0^1 \int_{B_t} (|\tilde{y}|^2 + (t - \frac{x_1}{R})^2)^{-\frac{N}{2}} (|\tilde{y}|^2 + t^2 - 2t)^{-s} d\tilde{y} dt \\ &= \omega_{N-2} \int_0^1 \int_{\sqrt{2t-t^2}}^{\infty} \tau^{N-2} (\tau^2 + (t - \frac{x_1}{R})^2)^{-\frac{N}{2}} (\tau^2 - [2t - t^2])^{-s} d\tau dt \\ &= \omega_{N-2} \int_0^1 (2t - t^2)^{-\frac{1}{2}-s} \int_1^{\infty} \sigma^{N-2} \left(\sigma^2 + \frac{(t - \frac{x_1}{R})^2}{2t - t^2}\right)^{-\frac{N}{2}} (\sigma^2 - 1)^{-s} d\sigma dt \\ &\leq \omega_{N-2} \int_0^1 t^{-\frac{1}{2}-s} \int_0^{\infty} (\rho + 1)^{\frac{N-3}{2}} \left(\rho + 1 + \frac{(t - \frac{x_1}{R})^2}{2t}\right)^{-\frac{N}{2}} \rho^{-s} d\rho dt \\ &\leq \omega_{N-2} \int_0^1 t^{-\frac{1}{2}-s} \left(1 + \frac{(t - \frac{x_1}{R})^2}{2t}\right)^{-\frac{1}{2}} \int_0^{\infty} (\rho + 1)^{-1} \rho^{-s} d\rho dt \end{aligned}$$

$$(3.11) \qquad \leq C \int_0^1 t^{-\frac{1}{2}-s} \left(1 + \frac{(t - \frac{x_1}{R})^2}{2t}\right)^{-\frac{1}{2}} dt,$$

whereas

$$\left(1 + \frac{(t - \frac{x_1}{R})^2}{2t}\right)^{-\frac{1}{2}} \le \sqrt{2t} \left(2t + (t - \frac{x_1}{R})^2\right)^{-\frac{1}{2}} \le \min\left\{1, \frac{\sqrt{2tR}}{x_1}\right\} \quad \text{for } t \in (0, 1),$$

since $\frac{x_1}{R} \leq 1$ by assumption. Inserting this in (3.11) yields

$$\int_{A_1} |y - \frac{x}{R}|^{-N} (|y|^2 - 2y_1)^{-s} dy \le C \left(\frac{\sqrt{2R}}{x_1} \int_0^{\frac{x_1^2}{2R^2}} t^{-s} ds + \int_{\frac{x_1^2}{2R^2}}^1 t^{-\frac{1}{2}-s} dt\right) \le C \begin{cases} 1 + R^{2s-1} & \text{if } s \neq \frac{1}{2} \\ 1 + \log R^2 & \text{if } s = \frac{1}{2} \end{cases}$$

Combining the last inequality with (3.10), we deduce (3.9), as required. Thus the proof of (3.2) is finished.

To show (3.3), we may assume without loss that $x = (x_1, 0, ..., 0)$ with $0 \le x_1 \le 1$. Moreover, we let C > 0 denote constants depending on N, s, $||f||_{L^{\infty}(\mathbb{R}^N_+)}$ and $||u||_{L^{\infty}(\mathbb{R}^N_+)}$ but not on x. By Corollary 2.9, we have

$$u(x) = \int_{\mathbb{R}^N \setminus B_1^+} \Gamma_1^+(x, y) u(y) \, dy + v(x) \qquad \text{with} \quad v(x) = \int_{B_1^+} G_1^+(x, y) f(y) \, dy.$$

By Lemma 2.10, it suffices to show (3.3) for v in place of u. For this we estimate, similarly as in (3.8),

(3.12)
$$0 \le v(x) \le C \int_{\mathbb{R}^N_+ \setminus B^+_1} \Gamma^+_1(x, y) dy \le C (2x_1 - x_1^2)^s \int_{\mathbb{R}^N_+ \setminus B^+_1} |x - y|^{-N} (|y|^2 - 2y_1)^{-s} dy \le C x_1^s \int_{\mathbb{R}^N_+ \setminus B^+_1} |y - x|^{-N} (|y|^2 - 2y_1)^{-s} dy.$$

Replacing $\frac{x}{R}$ with $x = (x_1, 0, \dots, 0)$ in (3.9), we also have, as $x_1 \to 0^+$,

(3.13)
$$\int_{\mathbb{R}^{N}_{+} \setminus B^{+}_{1}} |y - x|^{-N} (|y|^{2} - 2y_{1})^{-s} dy = \begin{cases} O(1) & \text{if } 0 < s < \frac{1}{2}; \\ O(x_{1}^{1-2s}) & \text{if } \frac{1}{2} < s < 1; \\ O(\log \frac{1}{x_{1}^{2}}) & \text{if } s = \frac{1}{2}. \end{cases}$$

Combining (3.12) and (3.13) yields constants C > 0, $\alpha \in (0, 1)$ such that (3.3) holds for v in place of u, and this finishes the proof.

