# THE ROLE OF THE MEAN CURVATURE IN A HARDY-SOBOLEV TRACE INEQUALITY

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**Abstract:** The Hardy-Sobolev trace inequality can be obtained via Harmonic extensions on the half-space of the Stein and Weiss weighted Hardy-Littlewood-Sobolev inequality. In this paper we consider a bounded domain and study the influence of the boundary mean curvature in the Hardy-Sobolev trace inequality on the underlying domain. We prove existence of minimizers when the mean curvature is negative at the singular point of the Hardy potential.

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## 1. Introduction

The weighted Stein and Weiss inequality (see [30]) states, in particular, that there exists a constant $C(N,s)>0$ such that

$$C(N,s)\left(\int_{\mathbb{R}^N}|x|^{-s}|u|^{q(s)}\right)^{\frac{2}{q(s)}}dx\le\int_{\mathbb{R}^N}|\xi|\widehat{u}^2d\xi\quad\forall u\in C^\infty_c(\mathbb{R}^N),$$

where $s\le 1$, $q(s)=\frac{2(N-s)}{N-1}$ and

$$\widehat{u}(\zeta)=\frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}e^{-\imath\xi\cdot x}u(x)dx$$

is the Fourier transform of $u$. We consider the Hardy-Sobolev trace constant which is given by

$$S(s):=\inf_{u\in C^\infty_c(\mathbb{R}^N)}\frac{\displaystyle\int_{\mathbb{R}^N}|\xi|\widehat{u}^2d\xi}{\left(\displaystyle\int_{\mathbb{R}^N}|x|^{-s}|u|^{q(s)}\right)^{\frac{2}{q(s)}}dx}.\tag{1.1}$$

Denote by

$$\mathbb{R}^{N+1}_+=\left\{z=(z^1,\tilde{z})\in\mathbb{R}^{N+1}\quad:\quad z^1>0\right\}$$

with boundary $\mathbb{R}^N\times\{0\}\equiv\mathbb{R}^N$. We denote and henceforth define $\mathcal{D}:=\mathcal{D}^{1,2}(\overline{\mathbb{R}^{N+1}_+})$ the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$u\mapsto\int_{\mathbb{R}^{N+1}_+}|\nabla u|^2dz.$$

Classical argument of harmonic extension, (see for instance [6] for generalizations) yields

$$S(s)=\inf_{u\in\mathcal{D}}\frac{\displaystyle\int_{\mathbb{R}^{N+1}_+}|\nabla u|^2dz}{\left(\displaystyle\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|u|^{q(s)}\,d\tilde z\right)^{\frac{2}{q(s)}}}. \tag{1.2}$$

Note that for $s=0$ then $q(0)=:2^\sharp$, the critical Sobolev exponent while $S(0)$ coincides with the Sobolev trace constant studied by Escobar [11] and Beckner [3] wiht applications in the Yamabe problem with prescribed mean curvature. Existence of symmetric decreasing minimizers for the quotient $S(s)$ in (1.1) were obtained by Lieb [ [24], Theorem 5.1]. We also quote the works [26, 27] for the existence of minimizers in critical Sobolev trace inequalities. If $s=1$, we recover $S(1)=2\frac{\Gamma^2(\frac{N+1}{4})}{\Gamma^2(\frac{N-1}{4})}$, the relativistic Hardy constant (see e.g. [19]) which is never achieved in $\mathcal{D}$. In this case, it is expected that there is no influence of the curvature in comparison with the works on Hardy inequalities with singularity at the boundary or in Riemannian manifolds, see [13, 31].

Let $\Omega$ be a smooth domain of $\mathbb{R}^{N+1}$, $N\geq 2$ with $0\in\partial\Omega$. We consider $(\partial\Omega,\tilde g)$ as a Riemaninan manifold, with Riemannian metric $\tilde g$ induced by $\mathbb{R}^{N+1}$ on $\partial\Omega$. Let $d$ denote the Riemannian distance in $(\partial\Omega,\tilde g)$. A classical argument of partitioning of unity (see Lemma 2.4 below) yields the existence of a constant $C(\Omega)>0$ such that the following inequality

$$C(\Omega)\left(\int_{\partial\Omega}d^{-s}(\sigma)|u|^{q(s)}d\sigma\right)^{\frac{2}{q(s)}}\leq\int_\Omega|\nabla u|^2dx+\int_\Omega|u|^2dx \qquad \forall u\in H^1(\Omega).$$

Our aims in this paper is to study the existence of minimizers for the following quotient:

$$S(s,\Omega):=\inf_{u\in H^1(\Omega)}\frac{\displaystyle\int_\Omega|\nabla u|^2dx+\int_\Omega|u|^2dx}{\left(\displaystyle\int_{\partial\Omega}d^{-s}(\sigma)|u|^{q(s)}d\sigma\right)^{\frac{2}{q(s)}}}, \tag{1.3}$$

for $s\in[0,1)$. Our main result is the following:

**Theorem 1.1.** *Let $\Omega$ be a smooth domain of $\mathbb{R}^{N+1}$, $N\geq 3$ with $0\in\partial\Omega$ and $s\in[0,1)$. Assume that the mean curvature of $\partial\Omega$ at 0 is negative. Then $S(s,\Omega)<S(s)$ and $S(s,\Omega)$ is achieved by a positive function $u\in H^1(\Omega)$ satisfying*

$$\begin{cases}-\Delta u+u=0 & \text{in } \Omega\\ \dfrac{\partial u}{\partial\nu}=S(s,\Omega)\,d^{-s}(\sigma)\,u^{q(s)-1} & \text{on } \partial\Omega,\end{cases}$$

*where $\nu$ is the unit outer normal of $\partial\Omega$.*

In the literature, several authors studied the influence of curvature in the Hardy-Sobolev inequalities in Euclidean space and in Riemmanian manifolds, see [9, 10, 14–18, 20, 22] and the references there in. For instance, consider the Hardy-Sobolev constant:

$$Q(s,\Omega) := \inf_{w \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla w|^2 dx}{\left(\displaystyle\int_\Omega |x|^{-s}|w|^{2(s)} dx\right)^{2/2(s)}}, \tag{1.4}$$

with $s \in (0,2)$ and $2(s) = \frac{2(N-s)}{N-2}$. The role of the local geometry $\partial\Omega$ at 0 in the study of minimizers for $Q(s,\Omega)$ was first investigated by Ghoussoub and Kang in [14]. In [14] the authors showed that if all the principal curvatures of $\partial\Omega$ at 0 are negative then $Q(s,\Omega) < Q(s,\mathbb{R}_+^{N+1})$ and it is achieved. This result were improved by Ghoussoub and Roberts assuming only that the mean curvature is negative at 0 while $N \geq 4$. Later Demyanov and Nazarov in [10] constructed domains in which $Q(s,\Omega)$ is achieved wile the mean curvature of $\partial\Omega$ at 0 is not negative. Actually, by the results in [10], extremals for $Q(s,\Omega)$ exists if $\Omega$ is "average concave in a neighborhood of the origin". Later on, in the same year, Ghoussoub and Robert [15,16] used refined blow-up analysis to prove existence of an extremal for $Q(s,\Omega)$ provided the mean curvature of $\partial\Omega$ is negative at 0.

