A note on Hardy's inequalities with boundary singularities

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Abstract. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 1$. In this paper we study the Hardy-Poincaré inequalities with weight function singular at the boundary of Ω. In particular we give sufficient conditions so that the best constant is achieved.

Key Words: Hardy inequality, extremals, p-Laplacian.

1 Introduction

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial\Omega$ and $p > 1$ a real number. In this note, we are interested in finding minima to the following quotient

$$\mu_{\lambda,p}(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega |u|^p \, dx}{\displaystyle\int_\Omega |x|^{-p} |u|^p \, dx}, \tag{1.1}$$

in terms of $\lambda \in \mathbb{R}$ and Ω. If $\lambda = 0$, we have the Ω-*Hardy constant*

$$\mu_{0,p}(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |x|^{-p} |u|^p \, dx} \tag{1.2}$$

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which is the best constant in the Hardy inequality for maps supported by Ω. The existence of extremals for $\mu_{\lambda,2}(\Omega)$ was studied in [10] while for $\mu_{0,2}(\Omega)$, one can see for instance [6], [5], [21] and [19] for $\mu_{0,N}(\Omega)$.
Given a unit vector ν of $\mathbb{R}^N$, we consider the half-space $H := \{x \in \mathbb{R}^N \,:\, x \cdot \nu \geq 0\}$. For $N = 1$, the following Hardy inequality is well known

$$\left(\frac{p-1}{p}\right)^p \int_0^\infty t^{-p}|u|^p\,dt \leq \int_0^\infty |u'|^p\,dt \quad \forall u \in W_0^{1,p}(0,\infty). \tag{1.3}$$

Moreover $\mu_{0,p}(H) = \left(\frac{p-1}{p}\right)^p$ is the H-Hardy constant and it is not achieved, see [15] for historical comments also.
For $N \geq 2$, it was recently proved by Nazarov [20] that the H-Hardy constant is not achieved and

$$\mu_{0,p}(H) := \inf_{V \in W_0^{1,p}(\mathbb{S}_+^{N-1})} \frac{\displaystyle\int_{\mathbb{S}_+^{N-1}} \left(\left(\frac{N-p}{p}\right)^2 |V|^2 + |\nabla_\sigma V|^2\right)^{\frac{p}{2}} d\sigma}{\displaystyle\int_{\mathbb{S}_+^{N-1}} |V|^p d\sigma}, \tag{1.4}$$

where $\mathbb{S}_+^{N-1}$ is an $(N-1)$-dimensional hemisphere. Notice that this problem always has a minimizer by the compact embedding $L^p(\mathbb{S}_+^{N-1}) \hookrightarrow W_0^{1,p}(\mathbb{S}_+^{N-1})$. The quantity $\mu_{0,p}(H)$ is explicitly known only in some special cases. Indeed, $\mu_{0,2}(H) = \frac{N^2}{4}$ while for $p = N$ then $\mu_{N,N}(H)$ is the first Dirichlet eigenvalue of the operator $-\mathrm{div}(|\nabla u|^{N-2}\nabla u)$ in $W_0^{1,N}(\mathbb{S}_+^{N-1})$ with the standard metric.
Problem (1.1) carries some similarities with the questions studied by Brezis and Marcus in [2], where the weight is the inverse-square of the distance from the boundary of Ω and $p = 2$. We also deal with this problem in the present paper for all $p > 1$ in Appendix A. We generalize here the existence result obtained by R.Musina and the author in [10] for any $p > 1$ and $N \geq 1$.

Theorem 1.1 *Let $p > 1$ and Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial\Omega$. There exits $\lambda^*(p,\Omega) \in [-\infty, +\infty)$ such that*

$$\mu_{\lambda,p}(\Omega) < \mu_{0,p}(H), \quad \forall \lambda > \lambda^*(p,\Omega). \tag{1.5}$$

The infinimum in (1.1) is attained for any $\lambda > \lambda^*(p,\Omega)$.

The existence of $\lambda^*(p,\Omega)$ comes from the fact that

$$\sup_{\lambda\in\mathbb{R}} \mu_{\lambda,p}(\Omega) = \mu_{0,p}(H),$$

see Lemma 2.2. Now observe that the mapping $\lambda \mapsto \mu_{\lambda,p}$ is non-increasing. Moreover, for bounded domains Ω, letting λ_1 be the first Dirichlet eigenvalue of the p-Laplace operator $-\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in $W^{1,p}_0(\Omega)$, it is plain that $\mu_{\lambda_1,p}(\Omega) = 0$. Then we define

$$\lambda^*(p,\Omega) := \inf\{\lambda \in \mathbb{R} \,:\, \mu_{\lambda,p}(\Omega) < \mu_{0,p}(H)\}$$

so that $\mu_{\lambda,p} < \mu_{0,p}(H)$ for all $\lambda > \lambda^*(p,\Omega)$. In particular $\lambda^*(p,\Omega) \le \lambda_1$. On the other hand there are various bounded smooth domains Ω with $0 \in \partial\Omega$ such that $\lambda^*(p,\Omega) \in [-\infty, 0)$, see Proposition 2.5 and Proposition 2.6. Furthermore if $N = 1$ then $\mu_{0,p}(\mathbb{R}\setminus\{0\}) = \left(\frac{p-1}{p}\right)^p = \mu_{0,p}(H)$ thus $\lambda^*(p,\Omega) \ge 0$.

It is obvious that if Ω is contained in a half-ball centered at the origin then $\mu_{0,p}(\Omega) = \mu_{0,p}(H)$ thus $\lambda^*(p,\Omega) \ge 0$ and in addition

$$\lambda^*(p,\Omega) = \inf_{u\in W^{1,p}_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx - \mu_{0,p}(H)\int_\Omega |x|^{-p}|u|^p\,dx}{\displaystyle\int_\Omega |u|^p\,dx}.$$

We have obtained the following result.

Theorem 1.2 *If Ω is contained in a half-ball centered at the origin then there exists a constant $c(N,p) > 0$ such that*

$$\lambda^*(p,\Omega) \ge \frac{c(N,p)}{diam(\Omega)^p}. \tag{1.6}$$

The constant $c(N,p)$ appearing in (1.6) has the property that $c(N,2)$ is the first Dirichlet eigenvalue of $-\Delta$ in the unit disc of $\mathbb{R}^2$. This type of estimates was first proved by Brezis-Vàzquez in [3] when $p = 2$, $N \ge 2$ and later on, extended to the case $1 < p < N$ by Gazzola-Grunau-Mitidieri in [13] when dealing with $\mu_{0,p}(\mathbb{R}^N\setminus\{0\}) := \left|\frac{N-p}{p}\right|^p$. More precisely they proved the existence of a positive constant $C(N,p)$ such that for any open subset Ω of $\mathbb{R}^N$, there holds

$$\int_\Omega |\nabla u|^p - \mu_{0,p}\left(\mathbb{R}^N\setminus\{0\}\right)\int_\Omega |x|^{-p}|u|^p \ge C(N,p)\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{p}{N}}\int_\Omega |u|^p \quad \forall u \in W^{1,p}_0(\Omega), \tag{1.7}$$

where $|\Omega|$ is the measure of Ω and ω_N the measure of the unit ball of $\mathbb{R}^N$. The constant $C(N,p)$ was explicitly given and $C(N,2) = c(N,2)$ as was obtained in [3]. The main ingredients to prove (1.7) is the Schwarz symmetrization and a "dimension reduction" via the transformation $x \mapsto \frac{u}{\omega}$, where $\omega(x) = |x|^{\frac{p-N}{p}}$ satisfies

$$\operatorname{div}(|\nabla\omega|^{p-2}\nabla\omega) + \mu_{0,p}\left(\mathbb{R}^N \setminus \{0\}\right)|x|^{-p}\omega^{p-1} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

For $p = 2$, the lower bound in (1.6) was obtained in [10] by a similar transformation and using the Poincaré inequality on $\mathbb{S}^{N-1}_+$. However, in view of (1.4), such argument do not apply here when $p \neq 2$ and $p \neq N$. By analogy, to reduce the dimension, we will consider the mapping $x \mapsto \frac{u}{v}$, where $v(x) := |x|^{\frac{p-N}{p}} V\left(\frac{x}{|x|}\right)$ is a weak solution to the equation

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \mu_{0,p}(H)\,|x|^{-p}|v|^{p-2}v = 0 \quad \text{in } \mathcal{D}'(H)$$

whenever V is a minimizer of (1.4). Then exploiting the strict convexity of the mapping $a \mapsto |a|^p$, estimate (1.6), for $p \geq 2$, follows immediately while the case $p \in (1,2)$ carries further difficulties as it can be seen in Section 2.2.