4 Proof of the monotonicity result

In this section we complete the proof of Theorem 1.1. We know from Theorem 3.1 that every bounded solution u of (1.1) obeys the integral representation

$$u(x) = \int_{\mathbb{R}^N_+} G^+_{\infty}(x, y) f(u(y)) \, dy \qquad \text{for all } x \in \mathbb{R}^N_+,$$

where G_{∞}^+ is the half-space Green function given by (3.1). We note the following simple estimate.

Lemma 4.1 Let L > 0. Then there exists a constant C = C(N, s, L) > 0 such that for every $x, y \in \mathbb{R}^N_+$ with $x_1, y_1 \leq L$ we have

(4.1)
$$G_{\infty}^{+}(x,y) \leq C \begin{cases} \min\{|x-y|^{2s-N}, |x-y|^{-N}\} & \text{if } N > 2s; \\ 1 & \text{if } N = 1 < 2s; \\ 1 + \log\frac{L}{|x-y|} & \text{if } N = 1 = 2s. \end{cases}$$

Proof. In the case N > 2s we have $\int_0^\infty \frac{z^{s-1}}{(z+1)^{N/2}} dz < \infty$, which immediately implies that $G_\infty^+(x,y) \le C|x-y|^{2s-N}$ for all $x, y \in \mathbb{R}^N_+$ with a constant C > 0 depending only on N and s. Moreover, for t > 0 we have

$$\int_0^t \frac{z^{s-1}}{(z+1)^{N/2}} \, dz < \frac{t^s}{s}$$

and

$$\int_0^t \frac{z^{s-1}}{(z+1)^{N/2}} dz \le \begin{cases} \frac{t^{s-N/2}}{s-N/2} & \text{if } N < 2s, \\ \frac{1}{s} + \log \max\{1,t\} & \text{if } N = 2s. \end{cases}$$

For $x, y \in \mathbb{R}^N_+$ with $x_1, y_1 \leq L$ we have also $\psi_{\infty}(x, y) \leq \frac{L^2}{|x-y|^2}$ and therefore

$$\int_0^{\psi_\infty(x,y)} \frac{z^{s-1}}{(z+1)^{N/2}} \, dz \le C |x-y|^{-2s}$$

as well as

$$\int_{0}^{\psi_{\infty}(x,y)} \frac{z^{s-1}}{(z+1)^{N/2}} \, dz \le C \begin{cases} |x-y|^{N-2s} & \text{if } N < 2s \\ 1+\ln\frac{L}{|x-y|} & \text{if } N < 2s \end{cases}$$

with a constant C > 0 depending only on N, s, L. This readily implies the assertion.

We need some notation. For $\lambda \geq 0$, we consider the set

$$\Sigma_{\lambda} = \{ x \in \mathbb{R}^N : 0 < x_1 < \lambda \} \subset \mathbb{R}^N_+.$$

From Lemma 4.1 we easily deduce the following.

Corollary 4.2 As
$$\lambda \to 0$$
, $\sup_{x \in \Sigma_{\lambda}} \int_{\Sigma_{\lambda}} G_{\infty}^{+}(x, y) dy \to 0.$

We also consider the reflection $x \mapsto x^{\lambda} := (2\lambda - x_1, x_2, \dots, x_N)$ at the hyperplane $\{x_1 = \lambda\}$. We need the following fact which also follows from Lemma 4.1 in a straightforward way.

Corollary 4.3 If $(\lambda_n)_n \subset (0,\infty)$ and $(z^n)_n \subset \mathbb{R}^N_+$ are sequences with

$$\lambda_n \to \lambda > 0 \quad and \quad z^n \to z \in \mathbb{R}^N_+ \qquad as \ n \to \infty,$$

then

$$\int_{\Sigma_{\lambda_n}} G^+_{\infty}(z^n, y^{\lambda_n}) \, dy \to \int_{\Sigma_{\lambda}} G^+_{\infty}(z, y^{\lambda}) \, dy \qquad \text{as } n \to \infty.$$

In the sequel we also consider $J_{\lambda} := \{x \in \mathbb{R}^N_+ : x_1 \ge 2\lambda\}$. We need the following "reflection inequalities" for the Green function.