Recently Chern and Lin in [9] proved that if the mean curvature of $\partial\Omega$ at 0 is negative then $Q(s,\Omega) < Q(s,\mathbb{R}_+^{N+1})$ for $N \geq 2$ and in these cases, $Q(s,\Omega)$ is attained. See also the recent work of Li and Lin [22] for generalizations.

We point out that the study of the effect of the curvature in the Hardy-Sobolev trace inequality seems to be quite rare in the literature while the Sobolev trace ($s = 0$) inequality have been intensively studied in the last years, see for instance [26, 27]. According to the authors level of information, the paper is one of the first dealing with this question. We would like to emphasize that our argument of proof (based on blow up analysis, in Proposition 3.1) is different from those in the papers cited above. The main observation is that, dealing with "pure" Hardy-Sobolev mnimization problem $s \in (0,1]$, one can depict a sequence of radii $r_n \to 0$ where, if blow up occur, then concentration can only happen in $B(0,r_n) \cap \partial\Omega$. Indeed, to prove existence of a minimizer for $S(s,\Omega)$, we consider a minimizing sequence $u_n$, given by Ekeland variational principle which is bound in $H^1(\Omega)$ and normalized so that

$$\int_{\partial\Omega} d^{-s}(\sigma)|u_n|^{q(s)} d\sigma = 1.$$

We suppose that $u_n$ converges weakly to 0 (that is blow up occurs). Then considering the Lévy concentration function $r \mapsto \int_{\partial\Omega \cap B_r(0)} d^{-s}(\sigma)|u_n|^{q(s)} d\sigma$, it easy to see, by continuity, that there exists a sequence of real number $r_n$ such that

$$\int_{\partial\Omega \cap B_{r_n}(0)} d^{-s}(\sigma)|u_n|^{q(s)} d\sigma = \frac{1}{2}.$$

Now because $0 < s$, we have $q(s) < 2^\sharp$ so that by compactness $u_n \to 0$ in $L^{q(s)}(\partial\Omega)$. Using this we show that up to a subsequence $r_n \to 0$ as $n \to \infty$. Now scaling $u_n$ with these parameters $r_n$ and making change of coordinates, we find out new sequence of functions $w_n$

for which their mass concentrate at a half ball centred at the origin. Namely

$$\int_{B^N_{r_0}} |\tilde{z}|^{-s} {w_n}^{q(s)} d\tilde{z} = \frac{1}{2}(1 + O(r_n)).$$

Cutting-off $w_n$ by a function $\eta_n$ near the origin and using further analysis, we see that $\eta_n w_n$ converges in $\mathcal{D}^{1,2}(\overline{\mathbb{R}^{N+1}_+})$ to a function $w \neq 0$ satisfying

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\frac{\partial w}{\partial z^1} = S(s, \Omega)|\tilde{z}|^{-s}|w|^{q(s)-2}w & \text{on } \partial\mathbb{R}^{N+1}_+, \\ \displaystyle\int_{\partial\mathbb{R}^{N+1}_+} |\tilde{z}|^{-s}|w|^{q(s)}\, d\tilde{z} \leq 1, & , \\ w \neq 0. \end{cases}$$

This then implies that $S(s, \Omega) \geq S(s)$.

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## 2. Tool box

2.1. **Existence of ground states in $\mathbb{R}^{N+1}_+$.** We start with a proof of existence of minimizers for $S(s)$, with $s \in (0, 1)$, which might be of interest in the study of Hardy-Sobolev inequalities and different from the one of [24].

**Theorem 2.1.** *Let $s \in (0, 1)$. Then $S(s)$ has a positive minimizer $w \in \mathcal{D}$ which satisfies*

$$\begin{cases} \Delta w = 0 & \textit{in } \mathbb{R}^{N+1}_+, \\ -\dfrac{\partial w}{\partial z^1} = S(s)w^{q(s)-1} & \textit{on } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} w^{q(s)}\, dx = 1. \end{cases}$$

**Proof.** Recall that $\mathcal{D} := \mathcal{D}^{1,2}(\overline{\mathbb{R}^{N+1}_+})$. Define the functionals $\Phi, \Psi : \mathcal{D} \to \mathbb{R}$ by

$$\Phi(w) := \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla w|^2 dx$$

and

$$\Psi(w) = \frac{1}{q(s)} \int_{\partial\mathbb{R}^{N+1}_+} |z|^{-s}|w|^{q(s)} dz.$$

By Ekeland variational principle there exits a minimizing sequence $w_n$ for the quotient $S := S(s)$ such that

$$\int_{\partial\mathbb{R}^{N+1}_+} |z|^{-s}|w_n|^{q(s)}dz = 1, \tag{2.1}$$

$$\Phi(w_n) \to \frac{1}{2}S$$

and

$$\Phi'(w_n) - S\Psi'(w_n) \to 0 \quad \text{in } \mathcal{D}', \tag{2.2}$$

where $\mathcal{D}'$ denotes the dual of $\mathcal{D}$. We have that

$$\int_{\mathbb{R}^{N+1}_+} |\nabla w_n|^2 dz \leq C. \tag{2.3}$$

We define the Levi-type concentration function: for $r > 0$

$$Q(r) := \int_{B^N_r} |\tilde{z}|^{-s}|w_n|^{q(s)}d\tilde{z}.$$

By continuity and (3.1) there exists $r_n > 0$ such that

$$Q(r_n) := \int_{B^N_{r_n}} |\tilde{z}|^{-s}|w_n|^{q(s)}d\tilde{z} = \frac{1}{2}.$$

Let $v_n(z) = r_n^{\frac{N-1}{2}} w_n(r_n z)$. It is easy to check that for every $s \in [0, 1]$

$$\int_{\mathbb{R}^{N+1}_+} |\nabla w_n|^2 dz = \int_{\mathbb{R}^{N+1}_+} |\nabla v_n|^2 dz, \quad \int_{\mathbb{R}^N} |\tilde{z}|^{-s}|w_n|^{q(s)}d\tilde{z} = \int_{\mathbb{R}^N} |\tilde{z}|^{-s}|v_n|^{q(s)}d\tilde{z}$$

and

$$\int_{B^N_1} |\tilde{z}|^{-s}|v_n|^{q(s)}d\tilde{z} = \frac{1}{2}. \tag{2.4}$$

Hence $v_n$ is a minimizing sequence. In particular $v_n \rightharpoonup v$ for some $v$ in $\mathcal{D}$. We wish to show that $v \neq 0$. If not then $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^{N+1}_+)$ and in $L^2_{loc}(\mathbb{R}^N)$. Let $\varphi \in C^\infty_c(B_1)$ and $\varphi \equiv 1$ on $B_{\frac{1}{2}}$. Using $\varphi^2 v_n$ as test in (2.2) and using standard integration by parts

