# UNIQUE CONTINUATION PROPERTY AND LOCAL ASYMPTOTICS OF SOLUTIONS TO FRACTIONAL ELLIPTIC EQUATIONS 

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#### Abstract

Asymptotics of solutions to fractional elliptic equations with Hardy type potentials is studied in this paper. By using an Almgren type monotonicity formula, separation of variables, and blow-up arguments, we describe the exact behavior near the singularity of solutions to linear and semilinear fractional elliptic equations with a homogeneous singular potential related to the fractional Hardy inequality. As a consequence we obtain unique continuation properties for fractional elliptic equations.


## 1. Introduction

The purpose of the present paper is to describe the asymptotic behavior of solutions to the following class of fractional elliptic semilinear equations with singular homogeneous potentials

$$
\begin{equation*}
(-\Delta)^{s} u(x)-\frac{\lambda}{|x|^{2 s}} u(x)=h(x) u(x)+f(x, u(x)), \quad \text { in } \Omega \tag{1}
\end{equation*}
$$

where $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ (see definition below) and $\Omega \subset \mathbb{R}^{N}$ is a bounded domain containing the origin,

$$
\begin{align*}
& N>2 s, \quad s \in(0,1), \quad \lambda<\Lambda_{N, s}:=2^{2 s} \frac{\Gamma^{2}\left(\frac{N+2 s}{4}\right)}{\Gamma^{2}\left(\frac{N-2 s}{4}\right)},  \tag{2}\\
& h \in C^{1}(\Omega \backslash\{0\}), \quad|h(x)|+|x \cdot \nabla h(x)| \leqslant C_{h}|x|^{-2 s+\varepsilon} \text { as }|x| \rightarrow 0,  \tag{3}\\
& \left\{\begin{array}{l}
f \in C^{1}(\Omega \times \mathbb{R}), \quad t \mapsto F(x, t) \in C^{1}(\Omega \times \mathbb{R}), \\
|f(x, t) t|+\left|f_{t}^{\prime}(x, t) t^{2}\right|+\left|\nabla_{x} F(x, t) \cdot x\right| \leqslant C_{f}|t|^{p} \text { for a.e. } x \in \Omega \text { and all } t \in \mathbb{R},
\end{array}\right. \tag{4}
\end{align*}
$$

where $2<p \leqslant 2^{*}(s)=\frac{2 N}{N-2 s}, F(x, t)=\int_{0}^{t} f(x, r) d r, C_{f}, C_{h}, \varepsilon>0$ are positive constants independent of $x \in \Omega$ and $t \in \mathbb{R}, \nabla_{x} F$ denotes the gradient of $F$ with respect to the $x$ variable, and $f_{t}^{\prime}(x, t)=\frac{\partial f}{\partial t}(x, t)$.

We recall that for any $\varphi \in C_{\mathrm{c}}^{\infty}\left(\mathbb{R}^{N}\right)$ and $s \in(0,1)$, the fractional Laplacian $(-\Delta)^{s} \varphi$ is defined as

$$
\begin{equation*}
(-\Delta)^{s} \varphi(x)=C(N, s) \mathrm{P} . \mathrm{V} \cdot \int_{\mathbb{R}^{N}} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2 s}} d y=C(N, s) \lim _{\rho \rightarrow 0^{+}} \int_{|x-y|>\rho} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2 s}} d y \tag{5}
\end{equation*}
$$

where P.V. indicates that the integral is meant in the principal value sense and

$$
C(N, s)=\pi^{-\frac{N}{2}} 2^{2 s} \frac{\Gamma\left(\frac{N+2 s}{2}\right)}{\Gamma(2-s)} s(1-s)
$$

The Dirichlet form associated to $(-\Delta)^{s}$ on $C_{\mathrm{c}}^{\infty}\left(\mathbb{R}^{N}\right)$ is given by

$$
\begin{equation*}
(u, v)_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)}=\frac{C(N, s)}{2} \int_{\mathbb{R}^{2 N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2 s}} d x d y=\int_{\mathbb{R}^{N}}|\xi|^{2 s} \overline{\widehat{v}(\xi)} \widehat{u}(\xi) d \xi \tag{6}
\end{equation*}
$$

[^0]where $\widehat{u}$ denotes the unitary Fourier transform of $u$. It defines a scalar product thanks to (8) or (9) below. From now on we define $\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ as the completion on $C_{\mathrm{c}}^{\infty}\left(\mathbb{R}^{N}\right)$ with respect to the norm induced by the scalar product (6).

By a weak solution to (1) we mean a function $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ such that

$$
\begin{equation*}
(u, \varphi)_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)}=\int_{\Omega}\left(\frac{\lambda}{|x|^{2 s}} u(x)+h(x) u(x)+f(x, u(x))\right) \varphi(x) d x, \text { for all } \varphi \in C_{c}^{\infty}(\Omega) \tag{7}
\end{equation*}
$$

We notice that the right hand side of (7) is well defined in view of assumptions (3)-(4), the Hardy-Littlewood-Sobolev inequality

$$
\begin{equation*}
S_{N, s}\|u\|_{L^{2^{*}(s)}\left(\mathbb{R}^{N}\right)}^{2} \leqslant\|u\|_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)}^{2} \tag{8}
\end{equation*}
$$

and the following Hardy inequality, due to Herbst in [19] (see also [27]),

$$
\begin{equation*}
\Lambda_{N, s} \int_{\mathbb{R}^{N}} \frac{u^{2}(x)}{|x|^{2 s}} d x \leqslant \int_{\mathbb{R}^{N}}|\xi|^{2 s}|\widehat{u}(\xi)|^{2} d \xi=\|u\|_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)}^{2}, \quad \text { for all } u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right) \tag{9}
\end{equation*}
$$

It should also be remarked that, in (7), we allow $p=2^{*}(s)$ and that $u$ is not prescribed outside $\Omega$.
One of the aim of this paper is to give the precise behaviour of a solution $u$ to (1). The rate and the shape of $u$ are given by the the eigenvalues and the eigenfunctions of the following eigenvalue problem

$$
\begin{cases}-\operatorname{div}_{\mathbb{S}^{N}}\left(\theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi\right)=\mu \theta_{1}^{1-2 s} \psi, & \text { in } \mathbb{S}_{+}^{N}  \tag{10}\\ -\lim _{\theta_{1} \rightarrow 0^{+}} \theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi \cdot \mathbf{e}_{1}=\kappa_{s} \lambda \psi, & \text { on } \partial \mathbb{S}_{+}^{N}\end{cases}
$$

where

$$
\begin{equation*}
\kappa_{s}=\frac{\Gamma(1-s)}{2^{2 s-1} \Gamma(s)} \tag{11}
\end{equation*}
$$

and

$$
\mathbb{S}_{+}^{N}=\left\{\left(\theta_{1}, \theta_{2}, \ldots, \theta_{N+1}\right) \in \mathbb{S}^{N}: \theta_{1}>0\right\}=\left\{\frac{z}{|z|}: z \in \mathbb{R}^{N+1}, z \cdot \mathbf{e}_{1}>0\right\}
$$

with $\mathbf{e}_{1}=(1,0, \ldots, 0)$; we refer to section 2.1 for a variational formulation of (10). From classical spectral theory (see section 2.1 for the details) problem (10) admits a diverging sequence of real eigenvalues with finite multiplicity

$$
\mu_{1}(\lambda) \leqslant \mu_{2}(\lambda) \leqslant \cdots \leqslant \mu_{k}(\lambda) \leqslant \cdots
$$

Moreover $\mu_{1}(\lambda)>-\left(\frac{N-2 s}{2}\right)^{2}$, see Lemma 2.2. We notice that if $\lambda=0$ then $\mu_{k}(0) \geqslant 0$ for all $k$.
Our first main result is the following theorem.
Theorem 1.1. Let $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ be a nontrivial solution to (1) in a bounded domain $\Omega \subset \mathbb{R}^{N}$ containing the origin as in (7) with $s, \lambda, h$ and $f$ satisfying assumptions (2), (3) and (4). Then there exists an eigenvalue $\mu_{k_{0}}(\lambda)$ of (10) and an eigenfunction $\psi$ associated to $\mu_{k_{0}}(\lambda)$ such that

$$
\begin{equation*}
\tau^{-\frac{2 s-N}{2}-\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}} u(\tau x) \rightarrow|x|^{-\frac{N-2 s}{2}+\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}} \psi\left(0, \frac{x}{|x|}\right) \quad \text { as } \tau \rightarrow 0^{+} \tag{12}
\end{equation*}
$$

in $C_{\mathrm{loc}}^{1, \alpha}\left(B_{1}^{\prime} \backslash\{0\}\right)$ for some $\alpha \in(0,1)$, where $B_{1}^{\prime}:=\left\{x \in \mathbb{R}^{N}:|x|<1\right\}$, and, in particular,

$$
\begin{equation*}
\tau^{-\frac{2 s-N}{2}-\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}} u\left(\tau \theta^{\prime}\right) \rightarrow \psi\left(0, \theta^{\prime}\right) \quad \text { in } C^{1, \alpha}\left(\mathbb{S}^{N-1}\right) \quad \text { as } \tau \rightarrow 0^{+} \tag{13}
\end{equation*}
$$

where $\mathbb{S}^{N-1}=\partial \mathbb{S}_{+}^{N}$.
We should mention that the above result is stated in such form for the sake of simplicity. In fact, for the eigenfunction $\psi$ in (13), we obtain precisely its components in any basis of the eigenspace corresponding to $\mu_{k_{0}}(\lambda)$, see Theorem 4.1 (below). We also remark that $\mu_{1}(\lambda)<0$ for $\lambda>0$ and it is determined implicitly by the usual Gamma function, see Proposition 2.3.

Analogous results were obtained in $[12,13,14]$ for corresponding equations involving the Laplacian (i.e. in the case $s=1$ ) and Hardy-type potentials for different kinds of problems: in $[12,14]$ for Schrödinger equations with electromagnetic potentials and in [13] for Schrödinger equations with inverse square many-particle potentials.

As a particular case of Theorem 1.1, if $\lambda=0$ we obtain that the convergence stated in 12 holds in $C^{1, \alpha}\left(B_{1}^{\prime}\right)$.

Corollary 1.2. Let $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ be a nontrivial weak solution to

$$
\begin{equation*}
(-\Delta)^{s} u(x)=h(x) u(x)+f(x, u(x)), \quad \text { in } \Omega, \tag{14}
\end{equation*}
$$

in a bounded domain $\Omega \subset \mathbb{R}^{N}$ with $s \in(0,1), h \in C^{1}(\Omega)$, and $f$ satisfying (4). Then, for every $x_{0} \in \Omega$, there exists an eigenvalue $\mu_{k_{0}}=\mu_{k_{0}}(0)$ of problem (10) with $\lambda=0$ and an eigenfunction $\psi$ associated to $\mu_{k_{0}}$ such that

$$
\begin{align*}
\tau^{-\frac{2 s-N}{2}-\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}}} u\left(x_{0}+\right. & \left.\tau\left(x-x_{0}\right)\right)  \tag{15}\\
& \rightarrow\left|x-x_{0}\right|^{-\frac{N-2 s}{2}+\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}}} \psi\left(0, \frac{x-x_{0}}{\left|x-x_{0}\right|}\right) \quad \text { as } \tau \rightarrow 0^{+},
\end{align*}
$$

in $C^{1, \alpha}\left(\left\{x \in \mathbb{R}^{N}: x-x_{0} \in B_{1}^{\prime}\right\}\right)$.
Our next result contains the so called strong unique continuation property which is a direct consequence of Theorem 1.1.
Theorem 1.3. Suppose that all the assumptions of Theorem 1.1 hold true. Let $u$ be a solution to (1) in a bounded domain $\Omega \subset \mathbb{R}^{N}$ containing the origin. If $u(x)=o\left(|x|^{n}\right)=o(1)|x|^{n}$ as $|x| \rightarrow 0$ for all $n \in \mathbb{N}$, then $u \equiv 0$ in $\Omega$.

From Theorem 1.2 a unique continuation principle related to sets of positive measures follows.
Theorem 1.4. Let $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ be a weak solution to (14) in a bounded domain $\Omega \subset \mathbb{R}^{N}$ with $s \in(0,1), h \in C^{1}(\Omega)$, and $f$ satisfying (4). If $u \equiv 0$ on a set $E \subset \Omega$ of positive measure, then $u \equiv 0$ in $\Omega$.

An interesting application of Theorem 1.4 is that the nodal sets of eigenfunctions for the fractional laplacian operator have zero Lebesgue measure. We should point out that this was a key assumption in [15], where the authors studied the existence of weak solutions to some non-local equations.

Recent research in the field of second order elliptic equations has devoted a great attention to the problem of unique continuation property in the presence of singular lower order terms, see e.g. [11, 20, 22]. Two different kinds of approach have been developed to treat unique continuation: a first one is due to Carleman [5] and is based on weighted priori inequalities, whereas a second one is due to Garofalo and Lin [17] and is based on local doubling properties proved by Almgren monotonicity formula. In the present paper we will follow the latter approach. Furthermore, in the spirit of $[12,13,14]$, the combination of monotonicity methods with blow-up analysis will enable us to prove not only unique continuation but also the precise asymptotics of solutions stated in Theorem 1.1.

As far as unique continuation from sets of positive measures is concerned, we mention [6] where it was proved for second order elliptic operators by combining strong unique continuation property with the De Giorgi inequality. Since the validity of a the De Giorgi type inequality for the fractional problem (or even its extension, see (17)) seems to be hard to prove, we will base the proof of Theorem 1.4 directly on the asymptotic of solutions proved in Theorem 1.1.

To explain our argument of proving the asymptotic behavior, let us we write (7) as

$$
\begin{equation*}
(-\Delta)^{s} u=G(x, u) \quad \text { in } \Omega \tag{16}
\end{equation*}
$$

The proof of our results is based on the study of the Almgren frequency function at the origin 0 : "ratio of the local energy over mass near the origin". Due to the non-locality of the Dirichlet form associated to $(-\Delta)^{s}$, it is not clear how to set up an Almgren's type frequency function using this energy as in the local case $s=1$. A way out for this difficulty is to use the Caffarelli-Silvestre extension [4] which can be seen as a local version of (16). The Caffarelli-Silvestre extension of a solution $u$ to (16) is a function $w$ defined on

$$
\mathbb{R}_{+}^{N+1}=\left\{z=(t, x): t \in(0,+\infty), x \in \mathbb{R}^{N}\right\}
$$

satisfying $w=u$ on $\Omega$ and solving in some weak sense (see Section 2 for more details) the boundary value problem

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla w\right)=0, & \text { in } \mathbb{R}_{+}^{N+1}  \tag{17}\\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial w}{\partial t}=\kappa_{s} G(x, w), & \text { on } \Omega\end{cases}
$$

Here and in the following, we write $z=(t, x) \in \mathbb{R}_{+}^{N+1}$ with $x \in \mathbb{R}^{N}$ and $t>0$, and we identify $\mathbb{R}^{N}$ with $\partial \mathbb{R}_{+}^{N+1}$, so that $\Omega$ is contained in $\partial \mathbb{R}_{+}^{N+1}$.

We then consider the Almgren's frequency function

$$
\mathcal{N}(r)=\frac{D(r)}{H(r)}
$$

where

$$
D(r)=\frac{1}{r^{N-2 s}}\left[\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}} G(x, w) w d x\right], \quad H(r)=\frac{1}{r^{N+1-2 s}} \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S
$$

being

$$
\begin{aligned}
& B_{r}^{+}=\left\{z=(t, x) \in \mathbb{R}_{+}^{N+1}:|z|<r\right\}, \quad B_{r}^{\prime}:=\left\{x \in \mathbb{R}^{N}:|x|<r\right\}, \\
& S_{r}^{+}=\left\{z=(t, x) \in \mathbb{R}_{+}^{N+1}:|z|=r\right\},
\end{aligned}
$$

and $d S$ denoting the volume element on $N$-dimensional spheres. We mention that an Almgren's frequency function for degenerate elliptic equations of the form (17) was first formulated in [4, Section 6].

As a first but nontrivial step, we prove that $\lim _{r \rightarrow 0} \mathcal{N}(r):=\gamma$ exists and it is finite, see Lemma 3.15. Next, we make a blow-up analysis by zooming around the origin the solution $w$ normalized also by $\sqrt{H}$. More precisely, setting $w_{\tau}(z)=\frac{w(\tau z)}{\sqrt{H(\tau)}}$, we have that $w_{\tau}$ converges, (in some Hölder and Sobolev spaces) to $\widetilde{w}$ solving the limiting equation

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla \widetilde{w}\right)=0, & \text { in } B_{1}^{+}, \\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial \widetilde{w}}{\partial t}=\frac{\kappa_{s} \lambda}{|z|^{2 s}} \widetilde{w}, & \text { on } B_{1}^{\prime} .\end{cases}
$$

To obtain this, the fact that $h$ is negligible with respect to the Hardy potential and $f$ is at most critical with respect to the Sobolev exponent (see assumptions (3) and (4)) plays a crucial role; we refer to Lemma 4.2 for more details.

The main point is that the Almgren's frequency for $\widetilde{w}$ is $\widetilde{\mathcal{N}}(r)=\lim _{\tau \rightarrow 0} \mathcal{N}(\tau r)=\gamma$, i.e. $\widetilde{\mathcal{N}}$ is constant; hence $\widetilde{w}$ and $\nabla \widetilde{w} \cdot \frac{z}{|z|}$ are proportional on $L^{2}\left(S_{r}^{+} ; t^{1-2 s}\right)$. As a consequence we obtain $\widetilde{w}(z)=\varphi(|z|) \psi\left(\frac{z}{|z|}\right)$. By separating variables in polar coordinates, we obtain that $\psi$ is an eigenfunction of (10) for some eigenvalue $\mu_{k_{0}}(\lambda)$. By the method of variation of constants and the fact that $\widetilde{w}$ has finite energy near the origin, we then prove that $\varphi(r)$ is proportional to $r^{\gamma}$ and that $\gamma=\frac{2 s-N}{2}+\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}$. We finally complete the proof by showing that $\lim _{r \rightarrow 0} r^{-2 \gamma} H(r)>0$, see Lemma 4.5.