Lemma 4.4 We have

(4.2)
$$\begin{cases} G_{\infty}^{+}(x^{\lambda}, y^{\lambda}) > G_{\infty}^{+}(x, y^{\lambda}) & and \\ G_{\infty}^{+}(x^{\lambda}, y^{\lambda}) - G_{\infty}^{+}(x, y) > G_{\infty}^{+}(x, y^{\lambda}) - G_{\infty}^{+}(x^{\lambda}, y) \end{cases}$$
 for all $x, y \in \Sigma_{\lambda}$

and

(4.3)
$$G_{\infty}^{+}(x^{\lambda}, y) - G_{\infty}^{+}(x, y) > 0 \quad \text{for } x \in \Sigma_{\lambda}, \ y \in J_{\lambda}.$$

Proof. We note that G(x, y) = H(s(x, y), t(x, y)), where

$$H: (0,\infty) \times [0,\infty) \to \mathbb{R}, \qquad H(r,t) = r^{s-\frac{N}{2}} \int_0^{\frac{t}{r}} \frac{z^{s-1}}{(z+1)^{N/2}} \, dz.$$

and

$$r(x,y) = |x-y|^2, \quad t(x,y) = 4x_1y_1 \quad \text{for } x, y \in \mathbb{R}^N_+.$$

By [4, Lemma 2] we have

(4.4)
$$\partial_r H(r,t) < 0, \quad \partial_t H(r,t) > 0 \quad \text{and} \quad \partial_r \partial_t H(r,t) < 0 \quad \text{for } r,t > 0.$$

Hence

(4.5)
$$H(r_1, t_1) > H(r_2, t_2) \quad \text{if } r_1 < r_2, \ t_1 > t_2,$$

and

$$H(r_1, t_4) - H(r_1, t_1) = \int_{t_1}^{t_4} \partial_t H(r_1, t) \, dt > \int_{t_1}^{t_4} \partial_t H(r_2, t) \, dt > \int_{\min\{t_2, t_3\}}^{\max\{t_2, t_3\}} \partial_t H(r_2, t) \, dt$$

$$(4.6) = |H(r_2, t_2) - H(r_2, t_3)| \quad \text{if } 0 < r_1 < r_2, \ 0 < t_1 < t_2, t_3 < t_4.$$

Now fix $\lambda > 0$. Then for $x, y \in \Sigma_{\lambda}$ we have

(4.7)
$$r(x,y) = r(x^{\lambda}, y^{\lambda}) < r(x, y^{\lambda}) = r(x^{\lambda}, y)$$

and

(4.8)
$$t(x^{\lambda}, y^{\lambda}) > \max\{t(x^{\lambda}, y), t(x, y^{\lambda})\} \ge \min\{t(x^{\lambda}, y), t(x, y^{\lambda})\} > t(x, y).$$

The inequalities given in Lemma 4.4 now follow from (4.5), (4.6), (4.7) and (4.8).

We now fix a solution u of (1.1), and we let $C_u > 0$ be a Lipschitz constant for f on $[0, ||u||_{L^{\infty}(\mathbb{R}^N)}]$, so that].

$$|f(t) - f(r)| \le C_u |t - r| \qquad \text{for all } r, t \in [0, ||u||_{L^{\infty}(\mathbb{R}^N)}]$$

Inequality (4.3) and the nonnegativity of f imply that

(4.9)
$$\int_{J_{\lambda}} [G_{\infty}^{+}(x^{\lambda}, y) - G_{\infty}^{+}(x, y)] f(u(y)) dy \ge 0 \quad \text{for } \lambda > 0 \text{ and } x \in \Sigma_{\lambda}.$$

We claim that the following reflection inequality holds for every $\lambda > 0$:

$$(\mathcal{C}_{\lambda}) \qquad \qquad u(x) \le u(x^{\lambda}) \qquad \text{for all } x \in \Sigma_{\lambda}.$$

As a first step, we prove

Lemma 4.5 There exists $\lambda_0 > 0$ such that (\mathcal{C}_{λ}) holds for $\lambda \in [0, \lambda_0]$.

Proof. By Corollary 4.2, we may fix $\lambda_0 > 0$ such that

(4.10)
$$\int_{\Sigma_{2\lambda_0}} G^+_{\infty}(x,y) \, dy < C_u^{-1} \quad \text{for every } x \in \Sigma_{\lambda_0},$$