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+} |\nabla(\varphi v_n)|^2 dz &= S(s)\int_{\mathbb{R}^N} |z|^{-s}|v_n|^{q(s)-2}|\varphi v_n|^2 d\tilde{z} + o(1) \\ &\leq \frac{S(s)}{2^{\frac{q(s)-2}{q(s)}}}\left(\int_{\mathbb{R}^N} |\tilde{z}|^{-s}|\varphi v_n|^{q(s)}d\tilde{z}\right)^{\frac{2}{q(s)}} + o(1), \end{aligned}$$

where we used (2.4). By (1.2) we deduce that

$$S(s)\left(\int_{\mathbb{R}^N} |\tilde{z}|^{-s}|\varphi v_n|^{q(s)}d\tilde{z}\right)^{\frac{2}{q(s)}} \leq \frac{S(s)}{2^{\frac{q(s)-2}{q(s)}}}\left(\int_{\mathbb{R}^N} |\tilde{z}|^{-s}|\varphi v_n|^{q(s)}d\tilde{z}\right)^{\frac{2}{q(s)}} + o(1).$$

Since $s \in (0, 1)$, we have $S(s) > \frac{S}{2^{\frac{q(s)-2}{q(s)}}}$ so that

$$o(1) = \int_{\mathbb{R}^N} |\tilde{z}|^{-s}|\varphi v_n|^{q(s)}d\tilde{z} = \int_{B^N_1} |\tilde{z}|^{-s}|v_n|^{q(s)}d\tilde{z} + o(1)$$

because $q(s) < 2^\sharp$. We are therefore in contradiction with (2.4). Therefore $v \neq 0$ is a minimizer. Standard arguments show that $v^+ = \max(v, 0)$ is also a minimizer and the proof is complete by the maximum principle. $\square$

### 2.2. **Symmetry and decay estimates of ground states.**

**Theorem 2.2.** *Let $N \geq 2$ and let $w \in \mathcal{D}$ such that $w > 0$ and*

$$
\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\partial_{z^1} w := -\frac{\partial w}{\partial z^1} = S(s)|\tilde{z}|^{-s} w^{q(s)-1} & \text{on } \partial\mathbb{R}^{N+1}_+. \end{cases} \tag{2.5}
$$

*Then we have:*
*(i) $w = w(z)$ only depends on $z^1$ and $|\tilde{z}|$, and $w$ is strictly decreasing in $|\tilde{z}|$.*
*(ii) $w(z) \leq \frac{C}{1+|z|^{N-1}}$ for all $z \in \overline{\mathbb{R}^{N+1}_+}$, for some positive constant $C$.*

**Proof.** (i) For simplicity, we write $S$, $q$, instead of $S(s)$, $q(s)$. We first show that

$$
w \text{ is strictly decreasing in } z^{N+1} \text{ in } \mathbb{R}^{N+1}_+ \setminus \{z^{N+1} = 0\}. \tag{2.6}
$$

This will be shown with a variant of the moving plane method, see [1,2,7,8,29]. For $\lambda > 0$, we consider the reflection $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, $z \mapsto z_\lambda$ at the hyperplane $\{z \in \mathbb{R}^{N+1} \,:\, z^{N+1} = \lambda\}$. Moreover, we let $H^\lambda := \{z \in \mathbb{R}^{N+1}_+ \,:\, z^{N+1} > \lambda\}$, and we define

$$
u^\lambda : \overline{\mathbb{R}^{N+1}_+ \cap H^\lambda} \to \mathbb{R}, \qquad u^\lambda(z) = w(z_\lambda) - w(z).
$$

Then $u^\lambda$ is harmonic in $\mathbb{R}^{N+1}_+ \cap H^\lambda$, and it satisfies

$$
u^\lambda(z) = 0 \quad \text{on } \mathbb{R}^{N+1}_+ \cap \partial H^\lambda
$$

as well as

$$
-\partial_{z^1} u^\lambda(z) = +S\Big(\frac{w^{q-1}(z_\lambda)}{|z_\lambda|^s} - \frac{w^{q-1}(z)}{|z|^s}\Big) \quad \text{on } \mathbb{R}^N \cap H^\lambda.
$$

Let $u^\lambda_- = \min\{u_\lambda, 0\}$. Then

$$
\begin{aligned}
\int_{H^\lambda} |\nabla u^\lambda_-|^2 \, dz &= \int_{H^\lambda} \nabla u^\lambda \nabla u^\lambda_- \, dz = -\int_{\mathbb{R}^N \cap H^\lambda} \partial_{z^1} u^\lambda u^\lambda_- \, d\sigma(z) \\
&= S \int_{\mathbb{R}^N \cap H^\lambda} u^\lambda_+ \Big(\frac{w^{q-1}}{|z_\lambda|^s} - \frac{w^{q-1}(z)}{|z|^s}\Big) \, d\sigma(z) \\
&\leq S \int_{\mathbb{R}^N \cap H^\lambda} u^\lambda_- |z|^{-s} [w^{q-1}(z) - w^{q-1}(z_\lambda)] \, d\sigma(z) \\
&\leq (q-1) S \int_{\mathbb{R}^N \cap H^\lambda} |u^\lambda_-(z)|^2 |z|^{-s} w^{q-2}(z) \, d\sigma(z).
\end{aligned}
$$

In the last step we used that if $w(z_\lambda) \leq w(z)$ then

$$
w^{q-1}(z) - w^{q-1}(z_\lambda) \leq (q-1) w^{q-2}(z)[w(z) - w(z_\lambda)] = -(q-1) u^\lambda_-(z) w^{q-2}(z)
$$

by the convexity of the function $t \mapsto t^{q-1}$ on $(0,\infty)$. Using Hölder's inequality, we conclude that

$$\int_{H^\lambda} |\nabla u_-^\lambda|^2 \le c(\lambda)\Big(\int_{\mathbb{R}^N\cap H^\lambda} |z|^{-s}|u_-^\lambda|^q \, d\sigma(z)\Big)^{\frac{2}{q}} \tag{2.7}$$

with

$$c(\lambda) = (q-1)S\Big(\int_{M_\lambda} |z|^{-s} w^q(z)\, d\sigma(z)\Big)^{\frac{q-2}{q}} \quad \text{and} \quad M_\lambda := \{z \in \mathbb{R}^N \cap H^\lambda \,:\, u(z) > u(z_\lambda)\}$$

for $\lambda > 0$. Since $c(\lambda) \to 0$ as $\lambda \to \infty$, we have $c(\lambda) < S$ and therefore $u_-^\lambda \equiv 0$ in $H_\lambda \cap \mathbb{R}^{N+1}_+$ for $\lambda > 0$ sufficiently large. As a consequence,

$$\lambda^* := \inf\{\lambda > 0 \,:\, w(z) \le w(z_{\lambda'}) \text{ for all } z \in H^{\lambda'} \cap \mathbb{R}^{N+1}_+ \text{ and all } \lambda' \ge \lambda\} < \infty.$$

We claim that $\lambda^* = 0$. Indeed, if, by contradiction, $\lambda^* > 0$, then $u^{\lambda^*}$ is a nonnegative harmonic function in $\mathbb{R}^{N+1}_+ \cap H^{\lambda^*}$ satisfying $u^{\lambda^*} = 0$ on $\mathbb{R}^{N+1}_+ \cap \partial H^\lambda$ and

$$-\partial_{z^1} u^{\lambda^*}(z) = \frac{w^{q-1}(z_{\lambda^*})}{|z_{\lambda^*}|^s} - \frac{w^{q-1}(z)}{|z|^s} \quad \text{on } \mathbb{R}^N \cap H^{\lambda^*},$$

where the last quantity is strictly positive whenever $w(z_{\lambda*}) > 0$. Consequently, unless $w \equiv 0$, $u^{\lambda^*}$ must be strictly positive in $\overline{\mathbb{R}^{N+1}_+} \cap H^{\lambda^*}$ by the strong maximum principle. We then choose a sufficiently large set $D$ compactly contained in $\mathbb{R}^N \cap H^{\lambda^*}$ such that

$$(q-1)S\Big(\int_{\mathbb{R}^N\cap H^{\lambda^*}\setminus D} |z|^{-s} w^q(z)\, d\sigma(z)\Big)^{\frac{q-2}{q}} < S.$$