The argument to prove the attainability of $\mu_{\lambda,p}(\Omega)$ is taken from de Valeriola-Willem [7]. It allows to show that, up to a subsequence, the gradient of the Palais-Smale sequences converges point-wise almost every where. Therefore an application of the Brezis-Lieb lemma with some simples arguments yields the existence of extremals.

2 Hardy inequality with one point singularity

Let $\mathcal{C}$ be a proper cone in $\mathbb{R}^N$, $N \geq 2$ and put $\Sigma := \mathcal{C} \cap \mathbb{S}^{N-1}$. It was shown in [20] that the $\mathcal{C}$-Hardy constant is not achieved and it is given by

$$\text{(2.1)} \qquad \mu_{0,p}(\mathcal{C}) = \inf_{V \in W^{1,p}_0(\Sigma)} \frac{\displaystyle\int_\Sigma \left(\left(\frac{N-p}{p}\right)^2 |V|^2 + |\nabla_\sigma V|^2\right)^{\frac{p}{2}} d\sigma}{\displaystyle\int_\Sigma |V|^p d\sigma}.$$

Letting $V \in W_0^{1,p}(\Sigma)$ be the positive minimizer to this quotient then the function

$$v(x) := |x|^{\frac{p-N}{p}} V\left(\frac{x}{|x|}\right) \tag{2.2}$$

satisfies

$$\int_{\mathcal{C}} |\nabla v|^{p-2}\nabla v \cdot \nabla h = \mu_{0,p}(\mathcal{C}) \int_{\mathcal{C}} |x|^{-p} v^{p-1} h \quad \forall h \in C_c^1(\mathcal{C}). \tag{2.3}$$

Notice that $\mu_{0,2}(\mathcal{C}) = \left(\frac{N-2}{2}\right)^2 + \lambda_1(\Sigma)$, where $\lambda_1(\Sigma)$ is the first Dirichlet eigenfunction of the Laplace operator on Σ endowed with the standard metric on $\mathbb{S}^{N-1}$. This was obtained in [21], [19] and [10].

2.1 Existence

In this Section we show that the condition $\mu_{\lambda,p}(\Omega) < \mu_{0,p}(H)$ is sufficient to guaranty the existence of a minimizer for $\mu_{\lambda,p}(\Omega)$.
We emphasize that throughout this section, Ω can to be taken to be an open set satisfying the uniform sphere condition at $0 \in \partial\Omega$. Namely there are balls $B_+ \subset \Omega$ and $B_- \subset \mathbb{R}^N \setminus \Omega$ such that $\partial B_+ \cap \partial B_- = \{0\}$. This holds if $\partial\Omega$ is of class C^2 at 0, see [[16] 14.6 Appendix]. We start with the following approximate local Hardy inequality.

Lemma 2.1 *Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial\Omega$ and let $p > 1$. Then for any $\varepsilon > 0$ there exits $r_\varepsilon > 0$ such that*

$$\mu_{0,p}(\Omega \cap B_{r_\varepsilon}(0)) \geq \mu_{0,p}(H) - \varepsilon, \tag{2.4}$$

where $B_r(0)$ is a ball of radius r centered at 0.

Proof. If $N = 1$ then (2.4) is an immediate consequence of (1.3). From now on we can assume that $N \geq 2$. We denote by $N_{\partial\Omega}$ the unit normal vector-field on $\partial\Omega$. Up to a rotation, we can assume that $N_{\partial\Omega}(0) = E_N$, so that the tangent plane of $\partial\Omega$ at 0 coincides with $\mathbb{R}^{N-1} = \mathrm{span}\{E_1, \dots, E_{N-1}\}$. Denote by $B_r^+ = \{y \in B_r(0) : y^N > 0\}$. For $r > 0$ small, we introduce the following system of coordinates centered at 0 (see [9]) via the mapping $F : B_r^+ \to \Omega$ given by

$$F(y) = \mathrm{Exp}_0(\tilde{y}) + y^N\, N_{\partial\Omega}\left(\mathrm{Exp}_0(\tilde{y})\right),$$

where $\tilde{y} = (y^1, \ldots, y^{N-1})$ and $\tilde{y} \mapsto \mathrm{Exp}_0(\tilde{y}) \in \partial\Omega$ is the exponential mapping of $\partial\Omega$ endowed with the metric induced by $\mathbb{R}^N$. This coordinates induces a metric on $\mathbb{R}^N$ given by $g_{ij}(y) = \langle \partial_i F(y), \partial_j F(y) \rangle$ for $i, j = 1, \ldots, N$. Let $u \in C_c^\infty(F(B_r^+))$ and put $v(y) = u(F(y))$ then

(2.5)

$$\int_{F(B_r^+)} |\nabla u|^p \, dx = \int_{B_r^+} |\nabla v|_g^p \sqrt{|g|} \, dy, \quad \int_{F(B_r^+)} |x|^{-p} |u|^p \, dx = \int_{B_r^+} |F(y)|^{-p} |v|^p \sqrt{|g|} \, dy,$$

with $|g|$ stands for the determinant of the g while $|\nabla v|_g^p = g(\nabla v, \nabla v)^{\frac{p}{2}}$. Since $|F(y)| = |y| + O(|y|^2)$ and $g_{ij}(y) = \delta_{ij} + O(|y|)$, we infer that

$$\frac{\displaystyle\int_{B_r^+} |\nabla v|_g^p \sqrt{|g|} \, dy}{\displaystyle\int_{B_r^+} |F(y)|^{-p} |v|^p \sqrt{|g|} \, dy} \geq (1 - Cr) \frac{\displaystyle\int_{B_r^+} |\nabla v|^p \, dy}{\displaystyle\int_{B_r^+} |y|^{-p} |v|^p \, dy},$$

for some constant $C > 0$ depending only on Ω and p. Furthermore since $\mu_{0,p}(B_r^+) \geq \mu_{0,p}(H)$, using (2.5) we conclude that

$$\mu_{0,p}(F(B_r^+)) \geq (1 - Cr)\mu_{0,p}(H).$$

$\square$

We are in position to prove (1.5) in the following

Lemma 2.2 *Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial\Omega$ and let $p > 1$. Then there exists $\lambda^*(p, \Omega) \in [-\infty, +\infty)$ such that*

$$\mu_{\lambda,p}(\Omega) < \mu_{0,p}(H) \quad \forall \lambda > \lambda^*(p, \Omega).$$

Proof. We first show that

$$\sup_{\lambda \in \mathbb{R}} \mu_{\lambda,p}(\Omega) = \mu_{0,p}(H). \tag{2.6}$$