For fixed $\lambda \in [0, \lambda_0]$, we consider the difference function

$$v: \Sigma_{\lambda} \to \mathbb{R}, \qquad v(x) = u(x^{\lambda}) - u(x)$$

and the set

$$W := \{ x \in \Sigma_{\lambda} : v(x) < 0 \}$$

For $x \in W$ we estimate, using Lemma 4.4 and (4.9),

$$\begin{split} 0 > v(x) &= \int_{\mathbb{R}^{N}_{+}} [G^{+}_{\infty}(x^{\lambda}, y) - G^{+}_{\infty}(x, y)] f(u(y)) \, dy = \int_{\Sigma_{\lambda}} \dots \, dy + \int_{\mathbb{R}^{N}_{+} \setminus \Sigma_{\lambda}} \dots \, dy \\ &= \int_{\Sigma_{\lambda}} \left([G^{+}_{\infty}(x^{\lambda}, y) - G^{+}_{\infty}(x, y)] f(u(y)) + [G^{+}_{\infty}(x^{\lambda}, y^{\lambda}) - G^{+}_{\infty}(x, y^{\lambda})] f(u(y^{\lambda})) \right) \, dy \\ &+ \int_{J_{\lambda}} [G(x^{\lambda}, y) - G^{+}_{\infty}(x, y)] f(u(y)) \, dy \geq \int_{\Sigma_{\lambda}} [G^{+}_{\infty}(x^{\lambda}, y^{\lambda}) - G^{+}_{\infty}(x, y^{\lambda})] [f(u(y^{\lambda})) - f(u(y))] \, dy \\ &\geq \int_{W} [G^{+}_{\infty}(x^{\lambda}, y^{\lambda}) - G^{+}_{\infty}(x, y^{\lambda})] [f(u(y^{\lambda})) - f(u(y))] \, dy \geq \int_{W} G^{+}_{\infty}(x^{\lambda}, y^{\lambda}) [f(u(y^{\lambda})) - f(u(y))] \, dy \\ &\geq C_{u} \int_{W} G^{+}_{\infty}(x^{\lambda}, y^{\lambda}) v(y) \, dy \geq -C_{u} \|v\|_{L^{\infty}(W)} \int_{W} G^{+}_{\infty}(x^{\lambda}, y^{\lambda}) \, dy \geq -C_{u} \|v\|_{L^{\infty}(W)} \int_{\Sigma_{2\lambda_{0}}} G^{+}_{\infty}(x^{\lambda}, y) \, dy \end{split}$$

Combining this with (4.10) we infer $||v||_{L^{\infty}(W)} \leq C ||v||_{L^{\infty}(W)}$ with some constant $C \in (0, 1)$, hence $v \equiv 0$ on W and therefore $W = \emptyset$ by definition. We therefore conclude that (\mathcal{C}_{λ}) holds. \Box

Next we put

$$\lambda_* := \sup\{\lambda > 0 : (\mathcal{C}_{\lambda'}) \text{ holds for all } \lambda' \le \lambda\}.$$

Then $\lambda_* \geq \lambda_0$. Using the continuity of u, it is easy to see that $(\mathcal{C}_{\lambda_*})$ holds. We suppose by contradiction that

$$(4.11) \qquad \qquad \lambda_* < \infty$$

Then there exists a sequence of numbers $\lambda_n > \lambda^*$, $n \in \mathbb{N}$ and points $x^n \in \Sigma_{\lambda_n}$ such that

(4.12)
$$u(x^n) > u((x^n)^{\lambda_n}) \qquad \text{for all } n$$

and

(4.13)
$$\lambda_n \to \lambda^*$$
 as $n \to \infty$.

We may further assume that

(4.14)
$$u(x^{n}) - u((x^{n})^{\lambda_{n}}) > \frac{1}{2} \sup_{x \in \Sigma_{\lambda_{n}}} \left(u(x) - u(x^{\lambda_{n}}) \right).$$

In the following we write $x = (x_1, \hat{x})$ for $x \in \mathbb{R}^N$ with $\hat{x} \in \mathbb{R}^{N-1}$. For $n \in \mathbb{N}$ we define the translated functions

$$u_n : \mathbb{R}^N \to \mathbb{R}, \qquad u_n(y) = u(y_1, \hat{x}^n + \hat{y})$$

and

$$v_n: \Sigma_{\lambda_n} \to \mathbb{R}, \qquad v_n(y) = u_n(y^{\lambda_n}) - u(y).$$

Then (4.12) is rewritten as

(4.15)
$$v_n(x_1^n, 0) = u_n(2\lambda_n - x_1^n, 0) - u_n(x_1^n, 0) < 0 \quad \text{for all } n.$$

We also consider the sets

$$W_n := \{ x \in \Sigma_{\lambda_n} : v_n(x) < 0 \}.$$

We let $\Lambda := 2 \max_n \lambda_n < \infty$. For abbreviation, we also put $z^n = (x_1^n, 0) \in \Sigma_{\lambda_n}$ for $n \in \mathbb{N}$. Using (4.14) and arguing similarly as in the proof of Lemma 4.5, we find

$$(4.16) - \|v_n\|_{L^{\infty}(W_n)} \ge 2v_n(z^n) \ge 2 \int_{W_n} \left(G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) - G^+_{\infty}(z^n, y^{\lambda_n}) \right) \left[f(u_n(y^{\lambda_n})) - f(u_n(y)) \right] dy$$

We now pass to a subsequence such that $x_1^n \to t \in [0, \lambda_*]$. Moreover, we note that the sequence $(u_n)_n$ is uniformly equicontinuous on compact subsets of \mathbb{R}^N . Indeed, this follows from Lemma 6.1 below and the boundary estimate (3.3) which holds uniformly for $u_n, n \in \mathbb{N}$ in place of u. Hence we may pass to a subsequence such that $u_n \to \overline{u}$ in $C^0_{loc}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ and $\overline{u} \equiv 0$ on $\mathbb{R}^N \setminus \mathbb{R}^N_+$. We distinguish three cases.