Then, for $\lambda < \lambda^*$ close to $\lambda^*$, we have $D \subset \mathbb{R}^N \cap H^\lambda$,

$$(q-1)S\Big(\int_{\mathbb{R}^N\cap H^{\lambda}\setminus D} |z|^{-s} w^q(z)\, d\sigma(z)\Big)^{\frac{q-2}{q}} < S.$$

and $u^\lambda > 0$ in $D$. As a consequence, $c(\lambda) < S$ for $\lambda < \lambda^*$ close to $\lambda^*$ because $M_\lambda \subset \mathbb{R}^N \cap H^\lambda \setminus D$. By (2.7) we have $u^\lambda \ge 0$ in $H^\lambda \cap \mathbb{R}^{N+1}_+$ for $\lambda < \lambda^*$ close to $\lambda^*$, contrary to the definition of $\lambda^*$. We therefore conclude that $\lambda^* = 0$, and this shows (2.6).
Repeating the same argument for the functions $z \mapsto w(z^1, A\tilde z)$, where $A \in O(N)$ is an $N$-dimensional rotation, we conclude that $w$ only depends on $z^1$ and $|\tilde z|$, and $w$ is strictly decreasing in $|\tilde z|$. This ends the proof of (i).
To prove (ii), we write the (2.5) as

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^{N+1} \\ -\partial_{z^1} w = a(z) w & \text{on } \mathbb{R}^N. \end{cases}$$

Where $a = S|z|^{-s} w^{q-2} \in L^p_{loc}(\mathbb{R}^N)$ for some $p > N$. Therefore by [21], we have that $w \in L^\infty_{loc}(\mathbb{R}^{N+1}_+)$. Now since (2.5) is invariant under Kelvin transform, we get immediately the result. $\square$

2.3. **Geometric preliminaries.** We let $E_i$, $i = 2, \dots, N+1$ be an orthonormal basis of $T_0\partial\Omega$, the tangent plane of $\partial\Omega$ at 0. We will consider the Riemaninan manifold $(\partial\Omega, \tilde{g})$ where $\tilde{g}$ is the Riemannian metric induced by $\mathbb{R}^{N+1}$ on $\partial\Omega$. We first introduce geodesic normal coordinates in a neighborhood (in $\partial\Omega$) of 0 with coordinates $y' = (y^2, \dots, y^{N+1}) \in \mathbb{R}^N$. We set

$$f(y') := \mathrm{Exp}_0^{\partial\Omega}\left(\sum_{i=2}^{N+1} y^i E_i\right).$$

It is clear that the geodesic distance $d$ satisfies

$$d(f(\tilde{y})) = |\tilde{y}|. \tag{2.8}$$

In addition the above choice of coordinates induces coordinate vector-fields on $\partial\Omega$:

$$Y_i(y') = f_*(\partial_{y^i}), \qquad \text{for } i = 2, \dots, N+1.$$

Let $\tilde{g}_{ij} = \langle Y_i, Y_j \rangle$, for $i, j = 2, \dots, N+1$, be the component of the metric $\tilde{g}$. We have near the origin

$$\tilde{g}_{ij} = \delta_{ij} + O(|y|^2).$$

We denote by $N_{\partial\Omega}$ the unit normal vector field along $\partial\Omega$ interior to $\Omega$. Up to rotations, we will assume that $N_{\partial\Omega}(0) = E_1$. For any vector field $Y$ on $T\partial\Omega$, we define $H(Y) = dN_{\partial\Omega}[Y]$. The mean curvature of $\partial\Omega$ at 0 is given by

$$H_{\partial\Omega}(0) = \sum_{i=2}^{N+1} \langle H(E_i), E_i \rangle.$$

Now consider a local parametrization of a neighbourhood of 0 in $\mathbb{R}^{N+1}$ defined as

$$F(y) := f(\tilde{y}) + y^1 N_{\partial\Omega}(f(\tilde{y})), \qquad y = (y^1, \tilde{y}) \in B_{r_0},$$

where $B_{r_0}$ is a small ball centred at 0. This yields the coordinate vector-fields in $\mathbb{R}^{N+1}$,

$$Y_i(y) \quad := \quad F_*(\partial_{y^i}) \qquad i = 1, \dots, N+1.$$

Let $g_{ij} = \langle Y_i, Y_j \rangle$, for $i, j = 1, \dots, N+1$, be the component of the flat metric $g$. It follows that

$$g_{ij} = \tilde{g}_{ij} + 2\langle H(Y_i), Y_j \rangle y^1 + O(|y|^2).$$

We have the following expansion of the metric. See for instance [12] for the proof.

**Lemma 2.3.** *In a small ball $B_{r_0}$ centered at 0,*

$$\begin{aligned} g_{ij} &= \delta_{ij} + 2\langle H(E_i), E_j \rangle y^1 + O(|y|^2); \\ g_{i1} &= 0; \\ g_{11} &= 1. \end{aligned}$$

We now prove that Hardy ($s = 1$) and Hardy-Sobolev trace inequality with singularity at the boundary and involving the geodesic boundary distance function hold.

**Lemma 2.4.** *Suppose that $s \in [0,1]$. Then there exists a positive constant $C(s,\Omega)$ such that We have*

$$C(s,\Omega)\Big(\int_{\partial\Omega} d^{-s}(\sigma)|u|^{q(s)}d\sigma\Big)^{\frac{2}{q(s)}} \leq \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx \qquad \forall u \in H^1(\Omega).$$

**Proof.** Let $u \in H^1(\Omega)$ and pick $\eta \in C_c^\infty(F(B_{r_0}))$ such that $\eta \equiv 1$ on $F(B_{\frac{r_0}{2}})$ and $\eta \equiv 0$ on $F(B_{r_0})$. Then, by using the Sobolev trace inequality, (1.2) and Young's inequality, we get

$$\begin{aligned}
\Big(\int_{\partial\Omega} d^{-s}(\sigma)|u|^{q(s)}d\sigma\Big)^{\frac{2}{q(s)}} &= \Big(\int_{\partial\Omega} d^{-s}(\sigma)|\eta u + (1-\eta)u|^{q(s)}d\sigma\Big)^{\frac{2}{q(s)}} \\
&\leq 2\Big(\int_{\partial\Omega} d^{-s}(\sigma)|\eta u|^{q(s)}d\sigma\Big)^{\frac{2}{q(s)}} + C\Big(\int_{\partial\Omega} |u|^{q(s)}d\sigma\Big)^{\frac{2}{q(s)}} \\
&\leq C\left(\int_{\mathbb{R}^N} |y|^{-s}|(\eta u)(F(y))|^{q(s)}\,dy\right)^{\frac{2}{q(s)}} + C\|u\|^2_{H^1(\Omega)} \\
&\leq C\frac{1}{S(s)}\int_{\mathbb{R}^{N+1}_+} |\nabla_y(\eta u)(F(y))|^2\,dy + C\|u\|^2_{H^1(\Omega)}.
\end{aligned}$$