Step 1: We claim that $\sup_{\lambda \in \mathbb{R}} \mu_{\lambda,p}(\Omega) \geq \mu_{0,p}(H)$.
For $r > 0$ small, we let $\psi \in C^\infty(B_r(0))$ with $0 \leq \psi \leq 1$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_{\frac{r}{2}}(0)$ and

$\psi \equiv 1$ in $B_{\frac{r}{4}}(0)$. For a fixed $\varepsilon > 0$ small, there holds

$$\begin{aligned}\int_\Omega |x|^{-p}|u|^p &= \int_\Omega |x|^{-p}|\psi u + (1-\psi)u|^p \\ &\leq (1+\varepsilon)\int_\Omega |x|^{-p}|\psi u|^p + c(\varepsilon)\int_\Omega |x|^{-p}(1-\psi)^p|u|^p \\ &\leq (1+\varepsilon)\int_\Omega |x|^{-p}|\psi u|^p + c(\varepsilon)\int_\Omega |u|^p.\end{aligned}$$

Now by (2.4)

$$(\mu_{0,p}(H)-\varepsilon)\int_\Omega |x|^{-p}|\psi u|^p \leq \int_\Omega |\nabla(\psi u)|^p$$

and hence

$$(\mu_{0,p}(H)-\varepsilon)\int_\Omega |x|^{-p}|u|^p \leq (1+\varepsilon)\int_\Omega |\nabla(\psi u)|^p + c(\varepsilon)\int_\Omega |u|^p. \tag{2.7}$$

Since $|\nabla(\psi u)|^p \leq (\psi|\nabla u| + |u||\nabla\psi|)^p$ we deduce that

$$|\nabla(\psi u)|^p \leq (1+\varepsilon)\psi^p|\nabla u|^p + c|u|^p|\nabla\psi|^p \leq (1+\varepsilon)|\nabla u|^p + c|u|^p.$$

Using (2.7), we conclude that

$$(\mu_{0,p}(H)-\varepsilon)\int_\Omega |x|^{-p}|u|^p \leq (1+\varepsilon)^2\int_\Omega |\nabla u|^p + c(\varepsilon)\int_\Omega |u|^p. \tag{2.8}$$

This implies that $\mu_{0,p}(H) \leq \sup_{\lambda\in\mathbb{R}} \mu_{\lambda,p}(\Omega)$ and the claim follows.
Step 2: We claim that $\sup_{\lambda\in\mathbb{R}} \mu_{\lambda,p}(\Omega) \leq \mu_{0,p}(H)$.
Denote by ν the unit interior normal of $\partial\Omega$. For $\delta \geq 0$ we consider the cone

$$\mathcal{C}_+^\delta := \left\{ x \in \mathbb{R}^N \mid x\cdot\nu > \delta|x| \right\}$$

and put $\Sigma_\delta = \mathcal{C}_+^\delta \cap \mathbb{S}^{N-1}$. For every $\eta > 0$, let $V \in C_c^\infty(\Sigma_0)$ such that

$$\frac{\displaystyle\int_{\Sigma_0}\left(\left(\frac{N-p}{p}\right)^2|V|^2 + |\nabla_\sigma V|^2\right)^{\frac{p}{2}} d\sigma}{\displaystyle\int_{\Sigma_0}|V|^p d\sigma} \leq \mu_{0,p}(H) + \eta.$$

On the other hand, there exists $\delta > 0$ small such that $supp\, V \subset \Sigma_\delta$. From this we conclude that

$$\mu_{0,p}(H) \leq \mu_{0,p}(\mathcal{C}_+^\delta) \leq \mu_{0,p}(H) + \eta. \tag{2.9}$$

Since $\partial\Omega$ is smooth at 0, for every $\delta > 0$, there exists $r_\delta > 0$ such that $\mathcal{C}^\delta_+ \cap B_r(0) \subset \Omega$ for all $r \in (0, r_\delta)$. Clearly by scale invariance, $\mu_{0,p}(\mathcal{C}^\delta_+ \cap B_r(0)) = \mu_{0,p}(\mathcal{C}^\delta_+)$. For $\varepsilon > 0$, we let $\phi \in W^{1,p}_0(\mathcal{C}^\delta_+ \cap B_r(0))$ such that

$$\frac{\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |\nabla\phi|^p \, dx}{\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |x|^{-p}|\phi|^p \, dx} \le \mu_{0,p}(\mathcal{C}^\delta_+) + \varepsilon.$$

From this we deduce that

$$\begin{aligned}
\mu_{\lambda,p}(\Omega) &\le \frac{\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |\nabla\phi|^p \, dx - \lambda \int_{\mathcal{C}^\delta_+ \cap B_r(0)} |\phi|^p \, dx}{\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |x|^{-p}|\phi|^p \, dx} \\
&\le \mu_0(\mathcal{C}^\delta_+) + \varepsilon + |\lambda| \frac{\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |\phi|^p \, dx}{\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |x|^{-p}|\phi|^p \, dx}.
\end{aligned}$$

Since $\displaystyle\int_{\mathcal{C}^\delta_+ \cap B_r(0)} |x|^{-p}|\phi|^p \, dx \ge r^{-p} \int_{\mathcal{C}^\delta_+ \cap B_r(0)} |\phi|^p \, dx$, we get

$$\mu_{\lambda,p}(\Omega) \le \mu_{0,p}(\mathcal{C}^\delta_+) + \varepsilon + r^p|\lambda|.$$

The claim follows immediately by (2.9). Therefore (2.6) is proved.
Finally as the map $\lambda \mapsto \mu_{\lambda,p}(\Omega)$ is non increasing while $\mu_{\lambda_1,p}(\Omega) = 0 < \mu_{0,p}(H)$, we can set

$$\lambda^*(p,\Omega) := \inf\{\lambda \in \mathbb{R} \,:\, \mu_{\lambda,p}(\Omega) < \mu_{0,p}(H)\}$$

so that $\lambda^*(p,\Omega) < \mu_{0,p}(H)$ for any $\lambda > \lambda^*(p,\Omega)$. □

Remark 2.3 *Observe that the proof of Lemma 2.2 highlights that*

$$\lim_{r\to 0} \mu_{0,p}(\Omega \cap B_r(0)) = \mu_{0,p}(H) = \lim_{\lambda\to-\infty} \mu_{\lambda,p}(\Omega).$$

Proof of Theorem 1.1

Let $\lambda > \lambda^*(p,\Omega)$ so that $\mu_{\lambda,p}(\Omega) < \mu_{0,p}(H)$. We define the mappings F, G :

$W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$F(u) = \int_\Omega |\nabla u|^p - \lambda \int_\Omega |u|^p$$

and

$$G(u) = \int_\Omega |x|^p |u|^p.$$

By Ekeland variational principal, there is a minimizing sequence $u_n \in W_0^{1,p}(\Omega)$ normalized so that

$$G(u_n) = 1, \quad \forall n \in \mathbb{N}$$

and with the properties that

$$F(u_n) \to \mu_{\lambda,p}(\Omega),$$

$$J(u_n) = F'(u_n) - \mu_{\lambda,p}(\Omega) G'(u_n) \to 0 \text{ in } (W_0^{1,p}(\Omega))'. \tag{2.10}$$

Up to a subsequence, we can assume that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^p(\Omega), \tag{2.11}$$

$u_n \to u$ in $L^p(\Omega)$ and $u_n \to u$ a.e. in Ω. Moreover by (2.8), we may assume that $|x|^{-1} u_n \rightharpoonup |x|^{-1} u$ in $L^p(\Omega)$. We set $\theta_n = u_n - u$ and

$$T(s) = \begin{cases} s & \text{if } |s| \le 1 \\ \frac{s}{|s|} & \text{if } |s| > 1. \end{cases}$$