Case 1: $t = \lambda_*$, i.e. $z^n \to (\lambda_*, 0)$. In this case (4.16) implies that

$$-\|v_n\|_{L^{\infty}(W_n)} \ge 2C_u \int_{W_n} \left(G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) - G^+_{\infty}(z^n, y^{\lambda_n}) \right) v_n(y) \, dy$$

$$\ge -2C_u \|v_n\|_{L^{\infty}(W_n)} \int_{\Sigma_{\lambda_n}} \left(G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) - G^+_{\infty}(z^n, y^{\lambda_n}) \right) \, dy$$

$$= -o(1) \|v_n\|_{L^{\infty}(W_n)},$$

where in the last step we used Corollary 4.3 and the fact that also $(z^n)^{\lambda_n} \to (\lambda_*, 0)$. Hence $||v_n||_{L^{\infty}(W_n)} = 0$ for *n* large, contrary to (4.15).

Case 2: $t < \lambda^*$ and $\bar{u} \not\equiv 0$.

We have

(4.17)

$$\int_{\mathbb{R}^N} u_n(x)(-\Delta)^s \varphi(x) \, dx = \int_{\mathbb{R}^N_+} f(u_n(x))\varphi(x) \, dx \quad \text{for all } \varphi \in C^2_c(\mathbb{R}^N_+).$$

It thus follows from the dominated convergence theorem that

$$\int_{\mathbb{R}^N} \bar{u}(x)(-\Delta)^s \varphi(x) \, dx = \int_{\mathbb{R}^N_+} f(\bar{u}(x))\varphi(x) \, dx \quad \text{for all } \varphi \in C^2_c(\mathbb{R}^N_+).$$

Hence, by Theorem 3.1, \bar{u} is represented as

$$\bar{u}(x) = \int_{\mathbb{R}^N_+} G^+_{\infty}(x, y) f(\bar{u}(y)) \, dy \qquad \text{for all } x \in \mathbb{R}^N_+$$

Since $u \neq 0$ and f(t) > 0 for t > 0, it then follows that \bar{u} and $f \circ \bar{u}$ are strictly positive on \mathbb{R}^N_+ . Moreover, by the locally uniform convergence $u_n \to \bar{u}$, we have

$$\bar{u}(x) \le \bar{u}(x^{\lambda_*})$$
 for all $x \in \Sigma_{\lambda_*}$.

Indeed this inequality is strict, since we have a strict inequality in (4.9) for \bar{u} in place of u and $\lambda = \lambda^*$, so that

$$\begin{split} \bar{u}(x^{\lambda_{*}}) &- \bar{u}(x) = \int_{\mathbb{R}^{N}_{+}} [G^{+}_{\infty}(x^{\lambda_{*}}, y) - G^{+}_{\infty}(x, y)] f(\bar{u}(y)) \, dy = \int_{\Sigma_{\lambda_{*}}} \dots \, dy + \int_{\mathbb{R}^{N}_{+} \setminus \Sigma_{\lambda_{*}}} \dots \, dy \\ &= \int_{\Sigma_{\lambda_{*}}} \left([G^{+}_{\infty}(x^{\lambda_{*}}, y) - G^{+}_{\infty}(x, y)] f(\bar{u}(y)) + [G^{+}_{\infty}(x^{\lambda_{*}}, y^{\lambda_{*}}) - G^{+}_{\infty}(x, y^{\lambda_{*}})] f(\bar{u}(y^{\lambda_{*}})) \right) \, dy \\ &+ \int_{J_{\lambda_{*}}} [G^{+}_{\infty}(x^{\lambda_{*}}, y) - G^{+}_{\infty}(x, y)] f(\bar{u}(y)) \, dy \\ \end{split}$$

$$(4.18) \qquad > \int_{\Sigma_{\lambda^{*}}} [G^{+}_{\infty}(x^{\lambda_{*}}, y^{\lambda_{*}}) - G^{+}_{\infty}(x, y^{\lambda_{*}})] f(\bar{u}(y^{\lambda_{*}})) - f(\bar{u}(y))] \, dy \ge 0 \quad \text{for } x \in \Sigma_{\lambda_{*}}. \end{split}$$

On the other hand, also by the locally uniform convergence and (4.15), we have

$$\bar{u}(2\lambda_* - t, 0) - \bar{u}(t, 0) = \lim_{n \to \infty} \left(u_n((z^n)^{\lambda_n}) - u(z_n) \right) \le 0$$

This is a contradiction.