We should mention that in the recent literature, a great attention has been addressed to nonlocal fractional diffusion and many papers have been devoted to the study of existence, nonexistence, regularity and qualitative properties of solutions to elliptic equations associated to fractional Laplace type operators, see e.g. $[2,3,4,7,9,10,24,25]$ and references therein. In particular, semilinear fractional elliptic equations involving the Hardy potential were treated in [9], where some existence and nonexistence results were obtained from lower bounds of positive solutions. Adapting the ideas in [12] in the nonlocal case of the present paper, we provide the behavior (therefore regularity) of solutions at the singularity. In particular, the asymptotics we have here show that the lower bound of positive solutions proved in [9] is sharp.

## 2. Preliminaries and notations

Notation. We list below some notation used throughout the paper.

- $\mathbb{S}^{N}=\left\{z \in \mathbb{R}^{N+1}:|z|=1\right\}$ is the unit $N$-dimensional sphere.
- $\mathbb{S}_{+}^{N}=\left\{\left(\theta_{1}, \theta_{2}, \ldots, \theta_{N+1}\right) \in \mathbb{S}^{N}: \theta_{1}>0\right\}:=\mathbb{S}^{N} \cap \mathbb{R}_{+}^{N+1}$.
- $d S$ denotes the volume element on $N$-dimensional spheres.
- $d S^{\prime}$ denotes the volume element on $(N-1)$-dimensional spheres.

Let $\mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)$ be the completion of $C_{\mathrm{c}}^{\infty}\left(\overline{\mathbb{R}_{+}^{N+1}}\right)$ with respect to the norm

$$
\|w\|_{\mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)}=\left(\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s}|\nabla w(t, x)|^{2} d t d x\right)^{1 / 2}
$$

We recall that there exists a well defined continuous trace map $\operatorname{Tr}: \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right) \rightarrow \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$, see e.g. [2].

For every $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$, let $\mathcal{H}(u) \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)$ be the unique solution to the minimization problem

$$
\begin{align*}
& \int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s}|\nabla \mathcal{H}(u)|^{2} d t d x  \tag{18}\\
&=\min \left\{\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s}|\nabla v|^{2} d t d x: v \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right), \operatorname{Tr}(v)=u\right\}
\end{align*}
$$

By Caffarelli and Silvestre [4] we have that

$$
\begin{equation*}
\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s} \nabla \mathcal{H}(u) \cdot \nabla \widetilde{\varphi} d t d x=\kappa_{s}(u, \operatorname{Tr} \widetilde{\varphi})_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)} \quad \text { for all } \widetilde{\varphi} \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right) \tag{19}
\end{equation*}
$$

where $\kappa_{s}$ is defined in (11).
We notice that combining (9), (18), (19) and (8), we obtain the following Hardy-trace inequality

$$
\begin{equation*}
\kappa_{s} \Lambda_{N, s} \int_{\mathbb{R}^{N}} \frac{(\operatorname{Tr} v)^{2}(x)}{|x|^{2 s}} d x \leqslant \int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s}|\nabla v|^{2} d t d x, \quad \text { for all } v \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right) \tag{20}
\end{equation*}
$$

and also the Sobolev-trace inequality

$$
\begin{equation*}
\kappa_{s} S_{N, s}\|\operatorname{Tr} v\|_{L^{2^{*}(s)}\left(\mathbb{R}^{N}\right)}^{2} \leqslant \int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s}|\nabla v|^{2} d t d x, \quad \text { for all } v \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right) \tag{21}
\end{equation*}
$$

2.1. Separation of variables in the extension operator. For every $R>0$, we define the space $H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ as the completion of $C^{\infty}\left(\overline{B_{R}^{+}}\right)$with respect to the norm

$$
\|w\|_{H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)}=\left(\int_{B_{R}^{+}} t^{1-2 s}\left(|\nabla w(t, x)|^{2}+w^{2}(t, x)\right) d t d x\right)^{1 / 2} .
$$

By direct calculations, we can obtain the following lemma concerning separation of variables in the extension operator.

Lemma 2.1. If $v \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ is such that $v(z)=f(r) \psi(\theta)$ for a.e. $z=(t, x) \in \mathbb{R}_{+}^{N+1}$, with $r=|z|<R$ and $\theta=\frac{z}{|z|} \in \mathbb{S}^{N}$, then

$$
\operatorname{div}\left(t^{1-2 s} \nabla v(z)\right)=\frac{1}{r^{N}}\left(r^{N+1-2 s} f^{\prime}\right)^{\prime} \theta_{1}^{1-2 s} \psi(\theta)+r^{-1-2 s} f(r) \operatorname{div}_{\mathbb{S}^{N}}\left(\theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi(\theta)\right)
$$

in the distributional sense, where $\theta_{1}=\frac{t}{r}=\theta \cdot \mathbf{e}_{1}$ with $\mathbf{e}_{1}=(1,0, \ldots, 0), \operatorname{div}_{\mathbb{S}^{N}} \quad\left(\right.$ respectively $\left.\nabla_{\mathbb{S}^{N}}\right)$ denotes the Riemannian divergence (respectively gradient) on the unit sphere $\mathbb{S}^{N}$ endowed with the standard metric.

Let us define $H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$ as the completion of $C^{\infty}\left(\overline{\mathbb{S}_{+}^{N}}\right)$ with respect to the norm

$$
\|\psi\|_{H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)}=\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left(\left|\nabla_{\mathbb{S}^{N}} \psi(\theta)\right|^{2}+\psi^{2}(\theta)\right) d S\right)^{1 / 2}
$$

We also denote

$$
L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right):=\left\{\psi: \mathbb{S}_{+}^{N} \rightarrow \mathbb{R} \text { measurable such that } \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \psi^{2}(\theta) d S<+\infty\right\}
$$

The following trace inequality on the unit half-sphere $\mathbb{S}_{+}^{N}$ holds.

Lemma 2.2. There exists a well defined continuous trace operator

$$
H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \rightarrow L^{2}\left(\partial \mathbb{S}_{+}^{N}\right)=L^{2}\left(\mathbb{S}^{N-1}\right)
$$

Moreover for every $\psi \in H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$

$$
\kappa_{s} \Lambda_{N, s}\left(\int_{\mathbb{S}^{N}-1}\left|\psi\left(\theta^{\prime}\right)\right|^{2} d S^{\prime}\right) \leqslant\left(\frac{N-2 s}{2}\right)^{2} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}|\psi(\theta)|^{2} d S+\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}^{N}} \psi(\theta)\right|^{2} d S
$$

where $d S^{\prime}$ denotes the volume element on the sphere $\mathbb{S}^{N-1}=\partial \mathbb{S}_{+}^{N}=\left\{\left(\theta_{1}, \theta^{\prime}\right) \in \mathbb{S}_{+}^{N}: \theta_{1}=0\right\}$.
Proof. Let $\psi \in C^{\infty}\left(\overline{\mathbb{S}_{+}^{N}}\right)$ and $f \in C_{\mathrm{c}}^{\infty}(0,+\infty)$ with $f \neq 0$. Rewriting (20) for $v(z)=f(r) \psi(\theta)$, $r=|z|, \theta=\frac{z}{|z|}$, we obtain that

$$
\begin{aligned}
\kappa_{s} \Lambda_{N, s} & \left(\int_{0}^{+\infty} r^{N-1-2 s} f^{2}(r) d r\right)\left(\int_{\mathbb{S}^{N-1}}\left|\psi\left(0, \theta^{\prime}\right)\right|^{2} d S^{\prime}\right) \\
\leqslant & \left(\int_{0}^{+\infty} r^{N+1-2 s}\left|f^{\prime}(r)\right|^{2} d r\right)\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}|\psi(\theta)|^{2} d S\right) \\
& +\left(\int_{0}^{+\infty} r^{N-1-2 s} f^{2}(r) d r\right)\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}_{+}^{N}} \psi(\theta)\right|^{2} d S\right)
\end{aligned}
$$

and hence, by optimality of the classical Hardy constant, see [18, Theorem 330],

$$
\begin{aligned}
\kappa_{s} \Lambda_{N, s} & \left(\int_{\mathbb{S}^{N-1}}\left|\psi\left(0, \theta^{\prime}\right)\right|^{2} d S^{\prime}\right) \\
& \leqslant\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}|\psi(\theta)|^{2} d S\right) \inf _{f \in C_{c}^{\infty}(0,+\infty)} \frac{\int_{0}^{+\infty} r^{N+1-2 s}\left|f^{\prime}(r)\right|^{2} d r}{\int_{0}^{+\infty} r^{N-1-2 s} f^{2}(r) d r}+\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}_{+}^{N}} \psi(\theta)\right|^{2} d S \\
& =\left(\frac{N-2 s}{2}\right)^{2} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}|\psi(\theta)|^{2} d S+\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}_{+}^{N}} \psi(\theta)\right|^{2} d S .
\end{aligned}
$$

By density of $C^{\infty}\left(\overline{\mathbb{S}_{+}^{N}}\right)$ in $H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$, we obtain the conclusion.
In view of Lemma 2.1, in order to construct an orthonormal basis of $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$ for expanding solutions to (17) in Fourier series, we are naturally lead to consider the eigenvalue problem (10), which admits the following variational formulation: we say that $\mu \in \mathbb{R}$ is an eigenvalue of problem (10) if there exists $\psi \in H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \backslash\{0\}$ (called eigenfunction) such that

$$
Q(\psi, v)=\mu \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \psi(\theta) v(\theta) d S, \quad \text { for all } v \in H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)
$$

where

$$
\begin{aligned}
& Q: H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \times H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \rightarrow \mathbb{R} \\
& Q(\psi, v)=\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi(\theta) \cdot \nabla v(\theta) d S-\lambda \kappa_{s} \int_{\mathbb{S}^{N-1}} \mathcal{T} \psi\left(\theta^{\prime}\right) \mathcal{T} v\left(\theta^{\prime}\right) d S^{\prime}
\end{aligned}
$$

By Lemma 2.2 the bilinear form $Q$ is continuous and weakly coercive on $H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$. Moreover the belonging of the weight $t^{1-2 s}$ to the second Muckenhoupt class ensures that the embedding $H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \hookrightarrow \hookrightarrow L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$ is compact (see [8] for weighted embeddings with Muckenhoupt $A_{2}$ weights). Then, from classical spectral theory (see e.g. [23, Theorem 6.16]), problem (10) admits a diverging sequence of real eigenvalues with finite multiplicity $\mu_{1}(\lambda) \leqslant \mu_{2}(\lambda) \leqslant \cdots \leqslant \mu_{k}(\lambda) \leqslant \cdots$ the first of which admits the variational characterization

$$
\begin{equation*}
\mu_{1}(\lambda)=\min _{\psi \in H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \backslash\{0\}} \frac{Q(\psi, \psi)}{\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \psi^{2}(\theta) d S} \tag{22}
\end{equation*}
$$

Furthermore, in view of Lemma 2.2, we have that

$$
\begin{equation*}
\mu_{1}(\lambda)>-\left(\frac{N-2 s}{2}\right)^{2} \tag{23}
\end{equation*}
$$

To each $k \geqslant 1$, we associate an $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$-normalized eigenfunction $\psi_{k} \in H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \backslash\{0\}$ corresponding to the $k$-th eigenvalue $\mu_{k}(\lambda)$, i.e. satisfying

$$
\begin{equation*}
Q\left(\psi_{k}, v\right)=\mu_{k}(\lambda) \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \psi_{k}(\theta) v(\theta) d S, \quad \text { for all } v \in H^{1}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \tag{24}
\end{equation*}
$$

In the enumeration $\mu_{1}(\lambda) \leqslant \mu_{2}(\lambda) \leqslant \cdots \leqslant \mu_{k}(\lambda) \leqslant \cdots$ we repeat each eigenvalue as many times as its multiplicity; thus exactly one eigenfunction $\psi_{k}$ corresponds to each index $k \in \mathbb{N}, k \geqslant 1$. We can choose the functions $\psi_{k}$ in such a way that they form an orthonormal basis of $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$.

We can also determine $\mu_{1}(\lambda)$ for $\lambda \in\left(0, \Lambda_{N, s}\right)$, where $\Lambda_{N, s}$ is the fractional Hardy constant defined in (2).
Proposition 2.3. For every $\alpha \in\left(0, \frac{N-2 s}{2}\right)$, we define

$$
\lambda(\alpha)=2^{2 s} \frac{\Gamma\left(\frac{N+2 s+2 \alpha}{4}\right)}{\Gamma\left(\frac{N-2 s-2 \alpha}{4}\right)} \frac{\Gamma\left(\frac{N+2 s-2 \alpha}{4}\right)}{\Gamma\left(\frac{N-2 s+2 \alpha}{4}\right)} .
$$

Then the mapping $\alpha \mapsto \lambda(\alpha)$ is continuous and decreasing. In addition we have that

$$
\mu_{1}(\lambda(\alpha))=\alpha^{2}-\left(\frac{N-2 s}{2}\right)^{2} \quad \text { for all } \alpha \in\left(0, \frac{N-2 s}{2}\right)
$$

Proof. It was proved in [9, Lemma 3.1] that, for every $\alpha \in\left(0, \frac{N-2 s}{2}\right)$, there exists a positive continuous function $\Phi_{\alpha}: \mathbb{R}_{+}^{N+1} \rightarrow \mathbb{R}$ such that

$$
\left\{\begin{array}{l}
\operatorname{div}\left(t^{1-2 s} \nabla \Phi_{\alpha}\right)=0 \quad \text { in } \mathbb{R}_{+}^{N+1}  \tag{25}\\
\Phi_{\alpha}=|x|^{\frac{2 s-N}{2}+\alpha} \quad \text { on } \mathbb{R}^{N} \backslash\{0\} \\
-t^{1-2 s} \frac{\partial \Phi_{\alpha}}{\partial t}=\kappa_{s} \lambda(\alpha)|x|^{-2 s} \Phi_{\alpha} \quad \text { on } \mathbb{R}^{N} \backslash\{0\}
\end{array}\right.
$$

Moreover $\Phi_{\alpha} \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ for every $R>0$ and $\Phi_{\alpha}$ is scale invariant, i.e.

$$
\Phi_{\alpha}(\tau z)=\tau^{\frac{2 s-N}{2}+\alpha} \Phi_{\alpha}(z), \quad \text { for all } \tau>0
$$

thus implying that, for all $z \in \mathbb{R}_{+}^{N+1}$,

$$
\begin{equation*}
c_{1}|z|^{\frac{2 s-N}{2}+\alpha} \leqslant \Phi_{\alpha}(z) \leqslant c_{2}|z|^{\frac{2 s-N}{2}+\alpha} \tag{26}
\end{equation*}
$$

for some positive constants $c_{1}, c_{2}$. It is also known (see for instance [16]) that the map $\alpha \mapsto \lambda(\alpha)$ is continuous and monotone decreasing.

We write

$$
\Phi_{\alpha}(z)=\sum_{k=1}^{\infty} \Phi_{\alpha}^{k}(|z|) \psi_{k}(z /|z|), \quad \Phi_{\alpha}^{k}(r)=\int_{\mathbb{S}_{+}^{N}} \psi_{k}(\theta) \Phi_{\alpha}(r \theta) d S
$$

In particular, since $\psi_{1}>0$, by (26) we have, for every $r \in(0, R)$,

$$
\begin{equation*}
c_{1}^{\prime}|r|^{\frac{2 s-N}{2}+\alpha} \leqslant \Phi_{\alpha}^{1}(r) \leqslant c_{2}^{\prime}|r|^{\frac{2 s-N}{2}+\alpha} \tag{27}
\end{equation*}
$$

for some positive constants $c_{1}^{\prime}, c_{2}^{\prime}$. Using (25), we have, weakly, for every $k \geqslant 1$ and $r \in(0, R)$,

$$
\left\{\begin{array}{l}
\frac{1}{r^{N}}\left(r^{N+1-2 s}\left(\Phi_{\alpha}^{k}\right)^{\prime}\right)^{\prime} \theta_{1}^{1-2 s} \psi_{k}(\theta)+r^{-1-2 s} \Phi_{\alpha}^{k} \operatorname{div}_{\mathbb{S}^{N}}\left(\theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi_{k}(\theta)\right)=0 \\
-\Phi_{\alpha}^{k} \lim _{\theta_{1} \rightarrow 0^{+}} \theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi_{k}(\theta) \cdot \mathbf{e}_{1}=\kappa_{s} \lambda(\alpha) \psi_{k}\left(0, \theta^{\prime}\right) \Phi_{\alpha}^{k}
\end{array}\right.
$$

Testing the above equation with $\psi_{1}>0$ and using also the fact that $\Phi_{\alpha}^{1}>0$, we obtain

$$
\left(\Phi_{\alpha}^{1}\right)^{\prime \prime}+\frac{N+1-2 s}{r}\left(\Phi_{\alpha}^{1}\right)^{\prime}-\frac{\mu_{1}(\lambda(\alpha))}{r^{2}} \Phi_{\alpha}^{1}=0
$$

and hence $\Phi_{\alpha}^{1}(r)$ is of the form

$$
\Phi_{\alpha}^{1}(r)=c_{3} r^{\sigma^{+}}+c_{4} r^{\sigma^{-}}
$$

for some $c_{3}, c_{4} \in \mathbb{R}$, where

$$
\sigma^{+}=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{1}(\lambda(\alpha))} \quad \text { and } \quad \sigma^{-}=-\frac{N-2 s}{2}-\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{1}(\lambda(\alpha))} .
$$