It follows that for every $r \ge 1$

$$\int_\Omega |T(\theta_n)|^r \to 0. \tag{2.12}$$

Moreover notice that

$$\begin{aligned} \int_\Omega \big(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\big) \cdot \nabla T(\theta_n) &= \langle J(u_n), T(\theta_n) \rangle + \mu_{\lambda,p}(\Omega) \int_\Omega |x|^{-p} |u|_n^{p-2} u_n T(\theta_n) \\ &\quad + \lambda \int_\Omega |u|_n^{p-2} u_n T(\theta_n) - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla T(\theta_n). \end{aligned}$$

Therefore by (2.10), (2.11) and (2.12) we infer that

$$\int_\Omega \big(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\big) \cdot \nabla T(\theta_n) \to 0.$$

Consequently by [7]-Theorem 1.1,

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p - \int_\Omega |\nabla \theta_n|^p\right) = \int_\Omega |\nabla u|^p. \tag{2.13}$$

By Brezis-Lieb Lemma [4]

$$1 - \lim_{n\to\infty}\int_\Omega |x|^{-p}|\theta_n|^p = \int_\Omega |x|^{-p}|u|^p. \tag{2.14}$$

Fix $\varepsilon > 0$ small. By (2.8) and Rellich, there exists λ_ε such that

$$(\mu_{0,p}(H) - \varepsilon)\int_\Omega |x|^{-p}|\theta_n|^p \le \int_\Omega |\nabla\theta_n|^p - \lambda_\varepsilon\int_\Omega |\theta_n|^p = \int_\Omega |\nabla\theta_n|^p + o(1).$$

Using this together with (2.13) and (2.14) we get

$$\begin{aligned}
\mu_{\lambda,p}(\Omega)\int_\Omega |x|^{-p}|u|^p &\le \int_\Omega |\nabla u|^p - \lambda\int_\Omega |u|^p \le \int_\Omega |\nabla u_n|^p - \int_\Omega |\nabla\theta_n|^p - \lambda\int_\Omega |u_n|^p + o(1)\\
&\le F(u_n) - (\mu_{0,p}(H) - \varepsilon)\int_\Omega |x|^{-p}|\theta_n|^p + o(1)\\
&\le \mu_{\lambda,p}(\Omega) - (\mu_{0,p}(H) - \varepsilon)\left(1 - \int_\Omega |x|^{-p}|u|^p\right) + o(1)\\
&\le \mu_{\lambda,p}(\Omega) - \mu_{0,p}(H) + \varepsilon + (\mu_{0,p}(H) - \varepsilon)\int_\Omega |x|^{-p}|u|^p + o(1).
\end{aligned}$$

Send $n \to \infty$ and then $\varepsilon \to 0$ to get

$$(\mu_{\lambda,p}(\Omega) - \mu_{0,p}(H))\int_\Omega |x|^{-p}|u|^p \le \mu_{\lambda,p}(\Omega) - \mu_{0,p}(H).$$

Hence $\int_\Omega |x|^{-p}|u|^p \ge 1$ because $\mu_{\lambda,p}(\Omega) - \mu_{0,p}(H) < 0$ and the proof is complete. □

As a consequence of the existence theorem, we have

Corollary 2.4 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \ge 2$, with $0 \in \partial\Omega$. Then*

$$\mu_{0,p}\left(\mathbb{R}^N \setminus \{0\}\right) = \left|\frac{N-p}{p}\right|^p < \mu_{0,p}(\Omega) \le \mu_{0,p}(H)\,.$$

Proof. By (2.6) $0 < \mu_{0,p}(\Omega) \le \mu_{0,p}(H)$. If the strict inequality holds, then there exists a positive minimizer $u \in W_0^{1,p}(\Omega)$ for $\mu_{0,p}(\Omega)$ by Theorem 1.1. But then $\mu_{0,p}\left(\mathbb{R}^N \setminus \{0\}\right) < \mu_{0,p}(\Omega)$, because otherwise a null extension of u outside Ω would achieve the Hardy constant in $\mathbb{R}^N \setminus \{0\}$ which is not possible. □

As mentioned earlier, we shall show that there are smooth bounded domains in $\mathbb{R}^N$ such that $\lambda^*(p,\Omega) \in [-\infty, 0)$. These domains might be taken to be convex or even flat at 0. For that we let $\nu \in \mathbb{S}^{N-1}$ and $\delta, r, R > 0$. We consider the sector

$$\mathcal{C}^{\delta}_{r,R} := \left\{ x \in \mathbb{R}^N \mid x \cdot \nu > -\delta |x| \ , r < |x| < R \right\}. \tag{2.15}$$

Proposition 2.5 *Let $N \geq 2$ and $p > 1$. Then for all $\delta \in (0,1)$, there exist $r, R > 0$ such that if a domain Ω contains $\mathcal{C}^{\delta}_{r,R}$ then $\mu_{0,p}(\Omega) < \mu_{0,p}(H)$.*

Proof. Consider the cone

$$\mathcal{C}^{\delta} := \left\{ x \in \mathbb{R}^N \mid x \cdot \nu > -\delta |x| \ \right\}$$

Notice that by Harnack inequality $\mu(\mathcal{C}^{\delta}) < \mu(\mathcal{C}^{\delta'})$ for any $0 \leq \delta' < \delta < 1$. Thus for any $\delta \in (0,1)$, we can find $u \in C^{\infty}_c(\mathcal{C}^{\delta})$ such that

$$\frac{\displaystyle\int_{\mathcal{C}^{\delta}} |\nabla u|^p}{\displaystyle\int_{\mathcal{C}^{\delta}} |x|^{-p} |u|^p} < \mu_{0,p}(H).$$

Hence we choose $r, R > 0$ so that *supp* $u \subset \mathcal{C}^{\delta}_{r,R}$. □

By Corollary 2.4, starting from exterior domains, one can also build various example of (possibly annular) domains for which $\lambda^*(p,\Omega) < 0$. The following argument is taken in [Ghoussoub-Kang [14] Proposition 2.4]. If $U \subset \mathbb{R}^N$, $N \geq 2$, is a smooth exterior domain (the complement of a smooth bounded domain) with $0 \in \partial U$ then by scale invariance $\mu_{0,p}(U) = \mu_{0,p}(\mathbb{R}^N \setminus \{0\})$. We let $B_r(0)$ a ball of radius r centered at the 0 and define $\Omega_r := B_r(0) \cap U$ then clearly the map $r \mapsto \mu(\Omega_r)$ is decreasing with

$$\mu_{0,p}\left(\mathbb{R}^N \setminus \{0\}\right) = \inf_{r>0} \mu_{0,p}(\Omega_r) \quad \text{ and } \quad \mu_{0,p}(H) = \sup_{r>0} \mu_{0,p}(\Omega_r). \tag{2.16}$$

We have the following result for which the proof is similar to the one given in [14] by Corollary 2.4 and Harnack inequality.

Proposition 2.6 *There exists $r_0 > 0$ such that the mapping $r \mapsto \mu_{0,p}(\Omega_r)$ is left-continuous and strictly decreasing on $(r_0, +\infty)$. In particular*

$$\mu_{0,p}(\mathbb{R}^N \setminus \{0\}) < \mu_{0,p}(\Omega_r) < \mu_{0,p}(H), \quad \forall r \in (r_0, +\infty).$$

2.2 Remainder term

We know that for domains Ω contained in a half-ball $\lambda^*(p,\Omega)\geq 0$. Our aim in this section is to obtain positive lower bound for $\lambda^*(p,\Omega)$ by providing a remainder term for Hard's inequality in these domains. In [13], Gazzola-Grunau-Mitidieri proved the following improved Hardy inequality for $1<p<N$:

$$\int_\Omega |\nabla u|^p - \mu_{0,p}\left(\mathbb{R}^N\setminus\{0\}\right)\int_\Omega |x|^{-p}|u|^p \geq C(N,p)\left(\frac{\omega_N}{|\Omega|}\right)^{\frac{p}{N}}\int_\Omega |u|^p\,, \tag{2.17}$$

that holds for any bonded domain Ω of $\mathbb{R}^N$ and $u\in W_0^{1,p}(\Omega)$. Here the constant $C(N,p)>0$ is explicitly given while $C(N,2)$ is the first Dirichlet eigenvalue of $-\Delta$ of the unit disc in $\mathbb{R}^2$.