Case 3: $\bar{u} \equiv 0$. We use (4.16) and Lemma 4.4 to estimate, for r > 0,

(4.19)

$$- \|v_n\|_{L^{\infty}(W_n)} \ge 2 \int_{W_n} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) [f(u_n(y^{\lambda_n})) - f(u_n(y))] \, dy$$

$$= 2 \int_{W_n \cap B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) [f(u_n(y^{\lambda_n})) - f(u_n(y))] \, dy$$

$$+ 2 \int_{W_n \setminus B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) [f(u_n(y^{\lambda_n})) - f(u_n(y))] \, dy$$

where

$$\int_{W_n \setminus B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) [f(u(y^{\lambda_n})) - f(u(y))] \, dy \ge C_u \int_{W_n \setminus B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) v_n(u(y)) \, dy$$

$$(4.20) \ge -C_u \|v_n\|_{L^{\infty}(W_n)} \int_{\Sigma_\Lambda \setminus B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y) \, dy$$

Since $((z^n)^{\lambda_n})_n \subset \Sigma_{\Lambda}$ is a bounded sequence, Lemma 4.1 implies that

$$\sup_{n \in \mathbb{N}} \int_{\Sigma_{\Lambda} \setminus B_{r}(0)} G_{\infty}^{+}((z^{n})^{\lambda_{n}}, y) \, dy \to 0 \qquad \text{as } r \to \infty.$$

Hence we may fix r > 0 such that

(4.21)
$$\int_{\Sigma_{\Lambda} \setminus B_{r}(0)} G_{\infty}^{+}((z^{n})^{\lambda_{n}}, y) \, dy < \frac{1}{4C_{u}} \quad \text{for all } n \in \mathbb{N}.$$

Moreover, assumption (1.4) and the locally uniform convergence $u_n \to 0$ imply that there exists a sequence of numbers $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that

$$f(u_n(y^{\lambda_n})) - f(u_n(y)) \ge \varepsilon_n v_n(y)$$
 for $y \in W_n \cap B_r(0)$,

so that

$$\int_{W_n \cap B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) [f(u_n(y^{\lambda_n})) - f(u_n(y))] \, dy \ge -\varepsilon_n \|v_n\|_{L^{\infty}(W_n)} \int_{B_r(0)} G^+_{\infty}((z^n)^{\lambda_n}, y^{\lambda_n}) \, dy$$

$$\ge -C\varepsilon_n \|v_n\|_{L^{\infty}(W_n)}$$

with some constant C = C(r) > 0. Combining (4.19), (4.20), (4.21) and (4.22), we get

$$\|v_n\|_{L^{\infty}(W_n)} \le \left(\frac{1}{2} + 2C\varepsilon_n\right)\|v_n\|_{L^{\infty}(W_n)},$$

so we conclude that $||v_n||_{L^{\infty}}(W_n) = 0$ for large n, contradicting again (4.15). We have thus proved that property (\mathcal{C}_{λ}) holds for all $\lambda > 0$, which implies that u is increasing in x_1 . It thus remains to show that u is strictly increasing in x_1 if $u \neq 0$. This however follows since we can now derive inequality (4.18) for u in place of \bar{u} and all $\lambda > 0$ in place of λ^* . The proof of Theorem 1.1 is thus finished.

5 Proof of the Liouville Theorem in the half-space

This section is devoted to the proof of Theorem 1.2. Let $s \in (0,1)$ and assume that q > 1 if $N \leq 1 + 2s$ and $1 < q < \frac{N-1+2s}{N-1-2s}$ if N > 1 + 2s. Suppose by contradiction that there exists a bounded solution $u \neq 0$ of (1.5). Theorem 1.1 implies that u is strictly increasing in x_1 . In particular, we may define

$$\tilde{u} \in L^{\infty}(\mathbb{R}^{N-1}), \qquad \tilde{u}(x') = \lim_{x_1 \to \infty} \tilde{u}(x_1, x') \qquad \text{for } x' \in \mathbb{R}^{N-1}.$$

Here and in the following, we write $x = (x_1, x') \in \mathbb{R}^N$ with $x' \in \mathbb{R}^{N-1}$. Note that $\tilde{u} > 0$ on \mathbb{R}^{N-1} . For $\tau > 0$, define $u_{\tau} \in L^{\infty}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ by $u_{\tau}(x) = u(x + \tau e_1)$. Then

(5.1)
$$u_{\tau} \to u_{\infty}$$
 pointwise on \mathbb{R}^N with $u_{\infty} \in L^{\infty}(\mathbb{R}^N)$ defined by $u_{\infty}(x) = \tilde{u}(x')$.