Now by Lemma 2.3 and some integration by parts, we deduce that

$$\int_{\mathbb{R}^{N+1}_+} |\nabla_y(\eta u)(F(y))|^2\,dy \leq C\|u\|^2_{H^1(\Omega)}$$

and the proof is complete. □

## 2.4. Comparing $S(s,\Omega)$ and $S(s)$.

**Lemma 2.5.** *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lipschitz domain which is smooth at $0 \in \partial\Omega$. We have the following expansion*

$$S(s,\Omega) \quad \leq \quad S(s) + \varepsilon C_{N,\varepsilon} H_{\partial\Omega}(0) + O(\rho(\varepsilon)),$$

*where*

$$C_{N,\varepsilon} = \frac{N-2}{N}\int_{B^+_{\frac{r_0}{\varepsilon}}} z^1|\nabla_{\tilde{z}} w|^2 dz + \int_{B^+_{\frac{r_0}{\varepsilon}}} z^1|\partial_{z^1} w|^2 dz$$

*and*

$$\begin{aligned}
\rho(\varepsilon) \quad &= \quad \varepsilon^2\int_{B^+_{\frac{r_0}{\varepsilon}}} |z|^2|\nabla w|^2 dz + \varepsilon^2\int_{\frac{r_0}{2\varepsilon}<|z|<\frac{r_0}{\varepsilon}} w^2 dz + \varepsilon\int_{\frac{r_0}{2\varepsilon}<|\tilde{z}|<\frac{r_0}{\varepsilon}} w^2 d\tilde{z} \\
&+ \quad \varepsilon^2\int_{\partial' B^+_{\frac{r_0}{\varepsilon}}} |\tilde{z}|^{2-s}w^{q(s)}d\tilde{z} + \int_{\partial\mathbb{R}^{N+1}_+\setminus B_{\frac{r_0}{\varepsilon}}} |\tilde{z}|^{-s}w^{q(s)}d\tilde{z} + \varepsilon^2\int_{\mathbb{R}^{N+1}_+\cap B_{\frac{r_0}{\varepsilon}}} w^2 dz,
\end{aligned}$$

*where $\partial' B^+_r = \partial B^+_r \cap \partial\mathbb{R}^{N+1}_+$.*

**Proof.** Let $w \in \mathcal{D}$, $w > 0$ be the minimizer for $S(s)$ normalized so that

$$\int_{\partial\mathbb{R}^{N+1}_+} |\tilde{z}|^{-s}w^{q(s)} = 1.$$

Define

$$v_\varepsilon(F(y)) = \varepsilon^{\frac{1-N}{2}} w\left(\frac{y}{\varepsilon}\right) \quad y \in \overline{B^+_{r_0}}.$$

Let $\eta \in C^\infty_c(F(B_{r_0}))$ such that $\eta \equiv 1$ on $F(B_{\frac{r_0}{2}})$ and $\eta \le 1$ on $\mathbb{R}^{N+1}$. We let

$$u_\varepsilon(F(y)) = \eta(F(y)) v_\varepsilon(F(y)).$$

We have

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^2 dx &= \int_\Omega \eta^2 |\nabla v_\varepsilon|^2 dx - \int_\Omega (\Delta \eta)\eta v_\varepsilon^2 dx + \int_{\partial\Omega} \eta \frac{\partial \eta}{\partial \nu} v_\varepsilon^2 d\sigma \\
&\le \int_{\Omega \cap F(B_{r_0})} |\nabla v_\varepsilon|^2 dx + C \int_{\Omega \cap F(B_{r_0}) \setminus F(B_{\frac{r_0}{2}})} v_\varepsilon^2 dx + C \int_{\partial\Omega \cap F(B_{r_0}) \setminus F(B_{\frac{r_0}{2}})} v_\varepsilon^2 d\sigma \\
&= \int_{B^+_{\frac{r_0}{\varepsilon}}} g^{ij}(\varepsilon z) w_i w_j \sqrt{|g|}(\varepsilon z) dz + O(\rho_1(\varepsilon)),
\end{aligned}$$

where

$$\rho_1(\varepsilon) = \varepsilon^2 \int_{\frac{r_0}{2\varepsilon} < |z| < \frac{r_0}{\varepsilon}} w^2 dz + \varepsilon^2 \int_{\frac{r_0}{2\varepsilon} < |\tilde z| < \frac{r_0}{\varepsilon}} w^2 d\tilde z.$$

Notice that

$$g^{ij}(\varepsilon z) w_i w_j = |\nabla w|^2 - 2\varepsilon z^1 \langle H(\nabla_{\tilde z} w), \nabla_{\tilde z} w\rangle + O(\varepsilon^2 |z|^2 |\nabla w|^2)$$

and

$$\sqrt{|g|}(\varepsilon z) = 1 + \varepsilon z^1 H_{\partial\Omega}(0) + O(\varepsilon^2 |z|^2).$$

Using this with the fact that $w(z) = w(z^1, |\tilde z|)$, we get

$$\int_{B^+_{\frac{r_0}{\varepsilon}}} g^{ij}(\varepsilon z) w_i w_j \sqrt{|g|}(\varepsilon z) dz \quad \le \quad \int_{B^+_{\frac{r_0}{\varepsilon}}} |\nabla w|^2 dz + \varepsilon C_{N,\varepsilon} H_{\partial\Omega}(0) + O(\rho_2(\varepsilon)),$$

where

$$\rho_2(\varepsilon) = \varepsilon^2 \int_{B^+_{\frac{r_0}{\varepsilon}}} |z|^2 |\nabla w|^2 dz.$$

Hence we obtain

$$\int_\Omega |\nabla u_\varepsilon|^2 dx \le \int_{\mathbb{R}^{N+1}_+} |\nabla w|^2 \, dz + \varepsilon C_{N,\varepsilon} H_{\partial\Omega}(0) + O(\rho_2(\varepsilon)) + O(\rho_1(\varepsilon)).$$

On the other hand by (2.8), we have

$$\begin{aligned}
\int_{\partial\Omega} d^{-s}(\sigma) {u_\varepsilon}^{q(s)} &= \int_{\partial\mathbb{R}^{N+1}_+} d\left(\frac{F(0,\varepsilon\tilde z)}{\varepsilon}\right)^{-s} \eta^{q(s)}(\varepsilon\tilde z) w^{q(s)} \sqrt{|g|}(0,\varepsilon\tilde z) d\tilde z \\
&= \int_{\partial\mathbb{R}^{N+1}_+} |\tilde z|^{-s} (\varepsilon\tilde z) w^{q(s)} \sqrt{|g|}(0,\varepsilon\tilde z) d\tilde z \\
&\quad - \int_{\partial\mathbb{R}^{N+1}_+} |\tilde z|^{-s} (1 - \eta^{q(s)})(\varepsilon\tilde z) w^{q(s)} \sqrt{|g|}(0,\varepsilon\tilde z) d\tilde z \\
&= \int_{\partial\mathbb{R}^{N+1}_+} |\tilde z|^{-s} w^{q(s)} d\tilde z + O(\rho_3(\varepsilon)),
\end{aligned}$$

where

$$\rho_3(\varepsilon) \quad = \quad \varepsilon^2 \int_{\partial' B^+_{\frac{r_0}{\varepsilon}}} |\tilde{z}|^{2-s} w^{q(s)} d\tilde{z} + \int_{\partial \mathbb{R}^{N+1}_+ \setminus B_{\frac{r_0}{\varepsilon}}} |\tilde{z}|^{-s} w^{q(s)} d\tilde{z}.$$