We shall show that such type of inequality holds in the case where the singularity is placed at the boundary of the domain. To this end, we will use the function $v(x):=|x|^{\frac{p-N}{p}}V\left(\frac{x}{|x|}\right)$ defined in (2.2) to "reduce the dimension".

Throughout this section, we assume that $N\geq 2$ since the case $N=1$ was already proved by Tibodolm [22] Theorem 1.1. Indeed, he showed that

$$\int_0^1 |u'(r)|^p dr - \mu_{0,p}(H)\int_0^1 r^{-p}|u(r)|^p dr \geq (p-1)2^p\int_0^1 |u(r)|^p dr, \quad \forall u\in W_0^{1,p}(0,1).$$

We start with conic domains

$$\mathcal{C}_\Sigma = \{x = r\sigma\in\mathbb{R}^N \mid r\in(0,1),\ \sigma\in\Sigma\ \},$$

where Σ is a domain properly contained in $\mathbb{S}^{N-1}$ and having a Lipschitz boundary. We will denote by V the positive minimizer of (2.1) in Σ while $v(x):=|x|^{\frac{p-N}{p}}V\left(\frac{x}{|x|}\right)$ satisfies (2.3) in the infinite cone $\{x=r\sigma\in\mathbb{R}^N \mid r\in(0,+\infty),\ \sigma\in\Sigma\ \}$. Finally we remember that by Harnack inequality $\frac{1}{v}\in L^\infty_{loc}(\mathcal{C}_\Sigma)$.

Recall the following inequalities (see [17] Lemma 4.2) which will be useful in the remaining of the paper. Let $p\in[2,\infty)$ then for any $a,b\in\mathbb{R}^N$

$$|a+b|^p \geq |a|^p + \frac{1}{2^{p-1}-1}|b|^p + p|a|^{p-2}a\cdot b. \tag{2.18}$$

If $p\in(1,2)$ then for any $a,b\in\mathbb{R}^N$

$$|a+b|^p \geq |a|^p + c(p)\frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a\cdot b. \tag{2.19}$$

We first make the following observation.

Lemma 2.7 *Let $u \in C_c^\infty(\mathcal{C}_\Sigma)$, $u \geq 0$. Set $\psi = \frac{u}{v}$ then*
If $p \geq 2$

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p - \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} |x|^{-p}|u|^p \geq \frac{1}{2^{p-1}-1} \int_{\mathcal{C}_\Sigma} |v\nabla\psi|^p, \tag{2.20}$$

If $1 < p < 2$

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p - \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} |x|^{-p}|u|^p \geq c(p) \int_{\mathcal{C}_\Sigma} \frac{|v\nabla\psi|^2}{(|v\nabla\psi| + |\psi\nabla v|)^{2-p}}, \tag{2.21}$$

Proof. We prove only the case $p \geq 2$ as the case $p \in (1,2)$ goes similarly. Notice that $\nabla u = v\nabla\psi + \psi\nabla v$ then we use the inequality (2.18) with $a = v\nabla\psi$ and $b = \psi\nabla v$ to get

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p \geq \int_{\mathcal{C}_\Sigma} |\psi\nabla v|^p + p\int_{\mathcal{C}_\Sigma} |\psi\nabla v|^{p-2}\psi\nabla v \cdot (v\nabla\psi) + \frac{1}{2^{p-1}-1}\int_{\mathcal{C}_\Sigma} |v\nabla\psi|^p.$$

It is plain that

$$p|\psi\nabla v|^{p-2}\psi\nabla v \cdot (v\nabla\psi) = |\nabla v|^{p-2}\nabla v \cdot (v\nabla\psi^p) = |\nabla v|^{p-2}\nabla v \cdot \nabla(v\psi^p) - |\psi\nabla v|^p.$$

Inserting this in the first inequality and using (2.3) we deduce that

$$\begin{aligned}\int_{\mathcal{C}_\Sigma} |\nabla u|^p &\geq \frac{1}{2^{p-1}-1}\int_{\mathcal{C}_\Sigma} |v\nabla\psi|^p + \int_{\mathcal{C}_\Sigma} |\nabla v|^{p-2}\nabla v \cdot \nabla(v\psi^p)\\ &\geq \frac{1}{2^{p-1}-1}\int_{\mathcal{C}_\Sigma} |v\nabla\psi|^p + \mu_{0,p}(\mathcal{C}_\Sigma)\int_{\mathcal{C}_\Sigma} |x|^{-p}u^p.\end{aligned}$$

□

The improvement in the case $p \geq 2$ is an immediate consequence of the above lemma.

Lemma 2.8 *For all $p \geq 2$*

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p - \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} |x|^{-p}|u|^p \geq \frac{\Lambda_p}{2^{p-1}-1}\int_{\mathcal{C}_\Sigma} |u|^p, \quad \forall u \in C_c^\infty(\mathcal{C}_\Sigma),$$

where $\Lambda_p := \inf_{f \in C_c^1(0,1)} \frac{\int_0^1 r^{p-1}|f'|^p dr}{\int_0^1 r^{p-1}|f|^p dr}$.

Proof. Since $|\nabla|u|| \le |\nabla u|$, we may assume that $u \ge 0$. We only need to estimate the right hand side in (2.20). We use polar coordinates $x \mapsto (|x|, \frac{x}{|x|}) = (r, \sigma)$ and denote by ∂_r the radial direction. Then using (2.18),

$$\begin{aligned}
\int_{\mathcal{C}_\Sigma} |v\nabla\psi|^p &= \int_\Sigma \int_0^1 r^{p-1} V^p |\psi_r \partial_r + \nabla_\sigma \psi|^p \\
&\ge \int_\Sigma V^p \int_0^1 r^{p-1} |\psi_r|^p \ge \Lambda_p \int_\Sigma V^p \int_0^1 r^{p-1} |\psi|^p \\
&\ge \Lambda_p \int_\Sigma \int_0^1 u^p r^{N-1} = \Lambda_p \int_{\mathcal{C}_\Sigma} |u|^p.
\end{aligned}$$

The lemma readily follows from (2.20). $\square$

It is easy to see that by integration by parts $\Lambda_p \ge 1$ while for integer $p \in \mathbb{N}$ then Λ_p corresponds to the first Dirichlet eigenvalue of $-\Delta$ in the unit ball of $\mathbb{R}^p$.
We now turn to the case $p \in (1,2)$ which carries more difficulties. We shall need the following intermediate result.