Since the function $|z|^{\sigma_{k_{0}}^{-}} \psi_{1}\left(\frac{z}{|z|}\right) \notin H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$, we deduce that $c_{4}=0$ and thus for every $r \in(0, R)$

$$
\Phi_{\alpha}^{1}(r)=c_{3} r^{\sigma^{+}}
$$

This together with (27) implies that

$$
\mu_{1}(\lambda(\alpha))=\alpha^{2}-\left(\frac{N-2 s}{2}\right)^{2} \quad \text { for all } \alpha \in\left(0, \frac{N-2 s}{2}\right)
$$

as claimed.
2.2. Hardy type inequalities. From well-known weighted embedding inequalities and the fact that the weight $t^{1-2 s}$ belongs to the second Muckenhoupt class (see e.g. [8]), the embedding $H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right) \hookrightarrow L^{2}\left(B_{r}^{+} ; t^{1-2 s}\right)$ is compact. It can be also proved that both the trace operators

$$
\begin{align*}
& H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right) \hookrightarrow \hookrightarrow L^{2}\left(S_{r}^{+} ; \theta_{1}^{1-2 s}\right)  \tag{28}\\
& H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right) \hookrightarrow \hookrightarrow L^{2}\left(B_{r}^{\prime}\right) \tag{29}
\end{align*}
$$

are well defined and compact.
For sake of simplicity, in the following of this paper, we will often denote the trace of a function with the same letter as the function itself.
The following Hardy type inequality with boundary terms holds.
Lemma 2.4. For all $r>0$ and $w \in H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$, the following inequality holds

$$
\left(\frac{N-2 s}{2}\right)^{2} \int_{B_{r}^{+}} t^{1-2 s} \frac{w^{2}(z)}{|z|^{2}} d z \leqslant \int_{B_{r}^{+}} t^{1-2 s}\left(\nabla w(z) \cdot \frac{z}{|z|}\right)^{2} d z+\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S
$$

Proof. By scaling, it is enough to prove the stated inequality for $r=1$. Let

$$
V(z)=|z|^{\frac{2 s-N}{2}}, \quad z \in \mathbb{R}_{+}^{N+1} \backslash\{0\}
$$

We notice that $V$ satisfies

$$
\begin{equation*}
-\operatorname{div}\left(t^{1-2 s} \nabla V\right)=\left(\frac{N-2 s}{2}\right)^{2} t^{1-2 s}|z|^{-2} V(z) \quad \text { in } \mathbb{R}_{+}^{N+1} \backslash\{0\} \tag{30}
\end{equation*}
$$

Hence, letting $w \in C^{\infty}\left(\overline{B_{1}^{+}}\right)$, multiplying (30) with $\frac{w^{2}}{V}$, and integrating over $B_{1}^{+} \backslash B_{\delta}^{+}$with $\delta \in(0,1)$, we obtain

$$
\begin{aligned}
&\left(\frac{N-2 s}{2}\right)^{2} \int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s} \frac{w^{2}(z)}{|z|^{2}} d z \\
&=\int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s} \nabla V(z) \cdot \nabla\left(\frac{w^{2}}{V}\right)(z) d z-\int_{S_{1}^{+}} t^{1-2 s}(\nabla V \cdot \nu) \frac{w^{2}}{V} d S+\int_{S_{\delta}^{+}} t^{1-2 s}(\nabla V \cdot \nu) \frac{w^{2}}{V} d S \\
&=-(N-2 s) \int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s} \frac{w}{|z|}\left(\nabla w \cdot \frac{z}{|z|}\right) d z-\left(\frac{N-2 s}{2}\right)^{2} \int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s} \frac{w^{2}}{|z|^{2}} d z \\
&+\frac{N-2 s}{2} \int_{S_{1}^{+}} t^{1-2 s} w^{2} d S-\frac{N-2 s}{2} \frac{1}{\delta} \int_{S_{\delta}^{+}} t^{1-2 s} w^{2} d S
\end{aligned}
$$

where $\nu(z)=\frac{z}{|z|}$. Since, by Schwarz's inequality

$$
\begin{align*}
-(N-2 s) \int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s} & \frac{w}{|z|}\left(\nabla w \cdot \frac{z}{|z|}\right) d z  \tag{31}\\
& \leqslant\left(\frac{N-2 s}{2}\right)^{2} \int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s} \frac{w^{2}}{|z|^{2}} d z+\int_{B_{1}^{+} \backslash B_{\delta}^{+}} t^{1-2 s}\left(\nabla w \cdot \frac{z}{|z|}\right)^{2} d z
\end{align*}
$$

and

$$
\frac{1}{\delta} \int_{S_{\delta}^{+}} t^{1-2 s} w^{2} d S=O\left(\delta^{N-2 s}\right)=o(1) \quad \text { for } \delta \rightarrow 0^{+}
$$

letting $\delta \rightarrow 0$ we obtain the stated inequality for $r=1$ and $w \in C^{\infty}\left(\overline{B_{1}^{+}}\right)$. The conclusion follows by density of $C^{\infty}\left(\overline{B_{1}^{+}}\right)$in $H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$.

Lemma 2.5. For every $r>0$ and $w \in H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$, the following inequality holds

$$
\kappa_{s} \Lambda_{N, s} \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x \leqslant\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S+\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z .
$$

Proof. Let $w \in C^{\infty}\left(\overline{B_{r}^{+}}\right)$. Then, passing to polar coordinates and using Lemmas 2.2 and 2.4, we obtain

$$
\begin{aligned}
& \kappa_{s} \Lambda_{N, s} \int_{B_{r}^{\prime}} \frac{w^{2}(0, x)}{|x|^{2 s}} d x=\kappa_{s} \Lambda_{N, s} \int_{0}^{r} \rho^{N-1-2 s}\left(\int_{\mathbb{S}^{N-1}} w^{2}\left(0, \rho \theta^{\prime}\right) d S^{\prime}\right) d \rho \\
& \leqslant \int_{0}^{r} \rho^{N-1-2 s}\left(\left(\frac{N-2 s}{2}\right)^{2} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}|w(\rho \theta)|^{2} d S+\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}^{N}} w(\rho \theta)\right|^{2} d S\right) d \rho \\
&=\left(\frac{N-2 s}{2}\right)^{2} \int_{B_{r}^{+}} t^{1-2 s} \frac{w^{2}}{|z|^{2}} d z+\int_{0}^{r} \rho^{N-1-2 s}\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}^{N}} w(\rho \theta)\right|^{2} d S\right) d \rho \\
& \leqslant\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S \\
& \quad+\int_{0}^{r} \rho^{N+1-2 s}\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left(\frac{1}{\rho^{2}}\left|\nabla_{\mathbb{S}^{N}} w(\rho \theta)\right|^{2}+\left|\frac{\partial w}{\partial \rho}(\rho \theta)\right|^{2}\right) d S\right) d \rho \\
&=\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S+\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z .
\end{aligned}
$$

The conclusion follows by density of $C^{\infty}\left(\overline{B_{r}^{+}}\right)$in $H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$.
The following Sobolev type inequality with boundary terms holds.
Lemma 2.6. There exists $\widetilde{S}_{N, s}>0$ such that, for all $r>0$ and $w \in H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$,

$$
\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*}(s)}} \leqslant \widetilde{S}_{N, S}\left[\frac{N-2 s}{2 r} \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S+\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z\right] .
$$

Proof. By scaling, it is enough to prove the statement for $r=1$. Let $w \in C^{\infty}\left(\overline{B_{1}^{+}}\right)$and denote by $\widetilde{w}$ its fractional Kelvin transform defined as $\widetilde{w}(z)=|z|^{-(N-2 s)} w\left(\frac{z}{|z|^{2}}\right)$. Some computations (see also [10]) show that

$$
\begin{align*}
& \int_{B_{1}^{+}} t^{1-2 s}|\nabla w|^{2} d z+(N-2 s) \int_{S_{1}^{+}} t^{1-2 s} w^{2} d S=\int_{\mathbb{R}_{+}^{N+1} \backslash B_{1}^{+}} t^{1-2 s}|\nabla \widetilde{w}|^{2} d z  \tag{32}\\
& \int_{B_{1}^{\prime}}|w|^{2^{*}(s)} d x=\int_{\mathbb{R}^{N} \backslash B_{1}^{\prime}}|\widetilde{w}|^{2^{*}} d x \tag{33}
\end{align*}
$$

In particular the function

$$
v(z)= \begin{cases}w(z), & \text { if } z \in B_{1}^{+} \\ \widetilde{w}(z), & \text { if } z \in \mathbb{R}_{+}^{N+1} \backslash B_{1}^{+}\end{cases}
$$

belongs to $\mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)$, so that, by (21) we have

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{N}}|v|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*}(s)}} \leqslant \frac{1}{S_{N, s} \kappa_{s}} \int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s}|\nabla v|^{2} d z \tag{34}
\end{equation*}
$$

From (32), (33), and (34), it follows that

$$
\left(2 \int_{B_{1}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2 *^{*}(s)}} \leqslant \frac{2}{S_{N, s} \kappa_{s}}\left[\int_{B_{1}^{+}} t^{1-2 s}|\nabla w|^{2} d z+\frac{N-2 s}{2} \int_{S_{1}^{+}} t^{1-2 s} w^{2} d S\right]
$$

which, by density, yields the conclusion.
Combining Lemma 2.5 and Lemma 2.6 the following corollary follows.

Corollary 2.7. For all $r>0$ and $w \in H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$, the following inequalities hold

$$
\begin{align*}
\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z-\kappa_{s} \lambda \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x+\frac{N-2 s}{2 r} \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S &  \tag{35}\\
& \geqslant \kappa_{s}\left(\Lambda_{N, s}-\lambda\right) \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x
\end{align*}
$$

and

$$
\begin{align*}
\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z-\kappa_{s} \lambda \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x+\frac{N-2 s}{2 r} & \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S  \tag{36}\\
& \geqslant \frac{\Lambda_{N, s}-\lambda}{\left(1+\Lambda_{N, s}\right) \widetilde{S}_{N, s}}\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*}(s)}}
\end{align*}
$$

## 3. The Almgren type frequency function

Let $R>0$ be such that $B_{R}^{\prime} \subset \subset \Omega$ and $w \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ be a nontrivial solution to

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla w\right)=0, & \text { in } B_{R}^{+}  \tag{37}\\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial w}{\partial t}(t, x)=\kappa_{s}\left(\frac{\lambda}{|x|^{2 s}} w+h w+f(x, w)\right), & \text { on } B_{R}^{\prime}\end{cases}
$$

in a weak sense, i.e., for all $\varphi \in C_{c}^{\infty}\left(B_{R}^{+} \cup B_{R}^{\prime}\right)$, we have that

$$
\begin{equation*}
\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s} \nabla w \cdot \nabla \varphi d t d x=\kappa_{s} \int_{B_{R}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w+h w+f(x, w)\right) \varphi d x \tag{38}
\end{equation*}
$$

with $s, \lambda, h, f$ as in assumptions (2), (3), and (4).
For every $r \in(0, R]$ we define

$$
\begin{equation*}
D(r)=\frac{1}{r^{N-2 s}}\left[\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{2}+h w^{2}+f(x, w)\right) d x\right] \tag{39}
\end{equation*}
$$

and

$$
\begin{equation*}
H(r)=\frac{1}{r^{N+1-2 s}} \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S=\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} w^{2}(r \theta) d S \tag{40}
\end{equation*}
$$

The main result of this section is the existence of the limit as $r \rightarrow 0^{+}$of the Almgren's frequency function (see [17] and [1]) associated to $w$

$$
\begin{equation*}
\mathcal{N}(r)=\frac{D(r)}{H(r)}=\frac{r\left[\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{2}+h(x) w^{2}+f(x, w) w\right) d x\right]}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S} \tag{41}
\end{equation*}
$$

We notice that, by Lemma 2.5, w(0, $) \in L^{2}\left(B_{R}^{\prime} ;|x|^{-2 s}\right)$ and so the $L^{1}(0, R)$-function

$$
r \mapsto \int_{S_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d S, \quad \text { respectively } \quad r \mapsto \int_{\partial B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d S^{\prime}
$$

is the weak derivative of the $W^{1,1}(0, R)$-function

$$
r \rightarrow \int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z, \quad \text { respectively } \quad r \mapsto \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x
$$

In particular, for a.e. $r \in(0, R), \frac{\partial w}{\partial \nu} \in L^{2}\left(S_{r}^{+} ; t^{1-2 s}\right)$, where $\nu=\nu(z)=\frac{z}{|z|}$.
Next we observe the following integration by parts.
Lemma 3.1. For a.e. $r \in(0, R)$ and every $\widetilde{\varphi} \in C^{\infty}\left(\overline{B_{r}^{+}}\right)$

$$
\int_{B_{r}^{+}} t^{1-2 s} \nabla w \cdot \nabla \widetilde{\varphi} d z=\int_{S_{r}^{+}} t^{1-2 s} \frac{\partial w}{\partial \nu} \widetilde{\varphi} d S+\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w+h w+f(x, w)\right) \widetilde{\varphi} d x
$$

Proof. It follows by testing (38) with $\widetilde{\varphi}(z) \eta_{n}(|z|)$ where $\eta_{n}(\rho)=1$ if $\rho<r-\frac{1}{n}, \eta_{n}(r)=0$ if $\rho>r, \eta(\rho)=n(r-\rho)$ if $r-\frac{1}{n} \leqslant \rho \leqslant r$, passing to the limit, and noticing that a.e. $r \in(0, R)$ is a Lebesgue point for the $L^{1}(0, R)$ - function $r \mapsto \int_{S_{r}^{+}} t^{1-2 s} \frac{\partial w}{\partial \nu} \widetilde{\varphi} d S$.
3.1. Regularity estimates and Pohozaev-type identity. In this section, we will prove local regularity estimates for a general class of fractional elliptic equations in Lemma 3.3 below. This estimate will be useful for the blow-up analysis and also for establishing the Pohozaev identity in Theorem 3.7 below which is crucial in this paper. The proof of Lemma 3.3 uses mainly a result of Jin, Li and Xiong in [21] that we state here for sake of completeness.
Proposition 3.2 ([21] Proposition 2.4). Let $v \in H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$ be a weak solution to

$$
\left\{\begin{array}{lc}
\operatorname{div}\left(t^{1-2 s} \nabla v\right)=0, & \text { in } B_{1}^{+} \\
-\lim _{t \rightarrow 0^{+}} t^{1-2 s} v_{t}=c(x) v+b(x), & \text { on } B_{1}^{\prime}
\end{array}\right.
$$

with $c, b \in L^{p}\left(B_{1}^{\prime}\right)$ for some $p>\frac{N}{2 s}$. Then $v \in L_{\mathrm{loc}}^{\infty}\left(B_{1}^{+} \cup B_{1}^{\prime}\right)$ and there exists $C>0$ depending only on $N, s, p,\|c\|_{L^{p}\left(B_{1}^{\prime}\right)}$ such that

$$
\|v\|_{L^{\infty}\left([0,1 / 2) \times B_{1 / 2}^{\prime}\right)} \leqslant C\left(\|v\|_{H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)}+\|b\|_{L^{p}\left(B_{1}^{\prime}\right)}\right)
$$

Also $\left.v \in C_{\mathrm{loc}}^{0, \alpha}\left(B_{r}^{+} \cup B_{r}^{\prime}\right)\right)$ for some $\alpha \in(0,1)$ depending only on $N, s, p,\|c\|_{L^{p}\left(B_{1}^{\prime}\right)}$. In addition we have

$$
\|v\|_{C^{0, \alpha}\left([0,1 / 2) \times B_{1 / 2}^{\prime}\right)} \leqslant C\left(\|v\|_{L^{\infty}\left([0,1) \times B_{1}^{\prime}\right)}+\|b\|_{L^{p}\left(B_{1}^{\prime}\right)}\right)
$$

We now state the following technical but crucial result.
Lemma 3.3. (i) Let $r>0$ and $V \in L^{q}\left(B_{r}^{\prime}\right)$ for some $q>\frac{N}{2 s}$. For every $t_{0}, r_{0}>0$ such that $\left[0, t_{0}\right) \times B_{r_{0}}^{\prime} \Subset B_{r}^{+} \cup B_{r}^{\prime}$ there exist positive constants $A_{1}>0, \alpha \in(0,1)$ depending on $t_{0}, r_{0}, r,\|V\|_{L^{q}\left(B_{r}^{\prime}\right)}$ such that for every $v \in H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$ solving

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla v\right)=0, & \text { in } B_{r}^{+} \\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} v_{t}=V(x) v, & \text { on } B_{r}^{\prime}\end{cases}
$$

we have that $v \in C^{0, \alpha}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime}\right)$ and

$$
\begin{equation*}
\|v\|_{C^{0, \alpha}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime}\right)} \leqslant A_{1}\|v\|_{H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)} \tag{42}
\end{equation*}
$$

(ii) Let $v \in H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right) \cap C^{0, \alpha}\left(B_{r}^{+}\right)$for some $\alpha \in(0,1)$ be a function satisfying

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla v\right)=0, & \text { in } B_{r}^{+} \\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} v_{t}=g(x, v), & \text { on } B_{r}^{\prime}\end{cases}
$$

where $g \in C^{1}\left(B_{r}^{\prime} \times \mathbb{R}\right)$ and $|g(x, \rho)| \leqslant c\left(|\rho|+|\rho|^{p-1}\right)$ for some $2<p \leqslant 2^{*}(s)=\frac{2 N}{N-2 s}$, $c>0$, and every $x \in B_{r}^{\prime}$ and $\rho \in \mathbb{R}$. Let $t_{0}, r_{0}>0$ such that $\left[0, t_{0}\right) \times B_{r_{0}}^{\prime} \Subset B_{r}^{+}$. Then there exist positive constants $A_{2}, \beta$ depending only on $N, p, s, c, r, r_{0}, t_{0},\|v\|_{H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)}$ and $\|g\|_{C^{1}\left(B_{r_{0}}^{\prime} \times\left[0, A_{3}\right]\right)}$ where $A_{3}=\|v\|_{C^{0, \alpha}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime}\right)}$, with $\beta \in(0,1)$, such that

$$
\begin{gather*}
\left\|\nabla_{x} v\right\|_{C^{0, \beta}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime}\right)} \leqslant A_{2}  \tag{43}\\
\left\|t^{1-2 s} v_{t}\right\|_{C^{0, \beta}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime}\right)} \leqslant A_{2} \tag{44}
\end{gather*}
$$