The lemma then follows by putting $\rho(\varepsilon) = \rho_1(\varepsilon) + \rho_2(\varepsilon) + \rho_3(\varepsilon)$. □

**Proposition 2.6.** *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lipschitz domain which is smooth at $0 \in \partial\Omega$. Suppose that $N \geq 3$ and $s \in [0, 1)$. Assume that $H_{\partial\Omega}(0) < 0$. Then $S(s, \Omega) < S(s)$.*

**Proof.** Consider $w \in \mathcal{D}$ given by Theorem 2.1 the positive minimizer for $S(s)$. By Theorem 2.2 we have that $w(z) = \omega(z^1, |\tilde{z}|)$ and

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial w}{\partial z^1} = S(s)|\tilde{z}|^{-s} w^{q(s)-1} & \text{on } \partial\mathbb{R}^{N+1}_+ \\ \displaystyle\int_{\partial \mathbb{R}^{N+1}_+} |\tilde{z}|^{-s} w^{q(s)} = 1. \end{cases} \tag{2.9}$$

In addition, thanks to Theorem 2.2, we have

$$w(z) \leq \frac{C}{1 + |z|^{N-1}} \qquad \text{for all } z \in \mathbb{R}^{N+1}_+. \tag{2.10}$$

For $s = 0$, we consider the Escobar-Beckner (see [11], [3]) function

$$w(z) := c_n \frac{1}{(1 + |z|^2)^{\frac{N-1}{2}}}, \tag{2.11}$$

with $c_n = \frac{2}{N-1} |S^N|^{\frac{-1}{N}}$, which uniquely minimizes $S(0)$ up to translations.
Let $\varphi$ be a nonnegative radially symmetric cut-off function in $\mathbb{R}^{N+1}$ such that $\varphi \leq 1$ in $\mathbb{R}^{N+1}_+$, $\varphi \equiv 1$ on $B_{2r_0}$, $\varphi \equiv 0$ on $B_{3r_0}$ and $|\nabla\varphi| + |\Delta\varphi| \leq C$. Define $\varphi_\varepsilon(z) = \varphi(\varepsilon z)$ for all $z \in \mathbb{R}^{N+1}_+$. We multiply (2.9) by $|z| w \varphi_\varepsilon$ and integrate by part to get

$$\int_{B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon |z| |\nabla w|^2 = \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon |w|^2 + \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon |\tilde{z}|^{1-s} |w|^{q(s)} + \frac{1}{2} \int_{B^+_{\frac{3r_0}{\varepsilon}}} w^2 \Delta(|z| \varphi_\varepsilon).$$

By (2.10), provided $N \geq 3$ we have

$$\int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon |w|^2 + \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon |\tilde{z}|^{1-s} |w|^{q(s)} \approx C + \varepsilon^{N-2}. \tag{2.12}$$

We also have

$$\int_{B^+_{\frac{3r_0}{\varepsilon}}} w^2 \Delta(|z| \varphi_\varepsilon) \leq C\varepsilon^2 \int_{\frac{2r_0}{\varepsilon} < |z| < \frac{3r_0}{\varepsilon}} \varphi_\varepsilon |w|^2 + C\varepsilon \int_{\frac{2r_0}{\varepsilon} < |z| < \frac{3r_0}{\varepsilon}} \varphi_\varepsilon |w|^2 + C \int_{B^+_{\frac{3r_0}{\varepsilon}}} |z|^{-1} |w|^2$$

and thus

$$\int_{B^+_{\frac{3r_0}{\varepsilon}}} w^2 \Delta(|z| \varphi_\varepsilon) \approx C + \varepsilon^{N-2}.$$

Using this and (2.12) we deduce that

$$\int_{B^+_{\frac{r_0}{\varepsilon}}} |z||\nabla w|^2 \approx C + \varepsilon^{N-2}. \tag{2.13}$$

By using similar arguments as above (multiplying (2.9) by $|z|^2 w\varphi_\varepsilon$ and integrating by parts) we have

$$\begin{aligned}\int_{B^+_{\frac{3r_0}{\varepsilon}}} |z|^2|\nabla w|^2 &= \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon|\tilde z||w|^2 + \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} \varphi_\varepsilon|\tilde z|^{2-s}|w|^{q(s)} + \frac{1}{2}\int_{B^+_{\frac{3r_0}{\varepsilon}}} w^2\Delta(|z|^2\varphi_\varepsilon)\\ &\leq \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} |\tilde z||w|^2 + \int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} |\tilde z|^{2-s}|w|^{q(s)} + C\int_{B^+_{\frac{3r_0}{\varepsilon}}} w^2.\end{aligned}$$

By (2.10), the following estimates holds

$$\int_{\partial' B^+_{\frac{3r_0}{\varepsilon}}} |\tilde z||w|^2 d\tilde z \approx \int_{\mathbb{R}^{N+1}_+\cap B_{\frac{r_0}{\varepsilon}}} w^2 dz \approx C + \begin{cases}\varepsilon^{N-3}, & N>3\\ |\log\varepsilon|, & N=3.\end{cases}$$

Now provided $N \geq 3$, we have

$$\int_{\partial' B^+_{\frac{r_0}{\varepsilon}}} |\tilde z|^{2-s} w^{q(s)} d\tilde z \approx C + \begin{cases}\varepsilon^{N-2-s}, & s\in(0,1)\\ |\log\varepsilon|, & s=0.\end{cases}$$

We then deduce that

$$\int_{B^+_{\frac{r_0}{\varepsilon}}} |z|^2|\nabla w|^2 \approx C + \begin{cases}\varepsilon^{N-3}, & N>3 \text{ and } s\in(0,1)\\ |\log\varepsilon|, & N=3 \text{ or } s=0.\end{cases}$$

In addition, we have

$$\int_{\frac{r_0}{2\varepsilon}<|\tilde z|<\frac{r_0}{\varepsilon}} w^2 d\tilde z \approx \varepsilon^{N-2},$$

$$\int_{\frac{r_0}{2\varepsilon}<|z|<\frac{r_0}{\varepsilon}} w^2 dz \approx C + \begin{cases}\varepsilon^{N-3}, & N>3\\ |\log\varepsilon|, & N=3\end{cases}$$

and

$$\int_{\partial\mathbb{R}^{N+1}_+\setminus B_{\frac{r_0}{\varepsilon}}} |\tilde z|^{-s} w^{q(s)} d\tilde z \approx \varepsilon^{N-s}, \quad s\in[0,1).$$

Thanks to Lemma 2.5, and the above estimates we conclude that, provided $N \geq 4$ and $s \in [0,1)$,

$$S(s,\Omega) \leq S(s,0) + C_1\varepsilon H_{\partial\Omega}(0) + O(\varepsilon^2)$$

and if $N = 3$ or $s = 0$, we get

$$S(s,\Omega) \leq S(s) + C_1\varepsilon H_{\partial\Omega}(0) + O(\varepsilon^2|\log\varepsilon|),$$

with $C_1 > 0$. □

## 3. Existence of minimizer for $S(s,\Omega)$

It is clear from Proposition 2.6 that the proof of Theorem 1.1 is finalized by the two results in this section. However, we should emphasize that the argument following below works also for the pure Hardy case: $s = 1$.