Moreover, we have $(-\Delta)^s u_{\tau} = u_{\tau}^q$ in $H_{\tau} := \{x \in \mathbb{R}^N : x_1 > -\tau\}$. Hence (5.1) implies that $(-\Delta)^s u_{\infty} = u_{\infty}^q$ in \mathbb{R}^N in distributional sense. Indeed, let $\varphi \in C_c^2(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} u_{\infty}^q(x)\varphi(x)\,dx = \lim_{\tau \to \infty} \int_{H_{\tau}} u_{\tau}^q(x)\varphi(x)\,dx = \lim_{\tau \to \infty} \int_{\mathbb{R}^N} u_{\tau}(x)[(-\Delta)^s\varphi](x)\,dx$$
$$= \int_{\mathbb{R}^N} u_{\infty}(x)[(-\Delta)^s\varphi](x)\,dx$$

by Lebesgue's theorem and the estimate (1.3). It then follows from the regularity results in [42] and [14] that also $(-\Delta)^s u_{\infty} = u_{\infty}^q$ in classical sense in \mathbb{R}^N . Moreover, $u_{\infty} > 0$ in \mathbb{R}^N since $\tilde{u} > 0$ in \mathbb{R}^{N-1} . In case N = 1, u_{∞} is a positive constant on \mathbb{R} which contradicts the equation $(-\Delta)^s u_{\infty} = u_{\infty}^q$. In case $N \ge 2$, we deduce that

(5.2)
$$(-\Delta)^s \tilde{u} = \tilde{u}^q \quad \text{in } \mathbb{R}^{N-1}.$$

Indeed,

$$\begin{split} \tilde{u}^{q}(x') &= u_{\infty}^{q}(x) = (-\Delta)^{s} u_{\infty}(x) = a_{N,s} \int_{\mathbb{R}^{N}} \frac{u_{\infty}(x+z) + u_{\infty}(x-z) - 2u_{\infty}(x)}{|z|^{N+2s}} \, dz \\ &= a_{N,s} \int_{\mathbb{R}^{N}} \frac{\tilde{u}(x'+z') + \tilde{u}(x'-z') - 2\tilde{u}(x')}{(|z'|^{2} + z_{1}^{2})^{\frac{N}{2} + s}} \, dz \\ &= a_{N,s} \int_{\mathbb{R}^{N-1}} \frac{\tilde{u}(x'+z') + \tilde{u}(x'-z') - 2\tilde{u}(x')}{|z'|^{N-1+2s}} \left[\frac{1}{|z'|} \int_{\mathbb{R}} \left[1 + \left(\frac{z_{1}}{|z'|}\right)^{2} \right]^{-\frac{N}{2} - s} \, dz_{1} \right] dz' \\ &= \frac{a_{N,s}}{a_{N-1,s}} [(-\Delta)^{s} \tilde{u}](x') \int_{\mathbb{R}} (1 + \lambda^{2})^{-\frac{N}{2} - s} \, d\lambda = [(-\Delta)^{s} \tilde{u}](x') \quad \text{ for } x' \in \mathbb{R}^{N-1}, \end{split}$$

since

$$\int_{\mathbb{R}} (1+\lambda^2)^{-\frac{N}{2}-s} d\lambda = B(\frac{1}{2}, \frac{N-1}{2}+s) = \frac{\sqrt{\pi}\Gamma(\frac{N-1}{2}+s)}{\Gamma(\frac{N}{2}+s)} = \frac{a_{N-1,s}}{a_{N,s}}$$

However, since $\tilde{u} > 0$, this is impossible by Theorem 1.3 applied to $N - 1 \ge 1$ in place of N. The proof is finished.

6 Appendix: Interior Hölder estimates for distributional solutions to the equation $(-\Delta)^s u = f$

In the proof of Theorem 1.1 in Section 4 we used the following lemma on local Hölder regularity. Since the proofs available in the literature are only concerned with the case N > 2s and additional assumptions (see for instance [42, Proposition 2.1.9]), we will give a proof for the convenience of the reader. The proof is similar to arguments in [31, page 263].