Lemma 2.9 *Let $p \in (1,2)$ and $u \in C_c^\infty(\mathcal{C}_\Sigma)$, $u \ge 0$. Setting $\psi = \frac{u}{v}$ then there exists a constant $c = c(p, \Sigma) > 0$ such that*

$$c \int_{\mathcal{C}_\Sigma} r |\psi \nabla v|^p \le \int_{\mathcal{C}_\Sigma} r^{(2-p)/2} |v \nabla \psi|^p.$$

Proof. Let $\tilde\psi := r^{\frac{1}{p}} \psi$ and use $\tilde\psi^p v$ as a test function in the weak equation (2.3). Then by Hölder

$$\begin{aligned}
\int_{\mathcal{C}_\Sigma} |\tilde\psi \nabla v|^p &\le \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} r^{-p} v^p \tilde\psi^p + p \int_{\mathcal{C}_\Sigma} |\tilde\psi \nabla v|^{p-1} |v \nabla \tilde\psi| \\
&\le \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} r^{-p} v^p \tilde\psi^p + p \left(\int_{\mathcal{C}_\Sigma} |\tilde\psi \nabla v|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathcal{C}_\Sigma} |v \nabla \tilde\psi|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore by Young's inequality, for $\varepsilon > 0$ small there exists a constant $C_\varepsilon > 0$ depending on p and Σ such that

$$(1 - \varepsilon c(p)) \int_{\mathcal{C}_\Sigma} |\tilde\psi \nabla v|^p \le C_\varepsilon \int_{\mathcal{C}_\Sigma} r^{-p} v^p \tilde\psi^p + C_\varepsilon \int_{\mathcal{C}_\Sigma} |v \nabla \tilde\psi|^p.$$

Recall that $\tilde{\psi} = r^{\frac{1}{p}}\psi$. Then since

$$|\nabla\tilde{\psi}|^p \le c(p)\left(r^{1-p}\psi^p + r|\nabla\psi|^p\right),$$

we conclude that there exists a constant $c = c(p, \Sigma)$ such that

$$c\int_{\mathcal{C}_\Sigma} r|\psi\nabla v|^p \le \int_{\mathcal{C}_\Sigma} r^{1-p}v^p\psi^p + \int_{\mathcal{C}_\Sigma} r^{(2-p)/p}|v\nabla\psi|^p, \tag{2.22}$$

we have used the fact that $r \le r^{(2-p)/p}$ for all $r \in (0,1)$. To estimate the first term in the right hand side in (2.22) we will use the 2-dimensional Hardy inequality. Through the polar coordinates $x \mapsto (r, \sigma)$

$$\begin{aligned}
\int_{\mathcal{C}_\Sigma} r^{1-p}v^p\psi^p &= \int_\Sigma V^p \int_0^1 r^{p-1}\left(\frac{\psi}{r}\right)^p r \\
&\le \int_\Sigma V^p \int_0^1 \left(\frac{\psi}{r}\right)^p r \\
&\le \left|\frac{p}{p-2}\right|^{-p} \int_\Sigma V^p \int_0^1 |\psi_r|^p r \\
&= \left|\frac{p}{p-2}\right|^{-p} \int_0^1 \int_\Sigma V^p v^{-p}|v\nabla\psi|^p r = \left|\frac{p}{p-2}\right|^{-p} \int_0^1 \int_\Sigma r^{N-p+1}|v\nabla\psi|^p.
\end{aligned}$$

To conclude, we notice that $r^{N-p+1} = r^{N-\frac{p}{2}}r^{(2-p)/p} \le r^{N-1}r^{(2-p)/p}$ as $p \in (1,2)$ so that

$$\int_{\mathcal{C}_\Sigma} r^{1-p}v^p\psi^p \le \left|\frac{p}{p-2}\right|^{-p} \int_{\mathcal{C}_\Sigma} r^{(2-p)/p}|v\nabla\psi|^p.$$

Inserting this in (2.22) the lemma follows immediately. $\square$

We are now in position to prove the improved Hardy inequality for $p \in (1,2)$.

Lemma 2.10 *Let $p \in (1,2)$. Then there exists a constant $c = c(p, \Sigma) > 0$ such that*

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p - \mu_{0,p}(\mathcal{C}_\Sigma)\int_{\mathcal{C}_\Sigma} |x|^{-p}|u|^p \ge c\int_{\mathcal{C}_\Sigma} |u|^p, \quad \forall u \in C_c^\infty(\mathcal{C}_\Sigma).$$

Proof. Here also we may assume that $u \ge 0$. We need to estimate the right hand side of (2.21). Let $r = |x|$ then by Hölder and Lemma 2.9, we have

$$\begin{aligned}
\int_{\mathcal{C}_\Sigma} r^{\frac{2-p}{2}} |v\nabla\psi|^p &= \int_{\mathcal{C}_\Sigma} \frac{|v\nabla\psi|^2}{(|v\nabla\psi| + |\psi\nabla v|)^{(2-p)p/2}} r^{\frac{2-p}{2}} (|v\nabla\psi| + |\psi\nabla v|)^{(2-p)p/2} \\
&\le \left(\int_{\mathcal{C}_\Sigma} \frac{|v\nabla\psi|^2}{(|v\nabla\psi| + |\psi\nabla v|)^{2-p}}\right)^{p/2} \left(\int_{\mathcal{C}_\Sigma} r\, ||v\nabla\psi| + |\psi\nabla v||^p\right)^{(2-p)/2} \\
&\le \left(\int_{\mathcal{C}_\Sigma} \frac{|v\nabla\psi|^2}{(|v\nabla\psi| + |\psi\nabla v|)^{2-p}}\right)^{p/2} \\
&\quad \times \left(2^{p-1} \int_{\mathcal{C}_\Sigma} r|v\nabla\psi|^p + 2^{p-1} \int_{\mathcal{C}_\Sigma} r|\psi\nabla v|^p\right)^{(2-p)/2} \\
&\le c\left(\int_{\mathcal{C}_\Sigma} \frac{|v\nabla\psi|^2}{(|v\nabla\psi| + |\psi\nabla v|)^{2-p}}\right)^{p/2} \left(\int_{\mathcal{C}_\Sigma} r^{\frac{2-p}{2}} |v\nabla\psi|^p\right)^{(2-p)/2},
\end{aligned}$$

where c a positive constant depending only on p and Σ and we have used once more the fact that $r \le r^{(2-p)/p}$ for all $r \in (0,1)$. Consequently by (2.21), we deduce that

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p - \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} |x|^{-p}|u|^p \ge c \int_{\mathcal{C}_\Sigma} r^{\frac{2-p}{2}} |v\nabla\psi|^p. \tag{2.23}$$

To proceed we estimate

$$\begin{aligned}
\int_\Sigma \int_0^1 u^p r^{N-1} &= \int_\Sigma V^p \int_0^1 r^{p-1}|\psi|^p \le c(p) \int_\Sigma V^p \int_0^1 r|\psi_r|^p \\
&\le c(p) \int_\Sigma V^p \int_0^1 r^{\frac{p}{2}} |\psi_r|^p \\
&\le c(p) \int_{\mathcal{C}_\Sigma} r^{\frac{2-p}{2}} |v\nabla\psi|^p.
\end{aligned}$$

The first inequality comes from the 2-dimensional embedding $W_0^{1,p} \subset L^{\frac{2p}{2-p}} \subset L^{\frac{p}{3-p}}$, one can see [[13] page 2155] for the proof. Putting this in (2.23) we conclude that there exists a positive constant $c = c(p, \Sigma)$ such that

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^p - \mu_{0,p}(\mathcal{C}_\Sigma) \int_{\mathcal{C}_\Sigma} |x|^{-p}|u|^p \ge c \int_{\mathcal{C}_\Sigma} |u|^p$$

which was the purpose of the lemma. □

The main result in this section is contained in the next theorem.