Remark 3.4. The dependence of the constant $A_{2}$ in Lemma 3.3 on $\|v\|_{H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)}$ is continuous; in particular we can take the same $A_{2}$ for a family of solutions which are uniformly bounded in $H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right) \cap C^{0, \alpha}\left(B_{r}^{+}\right)$.
Proof. Part (i) and (42) follows from Proposition 3.2.
To prove (ii), for $h \in \mathbb{R}^{N}$ with $|h| \ll 1$, we set $v^{h}(t, x)=\frac{v(t, x+h)-v(t, x)}{|h|}$ for every $x \in B_{r_{0} / 2}^{\prime}$. Then we have

$$
\left\{\begin{array}{lc}
\operatorname{div}\left(t^{1-2 s} \nabla v^{h}\right)=0, & \text { in } B_{r_{0} / 2}^{+} \\
-\lim _{t \rightarrow 0^{+}} t^{1-2 s} v_{t}^{h}=c_{h}(x) v^{h}+b_{h}, & \text { on } B_{r_{0} / 2}^{\prime}
\end{array}\right.
$$

where

$$
c_{h}(x)=\frac{g(x, v(0, x+h))-g(x, v(0, x))}{v(0, x+h)-v(0, x)} \chi_{\{v(0, x+h) \neq v(0, x)\}}(x)
$$

with $\chi_{A}$ being the characteristic function of a set $A$ and

$$
b_{h}(x)=\frac{g(x+h, v(0, x+h))-g(x, v(0, x+h))}{|h|} .
$$

Let $A_{3}=\|v\|_{C^{0, \alpha}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime}\right)}$. Then we have

$$
\left\|c_{h}\right\|_{L^{\infty}\left(B_{r_{0} / 4}^{\prime}\right)}^{\prime}+\left\|b_{h}\right\|_{L^{\infty}\left(B_{r_{0} / 4}^{\prime}\right)}^{\prime} \leqslant\left\|g_{\rho}\right\|_{L^{\infty}\left(B_{r_{0} / 2}^{\prime} \times\left[0, A_{3}\right]\right)}+\left\|\nabla_{x} g\right\|_{L^{\infty}\left(B_{r_{0} / 2}^{\prime} \times\left[0, A_{3}\right]\right)},
$$

for every small $h$. Applying once again Proposition 3.2,

$$
\begin{aligned}
\left\|v^{h}\right\|_{C^{0, \alpha}\left(\left[0, t_{0} / 8\right) \times B_{r_{0} / 8}^{\prime}\right)}^{\prime} & \leqslant C\left(\left\|v^{h}\right\|_{L^{\infty}\left(\left[0, t_{0} / 4\right) \times B_{r_{0} / 4}^{\prime}\right)}+\left\|b_{h}\right\|_{L^{\infty}\left(B_{r_{0} / 2}^{\prime}\right)}\right) \\
& \leqslant C\left\|v^{h}\right\|_{L^{2}\left(\left[0, t_{0} / 2\right) \times B_{r_{0} / 2}^{\prime} ; t^{1-2 s}\right)}+C\|\nabla g\|_{L^{\infty}\left(B_{r_{0} / 2}^{\prime} \times\left[0, A_{3}\right]\right)} \\
& \leqslant C\|\nabla v\|_{L^{2}\left(\left[0, t_{0}\right) \times B_{r_{0}}^{\prime} ; t^{1-2 s}\right)}+C\|\nabla g\|_{L^{\infty}\left(B_{r_{0} / 2}^{\prime} \times\left[0, A_{3}\right]\right)} \\
& \leqslant A_{2}
\end{aligned}
$$

for every small $h$. By the Arzelà-Ascoli Theorem, passing to the limit as $|h| \rightarrow 0$, we conclude that $z \mapsto \nabla_{x} v(z) \in C\left(\left[0, t_{0} / 8\right) \times B_{r_{0} / 8}^{\prime}\right)$ and estimate (43) holds for all $\beta \in(0, \alpha)$.

Since the map $x \mapsto g(x, v(0, x)) \in C^{0, \beta}\left(B_{r_{0} / 8}^{\prime}\right)$, estimate (44) follows from [[3], Lemma 4.5].
Because of the presence of the Hardy potential $|x|^{-2 s}$, we cannot expect the solution $w$ of (37) to be $L_{\text {loc }}^{q}$ near the origin for every $q$, but we can expect $w$ to be in a space better than $L_{\text {loc }}^{2^{*}(s)}$. This is what we will prove in the next result.

Lemma 3.5. Let $w$ be a solution to (37) in the sense of (38). Then there exist $p_{0}>2^{*}(s)$ and $R_{0} \in(0, R)$ such that $w \in L^{p_{0}}\left(B_{R_{0}}^{+}\right)$.

Proof. By (3), there are $\delta>0$ and $R_{\delta} \in(0, R)$ such that

$$
\begin{equation*}
\left(\lambda+|x|^{2 s}|h(x)|\right) \leqslant \lambda+\delta<\Lambda_{N, s}, \quad \text { for all } x \in B_{R_{\delta}}^{\prime} \tag{45}
\end{equation*}
$$

Let $\beta>1$. For all $L>0$, we define $F_{L}(\tau)=|\tau|^{\beta}$ if $|\tau|<L$ and $F_{L}(\tau)=\beta L^{\beta-1}|\tau|+(1-\beta) L^{\beta}$ if $|\tau| \geqslant L$. Put $G_{=} \frac{1}{\beta} F_{L} F_{L}^{\prime}$. It is easy to verify that, for all $\tau \in \mathbb{R}$,

$$
\begin{equation*}
\tau G_{L}(\tau) \leqslant \tau^{2} G_{L}^{\prime}(\tau), \quad \tau G_{L}(\tau) \leqslant\left(F_{L}(\tau)\right)^{2}, \quad\left(F_{L}^{\prime}(\tau)\right)^{2} \leqslant \beta G_{L}^{\prime}(\tau) \tag{46}
\end{equation*}
$$

Let $\eta \in C_{c}^{\infty}\left(B_{R_{\delta}}^{+} \cup B_{R_{\delta}}^{\prime}\right)$ be a radial cut-off function such that $\eta \equiv 1$ in $B_{R_{\delta} / 2}^{+}$. It is clear that $\zeta:=\eta^{2} G_{L}(w) \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ and $F_{L}(w) \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$. Using $\zeta$ as a test in (37), from (45) and integration by parts, we have that

$$
\begin{aligned}
& \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \eta^{2}|\nabla w|^{2} G_{L}^{\prime}(w) d t d x-\kappa_{s}(\lambda+\delta) \int_{B_{R_{\delta}}^{\prime}}|x|^{-2 s} \eta^{2} w G_{L}(w) d x \\
& \leqslant-2 \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \eta \nabla w \cdot \nabla \eta G_{L}(w) d t d x \\
&+c \int_{B_{R_{\delta}}^{\prime}} \eta^{2} w G_{L}(w) d x+c \int_{B_{R_{\delta}}^{\prime}} \eta^{2}|w|^{2^{*}(s)-2} w G_{L}(w) d x
\end{aligned}
$$

for some positive $c>0$ depending only on $C_{f}, s, p, N$. By Young's inequality and (46), we have that, for every $\sigma>0$,

$$
\left|2(\eta \nabla w) \cdot(w \nabla \eta) \frac{G_{L}(w)}{w}\right| \leqslant \frac{\sigma}{2} \eta^{2}|\nabla w|^{2} G_{L}^{\prime}(w)+\frac{2}{\sigma}|\nabla \eta|^{2}\left(F_{L}(w)\right)^{2},
$$

hence we obtain that

$$
\begin{aligned}
&\left(1-\frac{\sigma}{2}\right) \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \eta^{2}|\nabla w|^{2} G_{L}^{\prime}(w) d t d x-\kappa_{s}(\lambda+\delta) \int_{B_{R_{\delta}}^{\prime}}|x|^{-2 s} \eta^{2} w G_{L}(w) d x \\
& \leqslant c \int_{B_{R_{\delta}}^{\prime}} \eta^{2}\left(|w|^{2^{*}(s)-2}+1\right) w G_{L}(w) d x+\frac{2}{\sigma} \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla \eta|^{2} \psi^{2} d t d x
\end{aligned}
$$

where we have set $\psi=F_{L}(w)$. Therefore by (46)

$$
\begin{aligned}
\frac{1}{\beta}\left(1-\frac{\sigma}{2}\right) \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \eta^{2}|\nabla \psi|^{2} d t & d x-\kappa_{s}(\lambda+\delta) \int_{B_{R_{\delta}}^{\prime}}|x|^{-2 s}(\eta \psi)^{2} d x \\
& \leqslant c \int_{B_{R_{\delta}}^{\prime}}\left(|w|^{2^{*}(s)-2}+1\right)(\eta \psi)^{2} d x+\frac{2}{\sigma} \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla \eta|^{2} \psi^{2} d t d x
\end{aligned}
$$

Since $|\nabla(\eta \psi)|^{2} \leqslant(1+\sigma) \eta^{2}|\nabla \psi|^{2}+\left(1+\frac{1}{\sigma}\right) \psi^{2}|\nabla \eta|^{2}$, we have that

$$
\begin{aligned}
& \frac{1}{\beta(1+\sigma)}\left(1-\frac{\sigma}{2}\right) \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla(\eta \psi)|^{2} d t d x-\kappa_{s}(\lambda+\delta) \int_{B_{R_{\delta}}^{\prime}}|x|^{-2 s}(\eta \psi)^{2} d x \\
& \leqslant c \int_{B_{R_{\delta}}^{\prime}}\left(|w|^{2^{*}(s)-2}+1\right)(\eta \psi)^{2} d x+C\left(c, \beta, \sigma, R_{\delta}\right) \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \psi^{2} d t d x
\end{aligned}
$$

for some positive $C\left(c, \beta, \sigma, R_{\delta}\right)>0$ depending only on $c, \sigma, R_{\delta}$, and $\beta$. By Hölder inequality and Lemma 2.6, we have that

$$
\begin{aligned}
& \int_{B_{R_{\delta}}^{\prime}}\left(|w|^{2^{*}(s)-2}+1\right)(\eta \psi)^{2} d x \\
& \qquad \leqslant \widetilde{S}_{N, s}\left[\left(\int_{B_{R_{\delta}}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2^{*}(s)-2}{2^{*}(s)}}+\left|B_{R_{\delta}}^{\prime}\right|^{\frac{2 s}{N}}\right] \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla(\eta \psi)|^{2} d t d x .
\end{aligned}
$$

We deduce that

$$
\begin{equation*}
A \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla(\eta \psi)|^{2} d t d x-\kappa_{s}(\lambda+\delta) \int_{B_{R_{\delta}}^{\prime}}|x|^{-2 s}(\eta \psi)^{2} d x \leqslant \mathrm{const} \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \psi^{2} d t d x \tag{47}
\end{equation*}
$$

for some positive const $>0$ depending only on $C_{f}, s, p, N, \beta, \sigma, R_{\delta}$, where

$$
A=\frac{1}{(1+\sigma) \beta}\left(1-\frac{\sigma}{2}\right)-c \widetilde{S}_{N, s}\left[\left(\int_{B_{R_{\delta}}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2^{*}(s)-2}{2^{*(s)}}}+\left|B_{R_{\delta}}^{\prime}\right|^{\frac{2 s}{N}}\right]
$$

From Hardy inequality (20), we have that

$$
\begin{aligned}
& A \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla(\eta \psi)|^{2} d t d x-\kappa_{s}(\lambda+\delta) \int_{B_{R_{\delta}}^{\prime}}|x|^{-2 s}(\eta \psi)^{2} d x \\
& \geqslant\left(A-\frac{\lambda+\delta}{\Lambda_{N, s}}\right) \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla(\eta \psi)|^{2} d t d x
\end{aligned}
$$

and, by (45), we can choose $\beta$ sufficiently close to 1 and $\sigma, R_{\delta}$ sufficiently small such that

$$
A-\frac{\lambda+\delta}{\Lambda_{N, s}}>0
$$

Hence we have that

$$
C \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|\nabla(\eta \psi)|^{2} d t d x \leqslant \int_{B_{R_{\delta}}^{+}} t^{1-2 s} \psi^{2} d t d x
$$

for some constant $C>0$ depending on $f, h, s, p, N, \beta, \varepsilon, w, \lambda, \delta$. From Lemma 2.6 it follows that

$$
C \widetilde{S}_{N, s}^{-1}\left(\int_{B_{R_{\delta}}^{\prime}}\left|\eta F_{L}(w)\right|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*(s)}}} \leqslant \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|w|^{2 \beta} d t d x \quad \text { for all } L \geqslant 0
$$

Hence by taking the limit as $L \rightarrow+\infty$

$$
\begin{equation*}
C \widetilde{S}_{N, s}^{-1}\left(\int_{B_{R_{\delta} / 2}^{\prime}}|w|^{\beta 2^{*}(s)} d x\right)^{\frac{2}{2^{*(s)}}} \leqslant \int_{B_{R_{\delta}}^{+}} t^{1-2 s}|w|^{2 \beta} d t d x \tag{48}
\end{equation*}
$$

The conclusion follows since $\beta>1$ and $H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right) \hookrightarrow L^{q}\left(B_{R}^{+} ; t^{1-2 s}\right)$ for some $q>2$, see for instance [[8], Theorem 1.2].

Remark 3.6. From Lemma 3.5, we deduce that, if $w \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ is a weak solution to (37), then $\frac{\lambda}{|x|^{2 s}}+h+\frac{f(x, w)}{w} \in L_{\text {loc }}^{q}\left(B_{R}^{\prime} \backslash\{0\}\right)$ for some $\frac{N}{2 s}<q \leqslant \frac{p_{0}}{p-2}$ and hence, from Lemma 3.3, we conclude that $w \in C_{\mathrm{loc}}^{0, \alpha}\left(\overline{B_{r}^{+}} \backslash\{0\}\right), \nabla_{x} v \in C_{\mathrm{loc}}^{0, \beta}\left(\overline{B_{r}^{+}} \backslash\{0\}\right)$, and $t^{1-2 s} v_{t} \in C_{\mathrm{loc}}^{0, \beta}\left(\overline{B_{r}^{+}} \backslash\{0\}\right)$ for all $r \in(0, R)$ and some $0<\beta<\alpha<1$.
We next prove the following Pohozaev-type identity which will be used to compute $D^{\prime}$ (see Lemma 3.9 below) and therefore $\mathcal{N}^{\prime}$.

Theorem 3.7. Let $w$ solves (38). Then for a.e. $r \in(0, R)$ there holds

$$
\begin{align*}
& -\frac{N-2 s}{2}\left[\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z-\kappa_{s} \lambda \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x\right]  \tag{49}\\
& \quad+\frac{r}{2}\left[\int_{S_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d S-\kappa_{s} \lambda \int_{\partial B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d S^{\prime}\right] \\
& =r \int_{S_{r}^{+}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S-\frac{\kappa_{s}}{2} \int_{B_{r}^{\prime}}(N h+\nabla h \cdot x) w^{2} d x+\frac{r \kappa_{s}}{2} \int_{\partial B_{r}^{\prime}} h w^{2} d S^{\prime} \\
& \\
& \quad+r \kappa_{s} \int_{\partial B_{r}^{\prime}} F(x, w) d S^{\prime}-\kappa_{s} \int_{B_{r}^{\prime}}\left[\nabla_{x} F(x, w) \cdot x+N F(x, w)\right] d x
\end{align*}
$$

and

$$
\begin{align*}
\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d z-\kappa_{s} \lambda \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x &  \tag{50}\\
& =\int_{S_{r}^{+}} t^{1-2 s} \frac{\partial w}{\partial \nu} w d S+\kappa_{s} \int_{B_{r}^{\prime}}\left(h w^{2}+f(x, w) w\right) d x
\end{align*}
$$

Proof. We write our problem in the form

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla w\right)=0, & \text { in } B_{R}^{+} \\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} w_{t}=G(x, w), & \text { on } B_{R}^{\prime}\end{cases}
$$

where $G \in C^{1}\left(B_{R}^{\prime} \backslash\{0\} \times \mathbb{R}\right), G(x, \varrho)=\kappa_{s}\left(\frac{\lambda}{|x|^{2 s}} \varrho+h(x) \varrho+f(x, \varrho)\right)$.
We have, on $B_{R}^{+}$, the formula
(51) $\operatorname{div}\left(\frac{1}{2} t^{1-2 s}|\nabla w|^{2} z-t^{1-2 s}(z \cdot \nabla w) \nabla w\right)=\frac{N-2 s}{2} t^{1-2 s}|\nabla w|^{2}-(z \cdot \nabla w) \operatorname{div}\left(t^{1-2 s} \nabla w\right)$.