Lemma 6.1 Let $f \in L^{\infty}(B_1)$, and let $u \in \mathcal{L}^1_s \cap L^{\infty}_{loc}(B_1)$ such that $(-\Delta)^s u = f$ in B_1 . Then for $r \in (0,1)$ and every $\alpha \in (0,\min(1,2s))$ there exists a constant $C_{s,N,r,\alpha} > 0$ such that

(6.1)
$$\|u\|_{C^{0,\alpha}(B_r)} \le C_{s,N,r,\alpha} \left(\|u\|_{L^{\infty}(B_r)} + \|f\|_{L^{\infty}(B_1)} \right).$$

Proof. Let $\eta \in C_c^{\infty}(\mathbb{R}^N)$ be such that $\eta = 1$ on B_r , $\eta = 0$ on $\mathbb{R}^N \setminus B_1$ and $0 \le \eta \le 1$ on \mathbb{R}^N . Consider the Riesz potential, see [5] and [7],

$$\Phi(x,y) = \begin{cases} C_{N,s}|x-y|^{2s-N} & \text{for } N > 2s, \\ \frac{1}{\pi} \log \frac{1}{|x|} & \text{for } 1 = N = 2s, \\ -C_{1,s}|x-y|^{2s-N} & \text{for } N = 1 < 2s, \end{cases}$$

with a positive normalization constant $C_{N,s}$. Then the function $v(x) = \int_{\mathbb{R}^N} \Phi(x,y)(\eta f)(y) dy$ satisfies

(6.2)
$$(-\Delta)^s v(x) = \eta(x) f(x) \quad \text{for all } x \in \mathbb{R}^N.$$

The following inequalities hold for all $a, b > 0, \alpha \in (0, 1)$ and $m \in \mathbb{R}$ with $m + \alpha > 0$:

(6.3)
$$|b^{-m} - a^{-m}| \le \frac{|m|}{m+\alpha} |b - a|^{\alpha} \max(a^{-(m+\alpha)}, b^{-(m+\alpha)})$$

(6.4)
$$|\log b - \log a| \le \frac{1}{\alpha} |b - a|^{\alpha} \max(a^{-\alpha}, b^{-\alpha}).$$

These estimates were proved in [31, page 263] in the case $m \ge 1$, and a slight change of their argument yields the more general case considered here. We thus get, for every $x, y, z \in \mathbb{R}^N$,

$$\left||x-z|^{-m} - |y-z|^{-m}\right| \le \frac{|m|}{m+\alpha} |x-y|^{\alpha} \left(|x-z|^{-(m+\alpha)} + |y-z|^{-(m+\alpha)}\right)$$

and

$$\left|\log|x-z| - \log|y-z|\right| \le \frac{1}{\alpha}|x-y|^{\alpha}(|x-z|^{-\alpha} + |y-z|^{-\alpha}).$$

In the case $N \ge 2s$ we therefore have, for $\alpha \in (0, \min(2s, 1))$ and $x, y \in B_r$,

(6.5)

$$|v(x) - v(y)| \leq C_{s,\alpha} |x - y|^{\alpha} \int_{\mathbb{R}^{N}} |x - y|^{2s - N - \alpha} \eta(y) |f(y)| dy$$

$$\leq C_{s,\alpha} |x - y|^{\alpha} ||f||_{L^{\infty}(B_{1})} \int_{B(x,2)} |x - y|^{2s - N - \alpha} dy$$

$$\leq C_{s,\alpha} ||f||_{L^{\infty}(B_{1})} |x - y|^{\alpha}.$$

Moreover, in the case N = 1 < 2s we have, for $\alpha \in (0, \min(2s, 1))$ and $x, y \in B_r$,

(6.6)
$$|v(x) - v(y)| \le C_{s,\alpha} |x - y|^{\alpha} ||f||_{L^{\infty}(B_1)} \int_{B_1} |x - y|^{2s - 1 - \alpha} dy \le C_{s,\alpha} ||f||_{L^{\infty}(\Omega)} |x - y|^{\alpha}.$$

Hence combing (6.5) and (6.6), we conclude that

(6.7)
$$\|v\|_{C^{0,\alpha}(B_r)} \le C_{s,N} \|f\|_{L^{\infty}(B_1)},$$

for every $\alpha \in (0, \min(1, 2s))$.

Next we note that the function w := u - v satisfies $(-\Delta)^s w = 0$ in B_r by (6.2). Therefore, thanks to [7, Lemma 3.2], we get, for every $r' \in (0, r)$,

$$\|\nabla w\|_{L^{\infty}(B_{r}')} \leq C_{N,s,r'} \|w\|_{L^{\infty}(B_{1})} \leq C_{N,s,r'}(\|u\|_{L^{\infty}(B_{r})} + \|v\|_{L^{\infty}(B_{r})}).$$

From this, together with (6.7), we conclude that

$$\|u\|_{C^{0,\alpha}(B_{r'})} = \|w+v\|_{C^{0,\alpha}(B_{r'})} \le C_{s,N,r'}\left(\|u\|_{L^{\infty}(B_{r})} + \|f\|_{L^{\infty}(B_{1})}\right)$$

for every $\alpha \in (0, \min(1, 2s))$.

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