Theorem 2.11 *Let Ω be a domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. If Ω is contained in a half-ball centered at 0 then there exists a constant $c(N,p) > 0$ such that*

$$\int_\Omega |\nabla u|^p - \mu_{0,p}(H) \int_\Omega |x|^{-p}|u|^p \geq \frac{c(N,p)}{diam(\Omega)^p} \int_\Omega |u|^p \qquad \forall u \in W_0^{1,p}(\Omega).$$

Proof. Let $R = \mathrm{diam}(\Omega)$ be the diameter of Ω. Then Ω is contained in a half ball B_R^+ of radius R centered at the origin. From Lemma 2.8 and Lemma 2.10 we infer that

$$\int_{B_R^+} |\nabla u|^p - \mu_{0,p}(H) \int_{B_R^+} |x|^{-p}|u|^p \geq \frac{c(N,p)}{R^p} \int_{B_R^+} |u|^p \quad \forall u \in C_c^\infty(\Omega)$$

by homogeneity. The theorem readily follows by density. $\square$

We do not know whether $\mathrm{diam}(\Omega)$ might be replaced with $\omega_N|\Omega|^{\frac{1}{N}}$ as in [13] at least when Ω is convex and $p \geq 2$. There might exists also "logarithmic" improvement as was recently obtained in [11] inside cones and $p = 2$. One can see also the work of Barbatis-Filippas-Tertikas in [1] for domains containing the origin or when $|x|$ is replaced by the distance to the boundary.

A Hardy's inequality

We denote by d the distance function of Ω:

$$d(x) := \inf\{|x - \sigma| \,:\, \sigma \in \partial\Omega\}.$$

In this section, we study the problem of finding minima to the following quotient

$$\text{(A.1)} \qquad \nu_{\lambda,p}(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega |u|^p \, dx}{\displaystyle\int_\Omega d^{-p}|u|^p \, dx},$$

where $p > 1$ and $\lambda \in \mathbb{R}$ is a varying parameter. Existence of extremals to this problem was studied in [2] when $p = 2$ and in [18] with $\lambda = 0$. It is known (see for instance [18]) that $\nu_{0,p}(\Omega) \leq \mathbf{c}_p$ for any smooth bounded domain Ω while for convex domain Ω, the Hardy constant $\nu_{0,p}(\Omega)$ is not achieved and $\nu_{0,p}(\Omega) = \left(\frac{p-1}{p}\right)^p =: \mathbf{c}_p$. The main result in this section is contained in the following

Theorem A.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p > 1$, there exits $\tilde{\lambda}(p, \Omega) \in [-\infty, +\infty)$ such that*

$$\nu_{\lambda,p}(\Omega) < \left(\frac{p-1}{p}\right)^p, \quad \forall \lambda > \tilde{\lambda}(p, \Omega). \tag{A.2}$$

The infimum in (A.1) *is attained if $\lambda > \tilde{\lambda}(p, \Omega)$.*

We start with the following result which is stronger than needed. It was proved in [2] for $p = 2$ and in [12] when $2 \leq p < N$ as the authors were dealing with Hardy-Sobolev inequalities.

Lemma A.2 *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p \in (1, \infty)$. Then there exists $\beta = \beta(p, \Omega) > 0$ small such that*

$$\int_{\Omega_\beta} |\nabla u|^p \geq \boldsymbol{c}_p \int_{\Omega_\beta} d^{-p} |u|^p \quad \forall u \in H^1_0(\Omega), \tag{A.3}$$

where $\Omega_\beta := \{x \in \Omega \,:\, d(x) < \beta\}$.

Proof. Since $|\nabla |u|| \leq |\nabla u|$, we may assume that $u \geq 0$. Let $u \in C^\infty_c(\Omega)$ and put $v = d^{\frac{1-p}{p}} u$. Using (2.18) and (2.19), we get

$$|\nabla u|^p - \mathbf{c}_p d^{-p} |u|^p \geq c(p) d^{p-1} |\nabla v|^p + \left|\frac{p-1}{p}\right|^{p-1} \nabla d \cdot \nabla(v^p) \quad \text{ if } p \geq 2, \tag{A.4}$$

(A.5)

$$|\nabla u|^p - \mathbf{c}_p d^{-p} |u|^p \geq c(p) \frac{d |\nabla v|^2}{\left(\mathbf{c}_p^{\frac{1}{p}} |v| + d |\nabla v|\right)^{2-p}} + \left|\frac{p-1}{p}\right|^{p-1} \nabla d \cdot \nabla(v^p) \quad \text{ if } p \in (1, 2).$$

By integration by parts, we have

$$\int_{\Omega_\beta} \nabla d \cdot \nabla(v^p) = -\int_{\Omega_\beta} \Delta d |v|^p + \int_{\partial\Omega_\beta} |v|^p \geq -c \int_{\Omega_\beta} |v|^p + \int_{\partial\Omega_\beta} |v|^p,$$

for a positive constant depending only on Ω. Multiply the identity $\mathrm{div}(d \nabla d) = 1 + d\Delta d$ by v in integrate by parts to get

$$(1 + o(1)) \int_{\Omega_\beta} |v|^p = -p \int_{\Omega_\beta} d |v|^{p-1} \nabla d \cdot \nabla v + \int_{\partial\Omega_\beta} d |v|^p \leq c(p) \int_{\Omega_\beta} d |v|^{p-1} |\nabla v| + \int_{\partial\Omega_\beta} d |v|^p.$$

By Hölder and Young's inequalities

$$(1+o(1)-c\varepsilon)\int_{\Omega_\beta}|v|^p \le c_\varepsilon \int_{\Omega_\beta} d^p|\nabla v|^p + \int_{\partial\Omega_\beta} d|v|^p. \tag{A.6}$$

Case $p\ge 2$. Using (A.6) we infer that

$$(1+o(1)-c\varepsilon)\int_{\Omega_\beta}|v|^p \le c_\varepsilon\beta \int_{\Omega_\beta} d^{p-1}|\nabla v|^p + \beta\int_{\partial\Omega_\beta} |v|^p.$$

It follows from (A.4) that for $\varepsilon,\beta>0$ small

$$\int_{\Omega_\beta}|\nabla u|^p - \mathbf{c}_p\int_{\Omega_\beta} d^{-p}|u|^p \ge c\left(\int_{\Omega_\beta} d^{p-1}|\nabla v|^p + \int_{\partial\Omega_\beta}|v|^p\right)$$

as desired.

Case $p\in(1,2)$. By Hölder and Young's inequalities

$$\begin{aligned}\int_{\Omega_\beta} d^p|\nabla v|^p &= \int_{\Omega_\beta}\frac{d^p|\nabla v|^p}{\left(\mathbf{c}_p^{\frac{1}{p}}|v|+d|\nabla v|\right)^{\frac{p(2-p)}{2}}}\left(\mathbf{c}_p^{\frac{1}{p}}|v|+d|\nabla v|\right)^{\frac{p(2-p)}{2}}\\ &\le c_\varepsilon\int_{\Omega_\beta}\frac{d^2|\nabla v|^2}{\left(\mathbf{c}_p^{\frac{1}{p}}|v|+d|\nabla v|\right)^{2-p}} + \varepsilon c\int_{\Omega_\beta}|v|^p + \varepsilon c\int_{\Omega_\beta} d^p|\nabla v|^p\end{aligned}$$

and thus

$$(1-c\varepsilon)\int_{\Omega_\beta} d^p|\nabla v|^p \le c_\varepsilon\int_{\Omega_\beta}\frac{d^2|\nabla v|^2}{\left(\mathbf{c}_p^{\frac{1}{p}}|v|+d|\nabla v|\right)^{2-p}} + \varepsilon c\int_{\Omega_\beta}|v|^p.$$

Using this in (A.6) we obtain

$$(1+o(1)-c\varepsilon)\int_{\Omega_\beta}|v|^p \le c\beta\int_{\Omega_\beta}\frac{d|\nabla v|^2}{\left(\mathbf{c}_p^{\frac{1}{p}}|v|+d|\nabla v|\right)^{2-p}} + c\beta\int_{\partial\Omega_\beta}|v|^p.$$