Let $\rho<r<R$. Now we integrate by parts over the set $O_{\delta}:=\left(B_{r}^{+} \backslash \overline{B_{\rho}^{+}}\right) \cap\{(t, x), t>\delta\}$ with $\delta>0$. We have

$$
\begin{aligned}
& \frac{N-2 s}{2} \int_{O_{\delta}} t^{1-2 s}|\nabla w(z)|^{2} d z=-\frac{1}{2} \delta^{2-2 s} \int_{\sqrt[B^{r^{2}-\delta^{2}}]{ } \backslash B_{\sqrt{\rho^{2}-\delta^{2}}}}|\nabla w|^{2}(\delta, x) d x \\
& \quad+\delta^{2-2 s} \int_{B_{\sqrt{r^{2}-\delta^{2}}} \backslash B_{\sqrt{\rho^{2}-\delta^{2}}}}\left|w_{t}\right|^{2}(\delta, x) d x \\
& \quad+\frac{r}{2} \int_{S_{r}^{+} \cap\{t>\delta\}} t^{1-2 s}|\nabla w|^{2} d S-r \int_{S_{r}^{+} \cap\{t>\delta\}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S \\
& \quad-\frac{\rho}{2} \int_{S_{\rho}^{+} \cap\{t>\delta\}} t^{1-2 s}|\nabla w|^{2} d S+\rho \int_{S_{\rho}^{+} \cap\{t>\delta\}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S \\
& \quad+\int_{B_{\sqrt{r^{2}-\delta^{2}}} \backslash B_{\sqrt{\rho^{2}-\delta^{2}}}^{\prime}}\left(x \cdot \nabla_{x} w(\delta, x)\right) \delta^{1-2 s} w_{t}(\delta, x) d x
\end{aligned}
$$

We now claim that there exists a sequence $\delta_{n} \rightarrow 0$ such that

$$
\lim _{n \rightarrow \infty}\left[\frac{1}{2} \delta_{n}^{2-2 s} \int_{B_{r}^{\prime}}|\nabla w|^{2}\left(\delta_{n}, x\right) d x+\delta_{n}^{2-2 s} \int_{B_{r}^{\prime}}\left|w_{t}\right|^{2}\left(\delta_{n}, x\right) d x\right]=0
$$

If no such sequence exists, we would have

$$
\liminf _{\delta \rightarrow 0}\left[\frac{1}{2} \delta^{2-2 s} \int_{B_{r}^{\prime}}|\nabla w|^{2}(\delta, x) d x+\delta^{2-2 s} \int_{B_{r}^{\prime}}\left|w_{t}\right|^{2}(\delta, x) d x\right] \geqslant C>0
$$

and thus there exists $\delta_{0}>0$ such that

$$
\frac{1}{2} \delta^{2-2 s} \int_{B_{r}^{\prime}}|\nabla w|^{2}(\delta, x) d x+\delta^{2-2 s} \int_{B_{r}^{\prime}}\left|w_{t}\right|^{2}(\delta, x) d x \geqslant \frac{C}{2} \quad \text { for all } \delta \in\left(0, \delta_{0}\right)
$$

It follows that

$$
\frac{1}{2} \delta^{1-2 s} \int_{B_{r}^{\prime}}|\nabla w|^{2}(\delta, x) d x+\delta^{1-2 s} \int_{B_{r}^{\prime}}\left|w_{t}\right|^{2}(\delta, x) d x \geqslant \frac{C}{2 \delta} \quad \text { for all } \delta \in\left(0, \delta_{0}\right)
$$

and so integrating the above inequality on $\left(0, \delta_{0}\right)$ we contradict the fact that $w \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$.
Next, from the Dominated Convergence Theorem, Lemma 3.3 and Remark 3.6, we have that

$$
\lim _{\delta \rightarrow 0} \int_{B_{\sqrt{r^{2}-\delta^{2}}} \backslash B_{\sqrt{\rho^{2}-\delta^{2}}}}\left(x \cdot \nabla_{x} w(\delta, x)\right) \delta^{1-2 s} w_{t}(\delta, x) d x=-\int_{B_{r}^{\prime} \backslash B_{\rho}^{\prime}}\left(x \cdot \nabla_{x} w\right) G(x, w) d x
$$

We conclude that (replacing $O_{\delta}$ with $O_{\delta_{n}}$, for a sequence $\delta_{n} \rightarrow 0$ ) that

$$
\begin{align*}
& \frac{N-2 s}{2} \int_{B_{r}^{+} \backslash B_{\rho}^{+}} t^{1-2 s}|\nabla w(z)|^{2} d z=\frac{r}{2} \int_{S_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d S-r \int_{S_{r}^{+}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S  \tag{52}\\
& \quad-\frac{\rho}{2} \int_{S_{\rho}^{+}} t^{1-2 s}|\nabla w|^{2} d S-\rho \int_{S_{\rho}^{+}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S-\int_{B_{r}^{\prime} \backslash B_{\rho}^{\prime}}\left(x \cdot \nabla_{x} w\right) G(x, w) d x
\end{align*}
$$

Furthermore, integration by parts yields

$$
\begin{align*}
\int_{B_{r}^{\prime} \backslash B_{\rho}^{\prime}} & \left(x \cdot \nabla_{x} w\right) G(x, w) d x=-\frac{N-2 s}{2} \kappa_{s} \lambda \int_{B_{r}^{\prime} \backslash B_{\rho}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x  \tag{53}\\
& -\frac{\kappa_{s}}{2} \int_{B_{r}^{\prime} \backslash B_{\rho}^{\prime}}(N h(x)+\nabla h(x) \cdot x) w^{2} d x+\kappa_{s} \lambda \frac{r}{2} \int_{\partial B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d S^{\prime} \\
& +\frac{r \kappa_{s}}{2} \int_{\partial B_{r}^{\prime}} h(x) w^{2} d S^{\prime}-\kappa_{s} \lambda \frac{\rho}{2} \int_{\partial B_{\rho}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d S^{\prime}-\frac{\rho \kappa_{s}}{2} \int_{\partial B_{\rho}^{\prime}} h(x) w^{2} d S^{\prime} \\
& -\kappa_{s} \int_{B_{r}^{\prime} \backslash B_{\rho}^{\prime}}\left[\nabla_{x} F(x, w) \cdot x+N F(x, w)\right] d x \\
& +r \kappa_{s} \int_{\partial B_{r}^{\prime}} F(x, w) d S^{\prime}-\rho \kappa_{s} \int_{\partial B_{\rho}^{\prime}} F(x, w) d S^{\prime}
\end{align*}
$$

Since $w \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$, in view of Lemma 2.5 and (8), there exists a sequence $\rho_{n} \rightarrow 0$ such that

$$
\lim _{n \rightarrow \infty} \rho_{n}\left[\int_{S_{\rho_{n}}^{+}} t^{1-2 s}|\nabla w|^{2} d S+\int_{\partial B_{\rho_{n}}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d S^{\prime}+\int_{\partial B_{\rho_{n}}^{\prime}}|F(x, w)| d S^{\prime}\right]=0
$$

Hence, taking $\rho=\rho_{n}$ and letting $n \rightarrow \infty$ in (52) and (53), we obtain (49).
(50) follows from Lemma 3.1 and density of $C^{\infty}\left(\overline{B_{r}^{+}}\right)$in $H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$.
3.2. On the Almgren type frequency $\mathcal{N}$. In this section, we shall study the differentiability of $\mathcal{N}$, it's limit at 0 and provide estimates of $\mathcal{N}^{\prime}$.

Lemma 3.8. $H \in C^{1}(0, R)$ and

$$
\begin{align*}
H^{\prime}(r) & =\frac{2}{r^{N+1-2 s}} \int_{S_{r}^{+}} t^{1-2 s} w \frac{\partial w}{\partial \nu} d S, \quad \text { for every } r \in(0, R),  \tag{54}\\
H^{\prime}(r) & =\frac{2}{r} D(r), \quad \text { for every } r \in(0, R) . \tag{55}
\end{align*}
$$

Proof. Fix $r_{0} \in(0, R)$ and consider the limit

$$
\begin{equation*}
\lim _{r \rightarrow r_{0}} \frac{H(r)-H\left(r_{0}\right)}{r-r_{0}}=\lim _{r \rightarrow r_{0}} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \frac{|w(r \theta)|^{2}-\left|w\left(r_{0} \theta\right)\right|^{2}}{r-r_{0}} d S \tag{56}
\end{equation*}
$$

Since $w \in C^{1}\left(\mathbb{R}_{+}^{N+1}\right)$, then, for every $\theta \in \mathbb{S}_{+}^{N}$,

$$
\begin{equation*}
\lim _{r \rightarrow r_{0}} \frac{|w(r \theta)|^{2}-\left|w\left(r_{0} \theta\right)\right|^{2}}{r-r_{0}}=2 \frac{\partial w}{\partial \nu}\left(r_{0} \theta\right) w\left(r_{0} \theta\right) \tag{57}
\end{equation*}
$$

On the other hand, for any $r \in\left(r_{0} / 2, R\right)$ and $\theta \in \mathbb{S}_{+}^{N}$ we have
$\left|\frac{|w(r \theta)|^{2}-\left|w\left(r_{0} \theta\right)\right|^{2}}{r-r_{0}}\right| \leqslant 2 \sup _{B_{R}^{+} \backslash B_{\frac{r_{0}}{2}}^{+}}|w| \cdot \sup _{B_{R}^{+} \backslash B_{\frac{r_{0}}{2}}^{+}}\left|\frac{\partial w}{\partial \nu}\right| \leqslant 2 \sup _{B_{R}^{+} \backslash B_{\frac{r_{0}}{2}}^{+}}|w| \cdot \sup _{B_{R}^{+} \backslash B_{\frac{r_{0}}{2}}^{+}}\left(\left|\frac{t}{|z|} w_{t}\right|+\left|\frac{\nabla_{x} w \cdot x}{|z|}\right|\right)$
and hence, by (56), (57), Lemma 3.3 and the Dominated Convergence Theorem, we obtain that

$$
H^{\prime}\left(r_{0}\right)=2 \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \frac{\partial w}{\partial \nu}\left(r_{0} \theta\right) w\left(r_{0} \theta\right) d S(\theta)=\frac{2}{r_{0}^{N+1-2 s}} \int_{S_{r_{0}}^{+}} t^{1-2 s} w \frac{\partial w}{\partial \nu} d S
$$

The continuity of $H^{\prime}$ on the interval $(0, R)$ follows by the representation of $H^{\prime}$ given above, Lemma 3.3 , and the Dominated Convergence Theorem.

Finally, (55) follows from (54), (39), and (50).
The regularity of the function $D$ is established in the following lemma.
Lemma 3.9. The function $D$ defined in (39) belongs to $W_{\text {loc }}^{1,1}(0, R)$ and

$$
\begin{align*}
D^{\prime}(r)= & \frac{2}{r^{N+1-2 s}}\left[r \int_{S_{r}^{+}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S-\kappa_{s} \int_{B_{r}^{\prime}}\left(s h+\frac{1}{2}(\nabla h \cdot x)\right) w^{2} d x\right]  \tag{58}\\
& +\frac{\kappa_{s}}{r^{N+1-2 s}} \int_{B_{r}^{\prime}}\left((N-2 s) f(x, w) w-2 N F(x, w)-2 \nabla_{x} F(x, w) \cdot x\right) d x \\
& +\frac{\kappa_{s}}{r^{N-2 s}} \int_{\partial B_{r}^{\prime}}(2 F(x, w)-f(x, w) w) d S^{\prime}
\end{align*}
$$

in a distributional sense and for a.e. $r \in(0, R)$.
Proof. For any $r \in\left(0, r_{0}\right)$ let

$$
\begin{equation*}
I(r)=\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{2}+h w^{2}+f(x, w) w\right) d x \tag{59}
\end{equation*}
$$

From the fact that $w \in H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$, Lemma 2.5, and (8), we deduce that $I \in W^{1,1}(0, R)$ and

$$
\begin{equation*}
I^{\prime}(r)=\int_{S_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d S-\kappa_{s} \int_{\partial B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{2}+h w^{2}+f(x, w) w\right) d S^{\prime} \tag{60}
\end{equation*}
$$

for a.e. $r \in(0, R)$ and in the distributional sense. Therefore $D \in W_{\mathrm{loc}}^{1,1}(0, R)$ and, using (49), (59), and (60) into

$$
D^{\prime}(r)=r^{2 s-1-N}\left[-(N-2 s) I(r)+r I^{\prime}(r)\right]
$$

we obtain (58) for a.e. $r \in(0, R)$ and in the distributional sense.
Before going on, we recall that $w$ is nontrivial and satisfies (38). We prove now that, if $w \not \equiv 0$, $H(r)$ does not vanish for $r$ sufficiently small.
Lemma 3.10. There exists $R_{0} \in(0, R)$ such that $H(r)>0$ for any $r \in\left(0, R_{0}\right)$, where $H$ is defined by (40).
Proof. Clearly from assumption (2), there exists $R_{0} \in(0, R)$ such that

$$
\begin{equation*}
\frac{\lambda}{\Lambda_{N, s}}+\frac{C_{h} R_{0}^{\varepsilon}}{\Lambda_{N, s}}+C_{f} S_{N, s}^{-1}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2^{*}(s)-p}{2^{*}(s)}} R_{0}^{\frac{N\left(2^{*}(s)-p\right)}{2^{*}(s)}}\|w\|_{L^{2^{*}(s)}\left(B_{R_{0}}^{\prime}\right)}^{p-2}<1 \tag{61}
\end{equation*}
$$

where $\omega_{N-1}$ denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1}=\int_{\mathbb{S}^{N-1}} d S$.

Next suppose by contradiction that there exists $r_{0} \in\left(0, R_{0}\right)$ such that $H\left(r_{0}\right)=0$. Then $w=0$ a.e. on $S_{r_{0}}^{+}$. From (50) it follows that

$$
\int_{B_{r_{0}}^{+}} t^{1-2 s}|\nabla w|^{2} d z-\kappa_{s} \lambda \int_{B_{r_{0}}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x-\kappa_{s} \int_{B_{r_{0}}^{\prime}}\left(h(x) w^{2}+f(x, w) w\right) d x=0 .
$$

From Lemma 2.5, assumptions (3)-(4), Hölder's inequality, and (8), it follows that

$$
\begin{aligned}
0 & =\int_{B_{r_{0}}^{+}} t^{1-2 s}|\nabla w|^{2} d z-\kappa_{s} \lambda \int_{B_{r_{0}}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x-\kappa_{s} \int_{B_{r_{0}}^{\prime}}\left(h(x) w^{2}+f(x, w) w\right) d x \\
& \geqslant\left[1-\frac{\lambda}{\Lambda_{N, s}}-\frac{C_{h} R_{0}^{\varepsilon}}{\Lambda_{N, s}}-C_{f} S_{N, s}^{-1}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2^{*}(s)-p}{2^{*(s)}}} R_{0}^{\frac{N\left(2^{*}(s)-p\right)}{2^{*(s)}}}\|w\|_{L^{*}(s)\left(B_{R_{0}}^{\prime}\right)}^{p-2}\right] \int_{B_{r_{0}}^{+}} t^{1-2 s}|\nabla w|^{2} d z,
\end{aligned}
$$

which, together with (61), implies $w \equiv 0$ in $B_{r_{0}}^{+}$by Lemma 2.4. Classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [26]) allow to conclude that $w=0$ a.e. in $B_{R}^{+}$, a contradiction.
Letting $R_{0}$ be as in Lemma 3.10 and recalling (41), the Almgren type frequency function

$$
\begin{equation*}
\mathcal{N}(r)=\frac{D(r)}{H(r)} \tag{62}
\end{equation*}
$$

is well defined in $\left(0, R_{0}\right)$. Using Lemmas 3.8 and 3.9, we can now compute the derivative of $\mathcal{N}$.
Lemma 3.11. The function $\mathcal{N}$ defined in (62) belongs to $W_{\text {loc }}^{1,1}\left(0, R_{0}\right)$ and

$$
\begin{equation*}
\mathcal{N}^{\prime}(r)=\nu_{1}(r)+\nu_{2}(r) \tag{63}
\end{equation*}
$$

in a distributional sense and for a.e. $r \in\left(0, R_{0}\right)$, where

$$
\begin{equation*}
\nu_{1}(r)=\frac{2 r\left[\left(\int_{S_{r}^{+}} t^{1-2 s}\left|\frac{\partial w}{\partial \nu}\right|^{2} d S\right) \cdot\left(\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S\right)-\left(\int_{S_{r}^{+}} t^{1-2 s} w \frac{\partial w}{\partial \nu} d S\right)^{2}\right]}{\left(\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S\right)^{2}}, \tag{64}
\end{equation*}
$$

$\nu_{1} \geqslant 0$, and

$$
\begin{align*}
\nu_{2}(r)= & -\kappa_{s} \frac{\int_{B_{r}^{\prime}}(2 s h+\nabla h \cdot x)|w|^{2} d x}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S}+\kappa_{s} \frac{r \int_{\partial B_{r}^{\prime}}(2 F(x, w)-f(x, w) w) d S^{\prime}}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S}  \tag{65}\\
& +\kappa_{s} \frac{\int_{B_{r}^{\prime}}\left((N-2 s) f(x, w) w-2 N F(x, w)-2 \nabla_{x} F(x, w) \cdot x\right) d x}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S} .
\end{align*}
$$

Proof. From Lemmas 3.8, 3.10, and 3.9, it follows that $\mathcal{N} \in W_{\text {loc }}^{1,1}\left(0, R_{0}\right)$. From (55) it follows that

$$
\mathcal{N}^{\prime}(r)=\frac{D^{\prime}(r) H(r)-D(r) H^{\prime}(r)}{(H(r))^{2}}=\frac{D^{\prime}(r) H(r)-\frac{1}{2} r\left(H^{\prime}(r)\right)^{2}}{(H(r))^{2}}
$$

and the proof of the lemma easily follows from (54) and (58). Now it is easy to see that $\nu_{1} \geqslant 0$ by Schwarz's inequality.

We now prove that $\mathcal{N}(r)$ admits a finite limit as $r \rightarrow 0^{+}$. To this aim, the following estimate plays a crucial role.