**Proposition 3.1.** *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lipschitz domain which is smooth at $0 \in \partial\Omega$. Let $s \in (0,1]$ and $N \geq 2$. Assume that $S(s,\Omega) < S(s)$. Then there exists a minimizer for $S(s,\Omega)$.*

**Proof.** We define $\Phi, \Psi : H^1(\Omega) \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx\right)$$

and

$$\Psi(u) = \frac{1}{q(s)} \int_{\partial\Omega} d^{-s}(\sigma)|u|^{q(s)} d\sigma.$$

By Ekeland variational principle there exits a minimizing sequence $u_n$ for the quotient $S(s,\Omega) = S(s,\Omega)$ such that

$$\int_{\partial\Omega} d^{-s}(\sigma)|u_n|^{q(s)} d\sigma = 1, \tag{3.1}$$

$$\Phi(u_n) \to \frac{1}{2} S(s,\Omega) \tag{3.2}$$

and

$$\Phi'(u_n) - S(s,\Omega)\Psi'(u_n) \to 0 \quad \text{ in } (H^1(\Omega))', \tag{3.3}$$

with $(H^1(\Omega))'$ denotes the dual of $H^1(\Omega)$. We have that

$$\int_\Omega |\nabla u_n|^2 dx + \int_{\partial\Omega} |u_n|^2 d\sigma \leq Const. \quad \forall n \geq 1. \tag{3.4}$$

In particular $u_n \rightharpoonup u$ for some $u$ in $H^1(\Omega)$.
**Claim:** $u \neq 0$.
Assume by contradiction that $u = 0$ (that is blow up occur). By continuity, (3.1) and the fact that $s \in (0,1]$, there exits a sequence $r_n > 0$ such that

$$\int_{\partial\Omega\cap B_{r_n}} d^{-s}(\sigma)|u_n|^{q(s)} d\sigma = \frac{1}{2}. \tag{3.5}$$

We now show that, up to a subsequence, $r_n \to 0$. Indeed, by (3.1) and (3.5)

$$\int_{\partial\Omega\setminus B_{r_n}} d^{-s}(\sigma)|u_n|^{q(s)} d\sigma = \frac{1}{2}.$$

Since $q(s) < q(0) = 2^\sharp$ for $s > 0$, by compactness we have

$$r_n^s\, C \leq \int_{\partial\Omega\setminus B_{r_n}} |u_n|^{q(s)} d\sigma \leq \int_{\partial\Omega} |u_n|^{q(s)} d\sigma \to 0 \quad \text{ as } n \to \infty,$$

for some positive constant $C$.
Define $F_n(z) = \frac{1}{r_n} F(r_n z)$ for every $z \in B^+_{\frac{r_0}{r_n}}$ and put $(g_n)_{i,j} = \langle \partial_i F_n, \partial_j F_n \rangle$. Clearly

$$g_n \to g_{Euc} \quad C^1(K) \quad \text{ for every compact set } K \subset \mathbb{R}^{N+1}, \tag{3.6}$$

where $g_{Euc}$ denotes the Euclidean metric. Let

$$w_n(z) = r_n^{\frac{N-1}{2}} u_n(F(r_n z)) \quad \forall z \in B^+_{\frac{r_0}{r_n}}.$$

Then we get

$$\int_{B^N_{r_0}} |\tilde{z}|^{-s} {w_n}^{q(s)} d\tilde{z} = (1 + o(1)) \int_{B^N_{r_0}} |\tilde{z}|^{-s} {w_n}^{q(s)} \sqrt{|g_n|} d\tilde{z}.$$

Hence by (3.5) we have

$$\int_{B^N_{r_0}} |\tilde{z}|^{-s} {w_n}^{q(s)} d\tilde{z} = \frac{1}{2}(1 + c r_n). \tag{3.7}$$

Let $\eta \in C^\infty_c(F(B_{r_0}))$, $\eta \equiv 1$ on $F(B_{\frac{r_0}{2}})$ and $\eta \equiv 0$ on $\mathbb{R}^{N+1} \setminus F(B_{r_0})$. We define

$$\eta_n(z) = \eta(F(r_n z)) \quad \forall z \in \mathbb{R}^{N+1}.$$

We have that

$$\|\eta_n w_n\|_{\mathcal{D}} \le C \quad \forall n \in \mathbb{N}, \tag{3.8}$$

where as usual $\mathcal{D} = \mathcal{D}^{1,2}(\overline{\mathbb{R}^{N+1}})$. Therefore

$$\eta_n w_n \rightharpoonup w \quad \text{ in } \mathcal{D}.$$

We first show that $w \ne 0$. Assume by contradiction that $w \equiv 0$. Thus $w_n \to 0$ in $L^p_{loc}(\mathbb{R}^{N+1}_+)$ and in $L^p_{loc}(\partial\mathbb{R}^{N+1}_+)$ for every $1 \le p < 2^\sharp$. Let $\varphi \in C^\infty_c(B_{\frac{r_0}{2}})$ be a cut-off function such that $\varphi \equiv 1$ on $B_{\frac{r_0}{4}}$ and $\varphi \le 1$ in $\mathbb{R}^{N+1}$. Define

$$\varphi_n(F(y)) = \varphi(r_n^{-1} y).$$

We multiply (3.3) by $\varphi_n^2 u_n$ (which is bounded in $H^1(\Omega)$) and integrate by parts to get

$$\begin{aligned} \int_\Omega \nabla u_n \nabla(\varphi_n^2 u_n) dx &= S(s,\Omega) \int_{\partial\Omega} d^{-s}(\sigma) |\varphi_n u_n|^{q(s)-2} (\varphi_n u_n)^2 d\sigma + o(1) \\ &\le S(s,\Omega) \left( \int_{\partial\Omega} d^{-s}(\sigma) |\varphi_n u_n|^{q(s)} d\sigma \right)^{\frac{2}{q(s)}} + o(1), \end{aligned}$$

where we have used (3.1). In the coordinate system and after integration by parts, the above becomes

$$\int_{\mathbb{R}^{N+1}_+} |\nabla(\varphi w_n)|^2_{g_n} \sqrt{|g_n|} dz = S(s,\Omega) \left( \int_{\partial\mathbb{R}^{N+1}_+} |\tilde{z}|^{-s} |\varphi w_n|^{q(s)} \sqrt{|g_n|} d\tilde{z} \right)^{\frac{2}{q(s)}} + o(1).$$

Therefore, by (3.6), for some constant $c > 0$, we have

$$(1 - c r_n) \int_{\mathbb{R}^{N+1}_+} |\nabla(\varphi w_n)|^2 dz = S(s,\Omega) \left( \int_{\partial\mathbb{R}^{N+1}_+} |\tilde{z}|^{-s} |\varphi w_n|^{q(s)} d\tilde{z} \right)^{\frac{2}{q(s)}} + o(1). \tag{3.9}$$