By (A.5), we conclude that for $\varepsilon,\beta>0$ small

$$\int_{\Omega_\beta}|\nabla u|^p - \mathbf{c}_p\int_{\Omega_\beta} d^{-p}|u|^p \ge c\int_{\Omega_\beta}\frac{d|\nabla v|^2}{\left(\mathbf{c}_p^{\frac{1}{p}}|v|+d|\nabla v|\right)^{2-p}} + c\int_{\partial\Omega_\beta}|v|^p.$$

This ends the proof of the lemma. □

Lemma A.3 *Let Ω be a bounded domain of class C^2 in $\mathbb{R}^N$. Then there exists $\tilde{\lambda}(p,\Omega) \in [-\infty, +\infty)$ such that*

$$\nu_{\lambda,p}(\Omega) < \boldsymbol{c}_p \quad \forall \lambda > \tilde{\lambda}(p,\Omega).$$

Proof. The proof will be carried out in 2 steps.
Step 1: We claim that $\sup_{\lambda\in\mathbb{R}} \nu_{\lambda,p}(\Omega) \geq \mathbf{c}_p$.
For $\beta > 0$ we define

$$\Omega_\beta := \{x \in \Omega \,:\, d(x) < \beta\}.$$

Let $\psi \in C^\infty(\Omega_\beta)$ with $0 \leq \psi \leq 1$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus \Omega_{\frac{\beta}{2}}$ and $\psi \equiv 1$ in $\Omega_{\frac{\beta}{4}}$. For $\varepsilon > 0$ small, there holds

$$\begin{aligned}
\int_\Omega d^{-p}|u|^p &= \int_\Omega d^{-p}|\psi u + (1-\psi)u|^p \\
&\leq (1+\varepsilon)\int_\Omega d^{-p}|\psi u|^p + C\int_\Omega d^{-p}(1-\psi)^p|u|^p \\
&\leq (1+\varepsilon)\int_\Omega d^{-p}|\psi u|^p + C\int_\Omega |u|^p.
\end{aligned}$$

By (A.3), we infer that

$$\mathbf{c}_p \int_\Omega d^{-p}|\psi u|^p \leq \int_\Omega |\nabla(\psi u)|^p$$

and hence

$$\mathbf{c}_p \int_\Omega d^{-p}|u|^p \leq (1+\varepsilon)\int_\Omega |\nabla(\psi u)|^p + C\int_\Omega |u|^p. \tag{A.7}$$

Since $|\nabla(\psi u)|^p \leq (\psi|\nabla u| + |u||\nabla\psi|)^p$ we deduce that

$$|\nabla(\psi u)|^p \leq (1+\varepsilon)\psi^p|\nabla u|^p + C|u|^p|\nabla\psi|^p \leq (1+\varepsilon)|\nabla u|^p + C|u|^p.$$

Using (A.7), we conclude that

$$\mathbf{c}_p \int_\Omega d^{-p}|u|^p \leq (1+\varepsilon)^2\int_\Omega |\nabla u|^p + C(\varepsilon,\beta)\int_\Omega |u|^p.$$

This means that $\mathbf{c}_p \leq \sup_{\lambda\in\mathbb{R}} \nu_{\lambda,p}(\Omega)$.
Step 2: We claim that $\sup_{\lambda\in\mathbb{R}} \nu_{\lambda,p}(\Omega) \leq \mathbf{c}_p$.

Let $\beta > 0$ then by (1.3) and scale invariance we have $\mu_{0,p}(0,\beta) = \mathbf{c}_p$. Hence for $\varepsilon > 0$ there exits a function $\phi \in W_0^{1,p}(0,\beta)$ such that

$$\mathbf{c}_p + \varepsilon \geq \frac{\int_0^\beta |\phi'|^p \, ds}{\int_0^\beta s^{-p}\phi^p \, ds}. \tag{A.8}$$

Letting $u(x) = \phi(d(x))$, there exists a positive constant C depending only on Ω such that

$$\int_{\Omega_\beta} |\nabla u|^p = \int_0^\beta \int_{\partial\Omega_s} |\phi'(s)|^p \, d\sigma_s \leq (1 + C\beta)\, |\partial\Omega| \int_0^\beta |\phi'(s)|^p \, ds.$$

Furthermore

$$\int_{\Omega_\beta} d^{-p}|u|^p = \int_0^\beta \int_{\partial\Omega_s} s^{-p}|\phi(s)|^p \, d\sigma_s \geq (1 - C\beta)\, |\partial\Omega| \int_0^\beta |\phi(s)|^p \, ds.$$

By (A.8) we conclude that

$$\nu_{\lambda,p}(\Omega) \leq \frac{\displaystyle\int_{\Omega_\beta} |\nabla u|^p \ dx - \lambda \int_{\Omega_\beta} u^p \ dx}{\displaystyle\int_{\Omega_\beta} d^{-p}|u|^p \ dx} \leq (\mathbf{c}_p + \varepsilon)\frac{1 + C\beta}{1 - C\beta} + |\lambda| \frac{\displaystyle\int_{\Omega_\beta} |u|^p \ dx}{\displaystyle\int_{\Omega_\beta} d^{-p}|u|^p \ dx}.$$

Since $\int_\Omega d^{-p}|u|^2 \ dx \geq \beta^{-p} \int_{\Omega_\beta} |u|^p \ dx$, we get

$$\nu_{\lambda,p}(\Omega) \leq (\mathbf{c}_p + \varepsilon)\frac{1 + C\beta}{1 - C\beta} + \beta^p|\lambda|,$$

sending β to 0 we get the desired result. $\square$

Clearly the proof of Theorem A.1 goes similarly as the one of Theorem 1.1 and we skip it.

It was shown in [2] that $\tilde{\lambda}(2,\Omega) \in \mathbb{R}$ and that $\nu_{\lambda,p}(\Omega)$ is not achieved for any $\lambda \geq \tilde{\lambda}(2,\Omega)$. On the other hand by [8], there are domains for which $\tilde{\lambda}(2,\Omega) < 0$, see also [18].

We point out that if Ω is convex then by [22] there exists a constant $a(N,p) > 0$ (explicitly given) such that

$$\tilde{\lambda}(p,\Omega) \geq \frac{a(N,p)}{|\Omega|^{\frac{p}{N}}}.$$

We finish this section by showing that there are smooth bounded domains in $\mathbb{R}^N$ such that $\tilde{\lambda}(p,\Omega) \in [-\infty, 0)$. We let $U \subset \mathbb{R}^N$, $N \geq 2$ with $0 \in \partial U$ be an exterior domain and set $\Omega_r = B_r(0) \cap U$.

Proposition A.4 *Assume that $p > \frac{N+1}{2}$ then there exists $r > 0$ such that $\nu_{0,p}(\Omega_r) < \left(\frac{p-1}{p}\right)^p$.*

Proof. Clearly $\mu_{0,p}(\mathbb{R}^N \setminus \{0\}) = \left|\frac{N-p}{p}\right|^p < \left(\frac{p-1}{p}\right)^p$ provided $p > \frac{N+1}{2}$. Let $\varepsilon > 0$ such that $\left(\frac{p-1}{p}\right)^p > \left|\frac{N-p}{p}\right|^p + \varepsilon$ so by (2.16), there exits $r > 0$ such that

$$\mu_{0,p}(\Omega_r) < \left|\frac{N-p}{p}\right|^p + \varepsilon < \left(\frac{p-1}{p}\right)^p.$$

The conclusion readily follows since $\nu_{0,p}(\Omega_r) \leq \mu_{0,p}(\Omega_r)$ because $0 \in \partial\Omega_r$. $\square$

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