Lemma 3.12. Let $\mathcal{N}$ be the function defined in (62). There exist $\tilde{R} \in\left(0, R_{0}\right)$ and a constant $\bar{C}>0$ such that
(66) $\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{2}+h w^{2}+f(x, w) w\right) d x$

$$
\geqslant-\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S+\bar{C}\left(\int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x+\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2 *(s)}}\right)
$$

$$
\begin{align*}
\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}} & \left(\frac{\lambda}{|x|^{2 s}} w^{2}+h w^{2}+f(x, w) w\right) d x  \tag{67}\\
& \geqslant-\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S+\bar{C} \int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x
\end{align*}
$$

and

$$
\begin{equation*}
\mathcal{N}(r)>-\frac{N-2 s}{2} \tag{68}
\end{equation*}
$$

for every $r \in(0, \tilde{R})$.
Proof. From Corollary 2.7, (3), and (4), it follows that

$$
\begin{array}{rl}
\int_{B_{r}^{+}} t^{1-2 s}|\nabla w|^{2} & d t d x-\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{2}+h w^{2}+f(x, w) w\right) d x+\left(\frac{N-2 s}{2 r}\right) \int_{S_{r}^{+}} t^{1-2 s} w^{2} d S \\
\geqslant & \left(\frac{\kappa_{s}\left(\Lambda_{N, s}-\lambda\right)}{2}-C_{h} \kappa_{s} r^{\varepsilon}\right) \int_{B_{r}^{\prime}} \frac{w^{2}}{|x|^{2 s}} d x \\
& +\left(\frac{\Lambda_{N, s}-\lambda}{2\left(1+\Lambda_{N, s}\right) \widetilde{S}_{N, s}}-C_{f} \kappa_{s}\left|B_{r}^{\prime}\right|^{\frac{2^{*}(s)-p}{2^{*(s)}}}\|w\|_{L^{2 *(s)}\left(B_{r}^{\prime}\right)}^{p-2}\right)\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*}(s)}}
\end{array}
$$

for every $r \in\left(0, R_{0}\right)$. Since $\lambda<\Lambda_{N, s}$, from the above estimate it follows that we can choose $\underline{\tilde{R}} \in\left(0, R_{0}\right)$ sufficiently small such that estimate (66) holds for $r \in(0, \tilde{R})$ for some positive constant $\bar{C}>0$. The proof of (67) can be performed in a similar way, using Lemmas 2.5 and 2.6. Estimate (66), together with (39) and (40), yields (68).

Lemma 3.13. Let $\tilde{R}$ be as in Lemma 3.12 and $\nu_{2}$ as in (65). Then there exist a positive constant $C_{1}>0$ and a function $g \in L^{1}(0, \tilde{R}), g \geqslant 0$ a.e. in $(0, \tilde{R})$, such that

$$
\left|\nu_{2}(r)\right| \leqslant C_{1}\left[\mathcal{N}(r)+\frac{N-2 s}{2}\right]\left(r^{-1+\varepsilon}+r^{-1+\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}+g(r)\right)
$$

for a.e. $r \in(0, \tilde{R})$ and

$$
\int_{0}^{r} g(\rho) d \rho \leqslant \frac{1}{1-\alpha}\|w\|_{L^{p}\left(B_{\tilde{R}}^{\prime}\right)}^{p(1-\alpha)} r^{N\left(\frac{\alpha 2^{*}(s)-2}{2^{*}(s)}-\frac{p \alpha-2}{p_{0}}\right)}
$$

for all $r \in(0, \tilde{R})$ with some $\alpha$ satisfying $\frac{2}{p}<\alpha<1$.
Proof. From (3) and (66) we deduce that

$$
\begin{aligned}
\left.\left|\int_{B_{r}^{\prime}}(2 \operatorname{sh}(x)+\nabla h(x) \cdot x)\right| w\right|^{2} d x \mid & \leqslant 2 C_{h} r^{\varepsilon} \int_{B_{r}^{\prime}} \frac{|w|^{2}}{|x|^{2 s}} d x \\
& \leqslant 2 C_{h} \bar{C}^{-1} r^{\varepsilon+N-2 s}\left[D(r)+\frac{N-2 s}{2} H(r)\right]
\end{aligned}
$$

and, therefore, for any $r \in(0, \tilde{R})$, we have that

$$
\begin{align*}
\left|\frac{\int_{B_{r}^{\prime}}(2 s h(x)+\nabla h(x) \cdot x)|w|^{2} d x}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S}\right| & \leqslant 2 C_{h} \bar{C}^{-1} r^{-1+\varepsilon} \frac{D(r)+\frac{N-2 s}{2} H(r)}{H(r)}  \tag{69}\\
& =2 C_{h} \bar{C}^{-1} r^{-1+\varepsilon}\left[\mathcal{N}(r)+\frac{N-2 s}{2}\right]
\end{align*}
$$

By (4), Hölder's inequality, and (66), for some constant const $=$ const $\left(N, s, C_{f}\right)>0$ depending on $N, s, C_{f}$, and for all $r \in(0, \tilde{R})$, there holds

$$
\begin{aligned}
& \left|\int_{B_{r}^{\prime}}\left((N-2 s) f(x, w) w-2 N F(x, w)-2 \nabla_{x} F(x, w) \cdot x\right) d x\right| \\
& \leqslant \mathrm{const} \int_{B_{r}^{\prime}}\left(|w|^{2}+|w|^{2^{*}(s)}\right) d x \\
& \leqslant \mathrm{const}\left(\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2 s}{N}} r^{2 s}+\|w\|_{L^{2^{*}(s)\left(B_{\tilde{R}}^{\prime}\right)}}^{2^{*}(s)-2}\right)\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*}(s)}} \\
& \leqslant \frac{\mathrm{const}}{\bar{C}}\left(\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2 s}{N}} r^{2 s}+\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2 s\left(p_{0}-2^{*}(s)\right)}{N p_{0}}} r^{\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}\|w\|_{L^{p_{0}}\left(B_{\tilde{R}}^{\prime}\right)}^{2^{*}(s)-2}\right) r^{N-2 s}\left[D(r)+\frac{N-2 s}{2} H(r)\right]
\end{aligned}
$$

and hence
(70) $\left|\frac{\int_{B_{r}^{\prime}}\left((N-2 s) f(x, w) w-2 N F(x, w)-2 \nabla_{x} F(x, w) \cdot x\right) d x}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S}\right|$

$$
\leqslant \frac{\mathrm{const}}{\bar{C}}\left(\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2 s}{N}} r^{\frac{2 s 2^{*}(s)}{p_{0}}}+\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2 s\left(p_{0}-2^{*}(s)\right)}{N p_{0}}}\|w\|_{L^{p_{0}}\left(B_{\tilde{R}}^{\prime}\right)}^{2^{*}(s)-2}\right) r^{-1+\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}\left[\mathcal{N}(r)+\frac{N-2 s}{2}\right]
$$

Let us fix $\frac{2}{p}<\alpha<1$. By Hölder's inequality and (66),

$$
\begin{align*}
& \left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{\alpha}=\left(\int_{B_{r}^{\prime}}|w|^{p-\frac{2}{\alpha}}|w|^{\frac{2}{\alpha}} d x\right)^{\alpha}  \tag{71}\\
& \leqslant\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s) \frac{p \alpha-2}{2^{*}(s) \alpha-2}} d x\right)^{\frac{\alpha 2^{*}(s)-2}{2^{*}(s)}}\left(\int_{B_{r}^{\prime}}|w|^{2^{*}(s)} d x\right)^{\frac{2}{2^{*}(s)}} \\
& \leqslant\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\alpha 2^{*}(s)-2}{2^{*}(s)}-\frac{p \alpha-2}{p_{0}}} r^{N\left(\frac{\alpha 2^{*}(s)-2}{2^{*}(s)}-\frac{p \alpha-2}{p_{0}}\right)}\|w\|_{L^{p o}\left(B_{\tilde{R}}^{\prime}\right)}^{p \alpha-2} \frac{r^{N-2 s}}{\bar{C}}\left[D(r)+\frac{N-2 s}{2} H(r)\right] \\
& =\bar{C}^{-1}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\beta}{N}} r^{-1+\beta}\left[\mathcal{N}(r)+\frac{N-2 s}{2}\right]\left(\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S\right)
\end{align*}
$$

for all $r \in(0, \tilde{R})$, where $\beta=N\left(\frac{\alpha 2^{*}(s)-2}{2^{*}(s)}-\frac{p \alpha-2}{p_{0}}\right)>0$. From (4), (71), and (68), there exists some const $=\operatorname{const}\left(N, s, C_{f}\right)>0$ depending on $N, s, C_{f}$ such that, for all $r \in(0, \tilde{R})$,
(72) $\left|\frac{r \int_{\partial B_{r}^{\prime}}(2 F(x, w)-f(x, w) w) d S^{\prime}}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S}\right| \leqslant$ const $\frac{r \int_{\partial B_{r}^{\prime}}|w|^{p} d S^{\prime}}{\int_{S_{r}^{+}} t^{1-2 s} w^{2} d S}$

$$
\leqslant \frac{\text { const }}{\bar{C}}\left(\frac{\omega_{N-1}}{N}\right)^{\frac{\beta}{N}}\left[\mathcal{N}(r)+\frac{N-2 s}{2}\right] \frac{r^{\beta} \int_{\partial B_{r}^{\prime}}|w|^{p} d S^{\prime}}{\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{\alpha}}
$$

By a direct calculation, we have that

$$
\begin{equation*}
\frac{r^{\beta} \int_{\partial B_{r}^{\prime}}|w|^{p} d S^{\prime}}{\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{\alpha}}=\frac{1}{1-\alpha}\left[\frac{d}{d r}\left(r^{\beta}\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{1-\alpha}\right)-\beta r^{-1+\beta}\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{1-\alpha}\right] \tag{73}
\end{equation*}
$$

in the distributional sense and for a.e. $r \in(0, \tilde{R})$. Since

$$
\lim _{r \rightarrow 0^{+}} r^{\beta}\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{1-\alpha}=0
$$

we deduce that the function

$$
r \mapsto \frac{d}{d r}\left(r^{\beta}\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{1-\alpha}\right)
$$

is integrable over $(0, \tilde{R})$. Being

$$
r^{-1+\beta}\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{1-\alpha}=o\left(r^{-1+\beta}\right)
$$

as $r \rightarrow 0^{+}$, we have that also the function

$$
r \mapsto r^{-1+\beta}\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{\frac{p-2}{p}}
$$

is integrable over $(0, \tilde{R})$. Therefore, by (73), we deduce that

$$
\begin{equation*}
g(r):=\frac{r^{\beta} \int_{\partial B_{r}^{\prime}}|w|^{p} d S^{\prime}}{\left(\int_{B_{r}^{\prime}}|w|^{p} d x\right)^{\alpha}} \in L^{1}(0, \tilde{R}) \tag{74}
\end{equation*}
$$

and

$$
\begin{equation*}
0 \leqslant \int_{0}^{r} g(\rho) d \rho \leqslant \frac{1}{1-\alpha}\|w\|_{L^{p}\left(B_{r}^{\prime}\right)}^{p(1-\alpha)} r^{\beta} \tag{75}
\end{equation*}
$$

for all $r \in(0, \tilde{R})$. Collecting (69), (70), (72), (74), and (75), we obtain the stated estimate.
Lemma 3.14. Let $\tilde{R}$ be as in Lemma 3.12 and $\mathcal{N}$ as in (62). Then there exist a positive constant $C_{2}>0$ such that

$$
\begin{equation*}
\mathcal{N}(r) \leqslant C_{2} \tag{76}
\end{equation*}
$$

for all $r \in(0, \tilde{R})$.
Proof. By Lemma 3.11 and Lemma 3.13, we obtain

$$
\begin{equation*}
\left(\mathcal{N}+\frac{N-2 s}{2}\right)^{\prime}(r) \geqslant \nu_{2}(r) \geqslant-C_{1}\left[\mathcal{N}(r)+\frac{N-2 s}{2}\right]\left(r^{-1+\varepsilon}+r^{-1+\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}+g(r)\right) \tag{77}
\end{equation*}
$$

for a.e. $r \in(0, \tilde{R})$. Integration over $(r, \tilde{R})$ yields

$$
\mathcal{N}(r) \leqslant-\frac{N-2 s}{2}+\left(\mathcal{N}(\tilde{R})+\frac{N-2 s}{2}\right) \exp \left(C_{1}\left(\frac{\tilde{R}^{\varepsilon}}{\varepsilon}+\frac{p_{0} \tilde{R}^{\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}}{2 s\left(p_{0}-2^{*}(s)\right)}+\int_{0}^{\tilde{R}} g(\rho) d \rho\right)\right)
$$

for any $r \in(0, \tilde{R})$, thus proving estimate (76).
Lemma 3.15. The limit

$$
\gamma:=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)
$$

exists and is finite.
Proof. By Lemmas 3.13 and 3.14, the function $\nu_{2}$ defined in (65) belongs to $L^{1}(0, \tilde{R})$. Hence, by Lemma 3.11, $\mathcal{N}^{\prime}$ is the sum of a nonnegative function and of a $L^{1}$-function on $(0, \tilde{R})$. Therefore

$$
\mathcal{N}(r)=\mathcal{N}(\tilde{R})-\int_{r}^{\tilde{R}} \mathcal{N}^{\prime}(\rho) d \rho
$$

admits a limit as $r \rightarrow 0^{+}$which is necessarily finite in view of (68) and (76).
The function $H$ defined in (40) can be estimated as follows.
Lemma 3.16. Let $\gamma:=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)$ be as in Lemma 3.15 and $\tilde{R}$ as in Lemma 3.12. Then there exists a constant $K_{1}>0$ such that

$$
\begin{equation*}
H(r) \leqslant K_{1} r^{2 \gamma} \quad \text { for all } r \in(0, \tilde{R}) \tag{78}
\end{equation*}
$$

Moreover, for any $\sigma>0$ there exists a constant $K_{2}(\sigma)>0$ depending on $\sigma$ such that

$$
\begin{equation*}
H(r) \geqslant K_{2}(\sigma) r^{2 \gamma+\sigma} \quad \text { for all } r \in(0, \tilde{R}) \tag{79}
\end{equation*}
$$

Proof. By Lemma $3.15, \mathcal{N}^{\prime} \in L^{1}(0, \tilde{r})$ and, by Lemma $3.14, \mathcal{N}$ is bounded, then from (77) and (75) it follows that

$$
\begin{equation*}
\mathcal{N}(r)-\gamma=\int_{0}^{r} \mathcal{N}^{\prime}(\rho) d \rho \geqslant-C_{3} r^{\delta} \tag{80}
\end{equation*}
$$

for some constant $C_{3}>0$ and all $r \in(0, \tilde{R})$, where

$$
\begin{equation*}
\delta=\min \left\{\varepsilon, \frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}, N\left(\frac{\alpha 2^{*}(s)-2}{2^{*}(s)}-\frac{p \alpha-2}{p_{0}}\right)\right\}>0 \tag{81}
\end{equation*}
$$

Therefore by (55), (62), and (80) we deduce that, for all $r \in(0, \tilde{R})$,

$$
\frac{H^{\prime}(r)}{H(r)}=\frac{2 \mathcal{N}(r)}{r} \geqslant \frac{2 \gamma}{r}-2 C_{3} r^{-1+\delta}
$$

which, after integration over the interval $(r, \tilde{R})$, yields (78).
Since $\gamma=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)$, for any $\sigma>0$ there exists $r_{\sigma}>0$ such that $\mathcal{N}(r)<\gamma+\sigma / 2$ for any $r \in\left(0, r_{\sigma}\right)$ and hence

$$
\frac{H^{\prime}(r)}{H(r)}=\frac{2 \mathcal{N}(r)}{r}<\frac{2 \gamma+\sigma}{r} \quad \text { for all } r \in\left(0, r_{\sigma}\right)
$$

Integrating over the interval $\left(r, r_{\sigma}\right)$ and by continuity of $H$ outside 0 , we obtain (79) for some constant $K_{2}(\sigma)$ depending on $\sigma$.

## 4. The blow-up argument

The main result of this section, which also contains Theorem 1.1, is the following theorem.
Theorem 4.1. Let $w$ satisfy (38), with $s, \lambda, h, f$ as in assumptions (2), (3), and (4). Then, letting $\mathcal{N}(r)$ as in (41), there there exists $k_{0} \in \mathbb{N}, k_{0} \geqslant 1$, such that

$$
\begin{equation*}
\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k_{0}}(\lambda)} \tag{82}
\end{equation*}
$$

Furthermore, if $\gamma$ denotes the limit in (82), $m \geqslant 1$ is the multiplicity of the eigenvalue $\mu_{j_{0}}(\lambda)=$ $\mu_{j_{0}+1}(\lambda)=\cdots=\mu_{j_{0}+m-1}(\lambda)$ and $\left\{\psi_{i}\right\}_{i=j_{0}}^{j_{0}+m-1}\left(j_{0} \leqslant k_{0} \leqslant j_{0}+m-1\right)$ is an $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$ orthonormal basis for the eigenspace of problem (10) associated to $\mu_{k_{0}}(\lambda)$, then

$$
\begin{aligned}
& \tau^{-\gamma} w(0, \tau x) \rightarrow|x|^{\gamma} \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i} \psi_{i}\left(0, \frac{x}{|x|}\right) \quad \text { in } C_{\mathrm{loc}}^{1, \alpha}\left(B_{1}^{\prime} \backslash\{0\}\right) \quad \text { as } \tau \rightarrow 0^{+} \\
& \tau^{-\gamma} w(\tau \theta) \rightarrow \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i} \psi_{i}(\theta) \quad \text { in } C^{0, \alpha}\left(\mathbb{S}_{+}^{N}\right) \quad \text { as } \tau \rightarrow 0^{+} \\
& \tau^{-\gamma} w\left(0, \tau \theta^{\prime}\right) \rightarrow \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i} \psi_{i}\left(0, \theta^{\prime}\right) \quad \text { in } C^{1, \alpha}\left(\mathbb{S}^{N-1}\right) \quad \text { as } \tau \rightarrow 0^{+}
\end{aligned}
$$

and

$$
\tau^{1-\gamma} \nabla_{x} w\left(0, \tau \theta^{\prime}\right) \rightarrow \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i}\left(\gamma \psi_{i}\left(0, \theta^{\prime}\right) \theta^{\prime}+\nabla_{\mathbb{S}^{N-1}} \psi_{i}(0, \cdot)\left(\theta^{\prime}\right)\right) \quad \text { in } C^{0, \alpha}\left(\mathbb{S}^{N-1}\right) \quad \text { as } \tau \rightarrow 0^{+}
$$

for some $\alpha \in(0,1)$, where

$$
\begin{aligned}
\beta_{i}= & R^{-\gamma} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} w(R \theta) \psi_{i}(\theta) d S(\theta) \\
& +\kappa_{s} \int_{\mathbb{S}^{N-1}}\left[\int_{0}^{R} \frac{h\left(t \theta^{\prime}\right) w\left(0, t \theta^{\prime}\right)+f\left(t \theta^{\prime}, w\left(0, t \theta^{\prime}\right)\right)}{2 \gamma+N-2 s}\left(t^{2 s-\gamma-1}-\frac{t^{\gamma+N-1}}{R^{2 \gamma+N-2 s}}\right) d t\right] \psi_{i}\left(0, \theta^{\prime}\right) d S\left(\theta^{\prime}\right)
\end{aligned}
$$

for all $R>0$ such that $\overline{B_{R}^{\prime}}=\left\{x \in \mathbb{R}^{N}:|x| \leqslant R\right\} \subset \Omega$ and $\left(\beta_{j_{0}}, \beta_{j_{0}+1}, \ldots, \beta_{j_{0}+m-1}\right) \neq(0,0, \ldots, 0)$.