Hence by the Hardy-Sobolev trace inequality (1.2), we get

$$
\begin{aligned}
(3.10) \qquad (1-cr_n)S(s)&\left(\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|\varphi w_n|^{q(s)}d\tilde z\right)^{\frac{2}{q(s)}}\\
&\le S(s,\Omega)\left(\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|\varphi w_n|^{q(s)}d\tilde z\right)^{\frac{2}{q(s)}}+o(1).
\end{aligned}
$$

Since $S(s) > S(s,\Omega)$, we conclude that

$$
o(1)=\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|\varphi w_n|^{q(s)}d\tilde z=\int_{B^N_{r_0}}|\tilde z|^{-s}|w_n|^{q(s)}d\tilde z+o(1)
$$

because by assumption $q(s) < 2^\sharp$. This is clearly in contradiction with (3.7) thus $w \neq 0$. Now pick $\phi \in C^\infty_c(\mathbb{R}^{N+1}\setminus\{0\})$, and put $\phi_n(F(y)) = \phi(r_n^{-1}y)$ for every $y \in B_{r_0}$. For $n$ sufficiently large, $\phi_n \in C^\infty_c(\overline{\Omega})$ and it is bounded in $H^1(\Omega)$. We multiply (3.3) by $\phi_n$ and integrate by parts to get

$$
\int_\Omega \nabla u_n\nabla\phi_n dx = S(s,\Omega)\int_{\partial\Omega}d^{-s}(\sigma)|u_n|^{q(s)-2}u_n\phi_n d\sigma+o(1).
$$

Hence

$$
\int_{\mathbb{R}^{N+1}_+}\langle\nabla w_n,\nabla\phi\rangle_{g_n}\sqrt{|g_n|}dz=S(s,\Omega)\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|w_n|^{q(s)-2}w_n\phi\sqrt{|g_n|}d\tilde z+o(1).
$$

Since $\eta_n \equiv 1$ on $B_{\frac{r_0}{2r_n}}$ and the support of $\phi$ is contained in an annulus, for $n$ sufficiently large

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+1}_+}\langle\nabla(\eta_n w_n),\nabla\phi\rangle_{g_n}\sqrt{|g_n|}dz\\
&\qquad=S(s,\Omega)\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|\eta_n w_n|^{q(s)-2}\eta_n w_n\phi\sqrt{|g_n|}d\tilde z+o(1).
\end{aligned}
$$

Since also $g_n$ converges smoothly to the Euclidean metric on the support of $\phi$, by passing to the limit, we infer that, for all $\phi\in C^\infty_c(\mathbb{R}^{N+1}\setminus\{0\})$

$$
(3.11) \qquad \int_{\mathbb{R}^{N+1}_+}\nabla w\nabla\phi\, dz = S(s,\Omega)\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|w|^{q(s)-2}w\phi\, d\tilde z.
$$

Notice that $C^\infty_c(\mathbb{R}^{N+1}\setminus\{0\})$ is dense in $C^\infty_c(\mathbb{R}^{N+1})$ with respect to the $H^1(\mathbb{R}^{N+1})$ norm when $N\ge 2$, see e.g. [25]. Consequently since $w\in\mathcal{D}$, it follows that (3.11) holds for all $\phi\in C^\infty_c(\mathbb{R}^{N+1})$ by (1.2). We conclude that

$$
\begin{cases}
\Delta w = 0 & \text{in } \mathbb{R}^{N+1}_+,\\
-\frac{\partial w}{\partial z^1}=S(s,\Omega)|\tilde z|^{-s}|w|^{q(s)-2}w & \text{on } \partial\mathbb{R}^{N+1}_+,\\
\displaystyle\int_{\partial\mathbb{R}^{N+1}_+}|\tilde z|^{-s}|w|^{q(s)}\,d\tilde z\le 1, & ,\\
w\neq 0.
\end{cases}
$$

Multiplying this equation by $w$ and integrating by parts, leads to $S(s,\Omega) \geq S(s)$ by (1.2) which is a contradiction and thus $u = \lim u_n \neq 0$ is a minimizer for $S(s,\Omega)$.
□

In the following we study the existence of minimizers for the Sobolev trace inequality.

**Proposition 3.2.** *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lipschitz domain which is smooth at $0 \in \partial\Omega$ and $N \geq 2$. Assume that $S(0,\Omega) < S(0)$. Then there exists a minimizer for $S(0,\Omega)$.*

**Proof.** Recall the Sobolev trace inequality, proved by Li and Zhu in [23]: there exists a positive constant $C = C(\Omega)$ such that for all $u \in H^1(\Omega)$, we have

$$S(0)\left(\int_{\partial\Omega} |u|^{2^\sharp} d\sigma\right)^{2/2^\sharp} \leq \int_\Omega |\nabla u|^2 dx + C\int_{\partial\Omega} |u|^2 d\sigma. \tag{3.12}$$

Now we let $u_n$ be a minimizing sequence for $S(0)$, normalized as $\|u_n\|_{L^{2^\sharp}(\partial\Omega)} = 1$. We now show that $u = \lim u_n$ is not zero. Put $\theta_n := u_n - u$ so that $\theta_n \rightharpoonup 0$ in $H^1(\Omega)$ and $\theta_n \to 0$ in $L^2(\Omega), L^2(\partial\Omega)$. Moreover by Brezis-Lieb Lemma [4] and recalling (3.1), it holds that

$$1 - \lim_{n\to\infty} \int_{\partial\Omega} |\theta_n|^{2^\sharp} d\sigma = \int_{\partial\Omega} |u|^{2^\sharp} d\sigma. \tag{3.13}$$

By using (3.12), we have

$$\begin{aligned}
S(0,\Omega)\left(\int_{\partial\Omega} |u|^{2^\sharp} d\sigma\right)^{2/2^\sharp} &\leq \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} |u|^2 d\sigma \\
&\leq \int_\Omega |\nabla u_n|^2 dx + \int_{\partial\Omega} |u_n|^2 d\sigma - \int_\Omega |\nabla \theta_n|^2 dx + o(1) \leq \int_\Omega |\nabla u_n|^2 dx + \int_{\partial\Omega} |u_n|^2 d\sigma \\
&- S(0)\left(\int_{\partial\Omega} |\theta_n|^{2^\sharp} d\sigma\right)^{2/2^\sharp} + o(1) \\
&\leq S(0,\Omega) - S(0)\left(\int_{\partial\Omega} |\theta_n|^{2^\sharp} d\sigma\right)^{2/2^\sharp} + o(1).
\end{aligned}$$

We take the limit as $n \to \infty$ and use (3.13) to get

$$S(0,\Omega)\left(\int_{\partial\Omega} |u|^{2^\sharp} d\sigma\right)^{2/2^\sharp} \leq S(0,\Omega) - S(0)\left(1 - \int_{\partial\Omega} |u|^{2^\sharp} d\sigma\right)^{2/2^\sharp}.$$

Thanks to the concavity of the function $t \mapsto t^{2/2^\sharp}$, the above implies that $\int_{\partial\Omega} |u|^{2^\sharp} d\sigma \geq 1$ whenever $S(0,\Omega) < S(0)$. This completes the proof. □

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