To prove Theorem 4.1, we start by determining the asymptotic profile of blowing up renormalized solutions to (37).

Lemma 4.2. Let $w$ as in Theorem 4.1. Let $\gamma:=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)$ as in Lemma 3.15. Then
(i) there exists $k_{0} \in \mathbb{N}, k_{0} \geqslant 1$, such that $\gamma=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}$;
(ii) for every sequence $\tau_{n} \rightarrow 0^{+}$, there exist a subsequence $\left\{\tau_{n_{k}}\right\}_{k \in \mathbb{N}}$ and an eigenfunction $\psi$ of problem (10) associated to the eigenvalue $\mu_{k_{0}}(\lambda)$ such that $\|\psi\|_{L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)}=1$ and

$$
\frac{w\left(\tau_{n_{k}} z\right)}{\sqrt{H\left(\tau_{n_{k}}\right)}} \rightarrow|z|^{\gamma} \psi\left(\frac{z}{|z|}\right)
$$

strongly in $H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)$ and in $C_{\mathrm{loc}}^{0, \alpha}\left(\overline{B_{r}^{+}} \backslash\{0\}\right)$ for some $\alpha \in(0,1)$ and all $r \in(0,1)$ and

$$
\frac{w\left(0, \tau_{n_{k}} x\right)}{\sqrt{H\left(\tau_{n_{k}}\right)}} \rightarrow|x|^{\gamma} \psi\left(0, \frac{x}{|x|}\right)
$$

in $C_{\text {loc }}^{1, \alpha}\left(B_{1}^{\prime} \backslash\{0\}\right)$.
Proof. Let us set

$$
\begin{equation*}
w^{\tau}(z)=\frac{w(\tau z)}{\sqrt{H(\tau)}} \tag{83}
\end{equation*}
$$

We notice that $\int_{S_{1}^{+}} t^{1-2 s}\left|w^{\tau}\right|^{2} d S=1$. Moreover, by scaling and (76),

$$
\begin{align*}
& \int_{B_{1}^{+}} t^{1-2 s}\left|\nabla w^{\tau}(z)\right|^{2} d z-\kappa_{s} \int_{B_{1}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}}\left|w^{\tau}\right|^{2}+\tau^{2 s} h(\tau x)\left|w^{\tau}\right|^{2}\right.  \tag{84}\\
&\left.+\frac{\tau^{2 s}}{\sqrt{H(\tau)}} f\left(\tau x, \sqrt{H(\tau)} w^{\tau}\right) w^{\tau}\right) d x=\mathcal{N}(\tau) \leqslant C_{2}
\end{align*}
$$

for every $\tau \in(0, \tilde{R})$, whereas, from (67),

$$
\begin{align*}
\mathcal{N}(\tau) \geqslant \frac{\tau^{-N+2 s}}{H(\tau)}\left(-\left(\frac{N-2 s}{2 \tau}\right) \int_{S_{\tau}^{+}} t^{1-2 s} w^{2} d S+\right. & \left.\bar{C} \int_{B_{\tau}^{+}} t^{1-2 s}|\nabla w|^{2} d t d x\right)  \tag{85}\\
& =-\frac{N-2 s}{2}+\bar{C} \int_{B_{1}^{+}} t^{1-2 s}\left|\nabla w^{\tau}(z)\right|^{2} d z
\end{align*}
$$

for every $\tau \in(0, \tilde{R})$. From (84) and (85) we deduce that

$$
\begin{equation*}
\left\{w^{\tau}\right\}_{\tau \in(0, \tilde{R})} \quad \text { is bounded in } H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right) \tag{86}
\end{equation*}
$$

Therefore, for any given sequence $\tau_{n} \rightarrow 0^{+}$, there exists a subsequence $\tau_{n_{k}} \rightarrow 0^{+}$such that $w^{\tau_{n_{k}}} \rightharpoonup \widetilde{w}$ weakly in $H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$ for some $\widetilde{w} \in H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$. Due to compactness of the trace embedding (28), we obtain that $\int_{S_{1}^{+}} t^{1-2 s}|\widetilde{w}|^{2} d S=1$. In particular $\widetilde{w} \not \equiv 0$.

For every small $\tau \in(0, \tilde{R})$, $w^{\tau}$ satisfies

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla w^{\tau}\right)=0  \tag{87}\\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial w^{\tau}}{\partial t}=\kappa_{s}\left(\frac{\lambda}{|x|^{2 s}} w^{\tau}+\tau^{2 s} h(\tau x) w^{\tau}+\frac{\tau^{2 s}}{\sqrt{H(\tau)}} f\left(\tau x, \sqrt{H(\tau)} w^{\tau}\right)\right), & \text { in } B_{1}^{+}\end{cases}
$$

in a weak sense, i.e.

$$
\begin{align*}
\int_{B_{1}^{+}} t^{1-2 s} \nabla w^{\tau} & \cdot \nabla \widetilde{\varphi} d t d x  \tag{88}\\
& =\kappa_{s} \int_{B_{1}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}} w^{\tau}+\tau^{2 s} h(\tau x) w^{\tau}+\frac{\tau^{2 s}}{\sqrt{H(\tau)}} f\left(\tau x, \sqrt{H(\tau)} w^{\tau}\right)\right) \widetilde{\varphi}(0, x) d x
\end{align*}
$$

for all $\widetilde{\varphi} \in H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$ s.t. $\widetilde{\varphi}=0$ on $\mathbb{S}_{+}^{N}$ and, for such $\widetilde{\varphi}$, by (4) and Hölder's inequality,

$$
\begin{align*}
& \left.\frac{\tau^{2 s}}{\sqrt{H(\tau)}} \right\rvert\, \int_{B_{1}^{\prime}} f(\tau x,  \tag{89}\\
& \left.\quad \leqslant C_{f} \tau^{2 s} \int_{B_{1}^{\prime}}|w(0, \tau x)|^{\tau}(0, x)\right) \widetilde{\varphi}(0, x) d x \mid \\
& \leqslant C_{f}\|\operatorname{Tr} \widetilde{\varphi}\|_{L^{p}\left(B_{1}^{\prime}\right)}\left\|w^{\tau}\right\|_{L^{p}\left(B_{1}^{\prime}\right)}\|w\|_{L^{p}\left(B_{\tau}^{\prime}\right)}^{p-2} \tau^{\frac{(N-2 s)\left(2^{*}(s)-p\right)}{p}}=o(1) \quad \text { as } \tau \rightarrow 0^{+}
\end{align*}
$$

and, by (3) and Lemma 2.5,

$$
\begin{aligned}
& \text { (90) } \quad \tau^{2 s}\left|\int_{B_{1}^{\prime}} h(\tau x) w^{\tau} \widetilde{\varphi}(0, x) d x\right| \\
& \leqslant \frac{C_{h} \tau^{\varepsilon}}{\kappa_{s} \Lambda_{N, s}}\left(\int_{B_{1}^{+}} t^{1-2 s}\left|\nabla w^{\tau}(z)\right|^{2} d z+\frac{N-2 s}{2}\right)^{1 / 2}\left(\int_{B_{1}^{+}} t^{1-2 s}|\nabla \widetilde{\varphi}(z)|^{2} d z\right)^{1 / 2}=o(1) \text { as } \tau \rightarrow 0^{+} .
\end{aligned}
$$

From (89), (90), and weak convergence $w^{\tau_{n_{k}}} \rightharpoonup \widetilde{w}$ in $H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$, we can pass to the limit in (87) along the sequence $\tau_{n_{k}}$ and obtain that $\widetilde{w}$ weakly solves

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla \widetilde{w}\right)=0, & \text { in } B_{1}^{+},  \tag{91}\\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial \widetilde{w}}{\partial t}=\kappa_{s} \frac{\lambda}{\mid x^{2 s}} \widetilde{w}, & \text { on } B_{1}^{\prime} .\end{cases}
$$

From (4), letting $q=\frac{p_{0}}{p-2}>\frac{N}{2 s}$ with $p_{0}$ as in Lemma 3.5, we have that

$$
\left.\left\|\frac{\tau^{2 s}}{\sqrt{H(\tau)}} \frac{f\left(\tau x, \sqrt{H(\tau)} w^{\tau}(x)\right)}{w^{\tau}(x)}\right\|_{L^{q}\left(B_{2}^{\prime}\right)} \leqslant C_{f} \tau^{2 s-\frac{N}{q}}\left(\int_{B_{2 \tau}}|w(x)|^{p_{0}} d x\right)^{1 / q}\right)=O(1)
$$

as $\tau \rightarrow 0^{+}$. Therefore from Lemma 3.3 part (i) there holds

$$
\begin{equation*}
w^{\tau_{n_{k}}} \rightarrow \widetilde{w} \quad \text { in } C_{\mathrm{loc}}^{0, \alpha}\left(\overline{B_{r}^{+}} \backslash\{0\}\right), \tag{92}
\end{equation*}
$$

while Lemma 3.3 part (ii) and Remark 3.4 imply that

$$
\begin{equation*}
\nabla_{x} w^{\tau_{n_{k}}} \rightarrow \nabla_{x} \widetilde{w}, \quad \text { and } \quad t^{1-2 s} \frac{\partial w^{\tau_{n_{k}}}}{\partial t} \rightarrow t^{1-2 s} \frac{\partial \widetilde{w}}{\partial t} \quad \text { in } C_{\mathrm{loc}}^{0, \alpha} \overline{\left.B_{r}^{+} \backslash\{0\}\right)} \tag{93}
\end{equation*}
$$

for some $\alpha \in(0,1)$ and all $r \in(0,1)$. Reasoning as in (89), (90), we can prove that

$$
\begin{equation*}
\frac{\tau^{2 s}}{\sqrt{H(\tau)}} \int_{B_{1}^{\prime}} f\left(\tau x, \sqrt{H(\tau)} w^{\tau}\right) w^{\tau} d x=o(1) \quad \text { as } \tau \rightarrow 0^{+} \tag{94}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau^{2 s} \int_{B_{1}^{\prime}} h(\tau x)\left|w^{\tau}\right|^{2} d x=o(1) \text { as } \tau \rightarrow 0^{+} \tag{95}
\end{equation*}
$$

Multiplying equation (87) with $w^{\tau}$, integrating in $B_{r}^{+}$, and using (93), (94), (95), we easily obtain that $\left\|w^{\tau_{n_{k}}}\right\|_{H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)} \rightarrow\|\widetilde{w}\|_{H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right)}$ for all $r \in(0,1)$, and hence

$$
\begin{equation*}
w^{\tau_{n_{k}}} \rightarrow \widetilde{w} \quad \text { in } H^{1}\left(B_{r}^{+} ; t^{1-2 s}\right) \tag{96}
\end{equation*}
$$

for any $r \in(0,1)$.
For any $r \in(0,1)$ and $k \in \mathbb{N}$, let us define the functions

$$
\begin{gathered}
D_{k}(r)=\frac{1}{r^{N-2 s}}\left[\int_{B_{r}^{+}} t^{1-2 s}\left|\nabla w^{\tau_{n_{k}}}\right|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}}\left(\frac{\lambda}{|x|^{2 s}}\left|w^{\tau_{n_{k}}}\right|^{2}+\tau_{n_{k}}^{2 s} h\left(\tau_{n_{k}} x\right)\left|w^{\tau_{n_{k}}}\right|^{2}\right.\right. \\
\left.\left.\quad+\frac{\tau_{n_{k}}^{2 s}}{\sqrt{H\left(\tau_{n_{k}}\right)}} f\left(\tau_{n_{k}} x, \sqrt{H\left(\tau_{n_{k}}\right)} w^{\tau_{n_{k}}}\right) w^{\tau_{n_{k}}}\right) d x\right]
\end{gathered}
$$

and

$$
H_{k}(r)=\frac{1}{r^{N+1-2 s}} \int_{S_{r}^{+}} t^{1-2 s}\left|w^{\tau_{n_{k}}}\right|^{2} d S
$$

Direct calculations yield

$$
\begin{equation*}
\mathcal{N}_{k}(r):=\frac{D_{k}(r)}{H_{k}(r)}=\frac{D\left(\tau_{n_{k}} r\right)}{H\left(\tau_{n_{k}} r\right)}=\mathcal{N}\left(\tau_{n_{k}} r\right) \quad \text { for all } r \in(0,1) \tag{97}
\end{equation*}
$$

From (96), (94), and (95), it follows that, for any fixed $r \in(0,1)$,

$$
\begin{equation*}
D_{k}(r) \rightarrow \widetilde{D}(r) \tag{98}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{D}(r)=\frac{1}{r^{N-2 s}}\left[\int_{B_{r}^{+}} t^{1-2 s}|\nabla \widetilde{w}|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}} \frac{\lambda}{|x|^{2 s}} \widetilde{w}^{2} d x\right] \quad \text { for all } r \in(0,1) \tag{99}
\end{equation*}
$$

On the other hand, by compactness of the trace embedding (28), we also have

$$
\begin{equation*}
H_{k}(r) \rightarrow \widetilde{H}(r) \quad \text { for any fixed } r \in(0,1) \tag{100}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{H}(r)=\frac{1}{r^{N+1-2 s}} \int_{S_{r}^{+}} t^{1-2 s} \widetilde{w}^{2} d S \tag{101}
\end{equation*}
$$

From (35) it follows that $\widetilde{D}(r)>-\frac{N-2 s}{2} \widetilde{H}(r)$ for all $r \in(0,1)$. Therefore, if, for some $r \in(0,1)$, $\widetilde{H}(r)=0$ then $\widetilde{D}(r)>0$, and passing to the limit in (97) should give a contradiction with Lemma 3.15. Hence $H_{w}(r)>0$ for all $r \in(0,1)$ and the function

$$
\widetilde{\mathcal{N}}(r):=\frac{\widetilde{D}(r)}{\widetilde{H}(r)}
$$

is well defined for $r \in(0,1)$. From (97), (98), (100), and Lemma 3.15, we deduce that

$$
\begin{equation*}
\widetilde{\mathcal{N}}(r)=\lim _{k \rightarrow \infty} \mathcal{N}\left(\tau_{n_{k}} r\right)=\gamma \tag{102}
\end{equation*}
$$

for all $r \in(0,1)$. Therefore $\widetilde{\mathcal{N}}$ is constant in $(0,1)$ and hence $\widetilde{\mathcal{N}}^{\prime}(r)=0$ for any $r \in(0,1)$. By (91) and Lemma 3.11 with $h \equiv 0$ and $f \equiv 0$, we obtain

$$
\left(\int_{S_{r}^{+}} t^{1-2 s}\left|\frac{\partial \widetilde{w}}{\partial \nu}\right|^{2} d S\right) \cdot\left(\int_{S_{r}^{+}} t^{1-2 s} \widetilde{w}^{2} d S\right)-\left(\int_{S_{r}^{+}} t^{1-2 s} \widetilde{w} \frac{\partial \widetilde{w}}{\partial \nu} d S\right)^{2}=0 \quad \text { for all } r \in(0,1)
$$

which implies that $\widetilde{w}$ and $\frac{\partial \widetilde{w}}{\partial \nu}$ have the same direction as vectors in $L^{2}\left(S_{r}^{+} ; t^{1-2 s}\right)$ and hence there exists a function $\eta=\eta(r)$ such that $\frac{\partial \widetilde{w}}{\partial \nu}(r, \theta)=\eta(r) \widetilde{w}(r, \theta)$ for all $r \in(0,1)$ and $\theta \in \mathbb{S}_{+}^{N}$. After integration we obtain

$$
\begin{equation*}
\widetilde{w}(r, \theta)=e^{\int_{1}^{r} \eta(s) d s} \widetilde{w}(1, \theta)=\varphi(r) \psi(\theta), \quad r \in(0,1), \theta \in \mathbb{S}_{+}^{N} \tag{103}
\end{equation*}
$$

where $\varphi(r)=e^{\int_{1}^{r} \eta(s) d s}$ and $\psi(\theta)=\widetilde{w}(1, \theta)$. From (91), (103), and Lemma 2.1, it follows that, weakly,

$$
\left\{\begin{array}{l}
\frac{1}{r^{N}}\left(r^{N+1-2 s} \varphi^{\prime}\right)^{\prime} \theta_{1}^{1-2 s} \psi(\theta)+r^{-1-2 s} \varphi(r) \operatorname{div}_{\mathbb{S}^{N}}\left(\theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi(\theta)\right)=0 \\
-\lim _{\theta_{1} \rightarrow 0^{+}} \theta_{1}^{1-2 s} \nabla_{\mathbb{S}^{N}} \psi(\theta) \cdot \mathbf{e}_{1}=\kappa_{s} \lambda \psi\left(0, \theta^{\prime}\right)
\end{array}\right.
$$

Taking $r$ fixed we deduce that $\psi$ is an eigenfunction of the eigenvalue problem (10). If $\mu_{k_{0}}(\lambda)$ is the corresponding eigenvalue then $\varphi(r)$ solves the equation

$$
\frac{1}{r^{N}}\left(r^{N+1-2 s} \varphi^{\prime}\right)^{\prime}-\mu_{k_{0}}(\lambda) r^{-1-2 s} \varphi(r)=0
$$

i.e.

$$
\varphi^{\prime \prime}(r)+\frac{N+1-2 s}{r} \varphi^{\prime}-\frac{\mu_{k_{0}}(\lambda)}{r^{2}} \varphi(r)=0
$$

and hence $\varphi(r)$ is of the form

$$
\varphi(r)=c_{1} r^{\sigma_{k_{0}}^{+}}+c_{2} r^{\sigma_{k_{0}}^{-}}
$$

for some $c_{1}, c_{2} \in \mathbb{R}$, where

$$
\sigma_{k_{0}}^{+}=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k_{0}}(\lambda)} \quad \text { and } \quad \sigma_{k_{0}}^{-}=-\frac{N-2 s}{2}-\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}
$$

Since the function $|x|^{\sigma_{k_{0}}^{-}} \psi\left(\frac{x}{|x|}\right) \notin L^{2}\left(B_{1}^{\prime} ;|x|^{-2 s}\right)$ and hence $|z|^{\sigma_{k_{0}}^{-}} \psi\left(\frac{z}{|z|}\right) \notin H^{1}\left(B_{1}^{+} ; t^{1-2 s}\right)$ in virtue of Lemma 2.5, we deduce that $c_{2}=0$ and $\varphi(r)=c_{1} r^{\sigma_{k_{0}}^{+}}$. Moreover, from $\varphi(1)=1$, we obtain that $c_{1}=1$ and then

$$
\begin{equation*}
\widetilde{w}(r, \theta)=r^{\sigma_{k_{0}}^{+}} \psi(\theta), \quad \text { for all } r \in(0,1) \text { and } \theta \in \mathbb{S}_{+}^{N} \tag{104}
\end{equation*}
$$

It remains to prove part (i). From (104) and the fact that $\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \psi^{2}(\theta) d S=1$ it follows that

$$
\begin{aligned}
& \widetilde{D}(r)=\frac{1}{r^{N-2 s}}\left[\int_{B_{r}^{+}} t^{1-2 s}|\nabla \widetilde{w}|^{2} d t d x-\kappa_{s} \int_{B_{r}^{\prime}} \frac{\lambda}{|x|^{2 s}} \widetilde{w}^{2} d x\right] \\
& =r^{2 s-N}\left(\sigma_{k_{0}}^{+}\right)^{2} \int_{0}^{r} t^{N-1-2 s+2 \sigma_{k_{0}}^{+}} d t \\
& \quad+r^{2 s-N}\left(\int_{0}^{r} t^{N-1-2 s+2 \sigma_{k_{0}}^{+}} d t\right)\left(\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s}\left|\nabla_{\mathbb{S}^{N}} \psi(\theta)\right|^{2} d S-\lambda \kappa_{s} \int_{\mathbb{S}^{N-1}}\left|\mathcal{T} \psi\left(\theta^{\prime}\right)\right|^{2} d S^{\prime}\right) \\
& =\frac{\left(\sigma_{k_{0}}^{+}\right)^{2}+\mu_{k_{0}}(\lambda)}{N-2 s+2 \sigma_{k_{0}}^{+}} r^{2 \sigma_{k_{0}}^{+}}=\sigma_{k_{0}}^{+} r^{2 \sigma_{k_{0}}^{+}}
\end{aligned}
$$

and

$$
\widetilde{H}(r)=\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \widetilde{w}^{2}(r \theta) d S=r^{2 \sigma_{k_{0}}^{+}}
$$

and hence from (102) it follows that $\gamma=\widetilde{\mathcal{N}}(r)=\frac{\widetilde{D}(r)}{\widetilde{H}(r)}=\sigma_{k_{0}}^{+}$. This completes the proof of the lemma.
The following lemma describes the behavior of $H(r)$ as $r \rightarrow 0^{+}$.
Lemma 4.3. Let $w$ satisfy (37), $H$ be defined in (40), and let $\gamma:=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)$ as in Lemma 3.15. Then the limit

$$
\lim _{r \rightarrow 0^{+}} r^{-2 \gamma} H(r)
$$

exists and it is finite.
Proof. In view of (78) it is sufficient to prove that the limit exists. By (40), (55), and Lemma 3.15 we have

$$
\frac{d}{d r} \frac{H(r)}{r^{2 \gamma}}=-2 \gamma r^{-2 \gamma-1} H(r)+r^{-2 \gamma} H^{\prime}(r)=2 r^{-2 \gamma-1}(D(r)-\gamma H(r))=2 r^{-2 \gamma-1} H(r) \int_{0}^{r} \mathcal{N}^{\prime}(\rho) d \rho
$$

Integration over $(r, \tilde{R})$ yields

$$
\begin{equation*}
\frac{H(\tilde{R})}{\tilde{R}^{2 \gamma}}-\frac{H(r)}{r^{2 \gamma}}=\int_{r}^{\tilde{R}} 2 s^{-2 \gamma-1} H(\rho)\left(\int_{0}^{\rho} \nu_{1}(t) d t\right) d \rho+\int_{r}^{\tilde{R}} 2 \rho^{-2 \gamma-1} H(\rho)\left(\int_{0}^{\rho} \nu_{2}(t) d t\right) d \rho \tag{105}
\end{equation*}
$$

where $\nu_{1}$ and $\nu_{2}$ are as in (64) and (65). Since, by Schwarz's inequality, $\nu_{1} \geqslant 0$, we have that $\lim _{r \rightarrow 0^{+}} \int_{r}^{\tilde{R}} 2 \rho^{-2 \gamma-1} H(\rho)\left(\int_{0}^{\rho} \nu_{1}(t) d t\right) d \rho$ exists. On the other hand, by (78), Lemma 3.13, and (76), we deduce that

$$
\begin{aligned}
& \left|\rho^{-2 \gamma-1} H(\rho)\left(\int_{0}^{\rho} \nu_{2}(t) d t\right)\right| \leqslant K_{1} C_{1}\left(C_{2}+\frac{N-2 s}{2}\right) \rho^{-1} \int_{0}^{\rho}\left(t^{-1+\varepsilon}+t^{-1+\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}+g(t)\right) d t \\
& \quad \leqslant K_{1} C_{1}\left(C_{2}+\frac{N-2 s}{2}\right) \rho^{-1}\left(\frac{\rho^{\varepsilon}}{\varepsilon}+\frac{p_{0} \rho^{\frac{2 s\left(p_{0}-2^{*}(s)\right)}{p_{0}}}}{2 s\left(p_{0}-2^{*}(s)\right)}+\frac{1}{1-\alpha}\|w\|_{L^{p}\left(B_{\tilde{R}}^{\prime}\right)}^{p(1-\alpha)} \rho^{N\left(\frac{\alpha 2^{*}(s)-2}{2^{*(s)}}-\frac{p \alpha-2}{p_{0}}\right)}\right)
\end{aligned}
$$

for all $\rho \in(0, \widetilde{R})$, which proves that $\rho^{-2 \gamma-1} H(\rho)\left(\int_{0}^{\rho} \nu_{2}(t) d t\right) \in L^{1}(0, \widetilde{R})$. Hence both terms at the right hand side of (105) admit a limit as $r \rightarrow 0^{+}$thus completing the proof.
From Lemma 4.2, the following pointwise estimate for solutions to (1) and (38) can be derived.
Lemma 4.4. Let $w$ satisfying (38). Then there exist $C_{4}, C_{5}>0$ and $\bar{r} \in(0, \tilde{R})$ such that (i) $\sup _{S_{r}^{+}}|w|^{2} \leqslant \frac{C_{4}}{r^{N+1-2 s}} \int_{S_{r}^{+}} t^{1-2 s}|w(z)|^{2} d S$ for every $0<r<\bar{r}$,
(ii) $|w(z)| \leqslant C_{5}|z|^{\gamma}$ for all $z \in B_{\bar{r}}^{+}$and in particular $|w(0, x)| \leqslant C_{5}|x|^{\gamma}$ for all $x \in B_{\bar{r}}^{\prime}$, where $\gamma:=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)$ is as in Lemma 3.15.

Proof. We first notice that (ii) follows directly from (i) and (78). In order to prove (i), we argue by contradiction and assume that there exists a sequence $\tau_{n} \rightarrow 0^{+}$such that

$$
\sup _{\theta \in \mathbb{S}_{+}^{N}}\left|w\left(\frac{\tau_{n}}{2} \theta\right)\right|^{2}>n H\left(\frac{\tau_{n}}{2}\right)
$$

with $H$ as in (40), i.e.

$$
\sup _{x \in S_{1 / 2}^{+}}\left|w\left(\tau_{n} z\right)\right|^{2}>2^{N+1-2 s} n \int_{S_{1 / 2}^{+}} t^{1-2 s} w^{2}\left(\tau_{n} z\right) d S
$$

i.e., defining $w^{\tau}$ as in (83)

$$
\begin{equation*}
\sup _{x \in S_{1 / 2}^{+}}\left|w^{\tau_{n}}(z)\right|^{2}>2^{N+1-2 s} n \int_{S_{1 / 2}^{+}} t^{1-2 s}\left|w^{\tau_{n}}(z)\right|^{2} d S \tag{106}
\end{equation*}
$$

From Lemma 4.2, along a subsequence $\tau_{n_{k}}$ we have that $w^{\tau_{n_{k}}} \rightarrow|z|^{\gamma} \psi\left(\frac{z}{|z|}\right)$ in $C_{\text {loc }}^{0, \alpha}\left(S_{1 / 2}^{+}\right)$, for some $\psi$ eigenfunction of problem (10), hence passing to the limit in (106) gives rise to a contradiction.
We will now prove that $\lim _{r \rightarrow 0^{+}} r^{-2 \gamma} H(r)$ is strictly positive.
Lemma 4.5. Under the same assumption as in Lemmas 4.3 and 4.4, we have

$$
\lim _{r \rightarrow 0^{+}} r^{-2 \gamma} H(r)>0
$$

Proof. For all $k \geqslant 1$, let $\psi_{k}$ be as in (24), i.e. $\psi_{k}$ is a $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$-normalized eigenfunction of problem (10) associated to the eigenvalue $\mu_{k}(\lambda)$ and $\left\{\psi_{k}\right\}_{k}$ is an orthonormal basis of $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$. From Lemma 4.2 there exist $j_{0}, m \in \mathbb{N}, j_{0}, m \geqslant 1$ such that $m$ is the multiplicity of the eigenvalue $\mu_{j_{0}}(\lambda)=\mu_{j_{0}+1}(\lambda)=\cdots=\mu_{j_{0}+m-1}(\lambda)$ and

$$
\begin{equation*}
\gamma=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{i}(\lambda)}, \quad i=j_{0}, \ldots, j_{0}+m-1 \tag{107}
\end{equation*}
$$

Let us expand $w$ as

$$
w(z)=w(\tau \theta)=\sum_{k=1}^{\infty} \varphi_{k}(\tau) \psi_{k}(\theta)
$$

where $\tau=|z| \in(0, R], \theta=z /|z| \in \mathbb{S}_{+}^{N}$, and

$$
\begin{equation*}
\varphi_{k}(\tau)=\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} w(\tau \theta) \psi_{k}(\theta) d S \tag{108}
\end{equation*}
$$

The Parseval identity yields

$$
\begin{equation*}
H(\tau)=\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} w^{2}(\tau \theta) d S=\sum_{k=1}^{\infty} \varphi_{k}^{2}(\tau), \quad \text { for all } 0<\tau \leqslant R \tag{109}
\end{equation*}
$$

In particular, from (78) and (109) it follows that, for all $k \geqslant 1$,

$$
\begin{equation*}
\varphi_{k}(\tau)=O\left(\tau^{\gamma}\right) \quad \text { as } \tau \rightarrow 0^{+} \tag{110}
\end{equation*}
$$

Equations (38) and (24) imply that, for every $k$,

$$
-\varphi_{k}^{\prime \prime}(\tau)-\frac{N+1-2 s}{\tau} \varphi_{k}^{\prime}(\tau)+\frac{\mu_{k}(\lambda)}{\tau^{2}} \varphi_{k}(\tau)=\zeta_{k}(\tau), \quad \text { in }(0, R)
$$

where

$$
\begin{equation*}
\zeta_{k}(\tau)=\frac{\kappa_{s}}{\tau^{2-2 s}} \int_{\mathbb{S}^{N-1}}\left(h\left(\tau \theta^{\prime}\right) w\left(0, \tau \theta^{\prime}\right)+f\left(\tau \theta^{\prime}, u\left(\tau \theta^{\prime}\right)\right)\right) \psi_{k}\left(0, \theta^{\prime}\right) d S^{\prime} \tag{111}
\end{equation*}
$$

A direct calculation shows that, for some $c_{1}^{k}, c_{2}^{k} \in \mathbb{R}$,

$$
\begin{equation*}
\varphi_{k}(\tau)=\tau^{\sigma_{k}^{+}}\left(c_{1}^{k}+\int_{\tau}^{R} \frac{t^{-\sigma_{k}^{+}+1}}{\sigma_{k}^{+}-\sigma_{k}^{-}} \zeta_{k}(t) d t\right)+\tau^{\sigma_{k}^{-}}\left(c_{2}^{k}+\int_{\tau}^{R} \frac{t^{-\sigma_{k}^{-}+1}}{\sigma_{k}^{-}-\sigma_{k}^{+}} \zeta_{k}(t) d t\right) \tag{112}
\end{equation*}
$$

where
(113) $\sigma_{k}^{+}=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k}(\lambda)} \quad$ and $\quad \sigma_{k}^{-}=-\frac{N-2 s}{2}-\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k}(\lambda)}$.

From (3), (4), Lemma 4.4, (107), and the fact that, in view of (23),

$$
2 s+(p-2) \gamma=\frac{(N-2 s)\left(2^{*}(s)-p\right)}{2}+(p-2) \sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{j_{0}}(\lambda)}>0
$$

we deduce that, for all $i=j_{0}, \ldots, j_{0}+m-1$,

$$
\begin{equation*}
\zeta_{i}(\tau)=O\left(\tau^{-2+\tilde{\delta}+\sigma_{i}^{+}}\right) \quad \text { as } \tau \rightarrow 0^{+} \tag{114}
\end{equation*}
$$

with $\tilde{\delta}=\min \{\varepsilon, 2 s+(p-2) \gamma\}>0$. Consequently, the functions

$$
t \mapsto \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) \quad \text { and } \quad t \mapsto \frac{t^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{-}-\sigma_{i}^{+}} \zeta_{i}(t)
$$

belong to $L^{1}(0, R)$. Hence

$$
\tau^{\sigma_{i}^{+}}\left(c_{1}^{i}+\int_{\tau}^{R} \frac{\rho^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(\rho) d \rho\right)=o\left(\tau^{\sigma_{i}^{-}}\right) \quad \text { as } \tau \rightarrow 0^{+}
$$

and then, by (110), there must be

$$
c_{2}^{i}=-\int_{0}^{R} \frac{t^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{-}-\sigma_{i}^{+}} \zeta_{i}(t) d t
$$

Using (114), we then deduce that

$$
\begin{equation*}
\tau^{\sigma_{i}^{-}}\left(c_{2}^{i}+\int_{\tau}^{R} \frac{t^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{-}-\sigma_{i}^{+}} \zeta_{i}(t) d t\right)=\tau^{\sigma_{i}^{-}}\left(\int_{0}^{\tau} \frac{t^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t\right)=O\left(\tau^{\sigma_{i}^{+}+\tilde{\delta}}\right) \tag{115}
\end{equation*}
$$

as $\tau \rightarrow 0^{+}$. From (112) and (115), we obtain that, for all $i=j_{0}, \ldots, j_{0}+m-1$,

$$
\begin{equation*}
\varphi_{i}(\tau)=\tau^{\sigma_{i}^{+}}\left(c_{1}^{i}+\int_{\tau}^{R} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t+O\left(\tau^{\tilde{\delta}}\right)\right) \quad \text { as } \tau \rightarrow 0^{+} \tag{116}
\end{equation*}
$$

Let us assume by contradiction that $\lim _{\lambda \rightarrow 0^{+}} \lambda^{-2 \gamma} H(\lambda)=0$. Then, for all $i \in\left\{j_{0}, \ldots, j_{0}+m-1\right\}$, (107) and (109) would imply that

$$
\lim _{\tau \rightarrow 0^{+}} \tau^{-\sigma_{i}^{+}} \varphi_{i}(\tau)=0
$$

Hence, in view of (116),

$$
c_{1}^{i}+\int_{0}^{R} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t=0
$$

which, together with (114), implies

$$
\begin{equation*}
\tau^{\sigma_{i}^{+}}\left(c_{1}^{i}+\int_{\tau}^{R} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t\right)=\tau^{\sigma_{i}^{+}} \int_{0}^{\tau} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{-}-\sigma_{i}^{+}} \zeta_{i}(t) d t=O\left(\tau^{\sigma_{i}^{+}+\tilde{\delta}}\right) \tag{117}
\end{equation*}
$$

as $\tau \rightarrow 0^{+}$. Collecting (112), (115), and (117), we conclude that

$$
\varphi_{i}(\tau)=O\left(\tau^{\sigma_{i}^{+}+\tilde{\delta}}\right) \quad \text { as } \tau \rightarrow 0^{+} \quad \text { for every } i \in\left\{j_{0}, \ldots, j_{0}+m-1\right\}
$$

namely,

$$
\sqrt{H(\tau)}\left(w^{\tau}, \psi\right)_{L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)}=O\left(\tau^{\gamma+\tilde{\delta}}\right) \quad \text { as } \tau \rightarrow 0^{+}
$$

for every $\psi \in \mathcal{A}_{0}=\operatorname{span}\left\{\psi_{i}\right\}_{i=j_{0}}^{j_{0}+m-1}$, where $\mathcal{A}_{0}$ is the eigenspace of problem (10) associated to the eigenvalue $\mu_{j_{0}}(\lambda)=\mu_{j_{0}+1}(\lambda)=\cdots=\mu_{j_{0}+m-1}(\lambda)$. From (79), there exists $C(\tilde{\delta})>0$ such that $\sqrt{H(\tau)} \geqslant C(\tilde{\delta}) \tau^{\gamma+\frac{\delta}{2}}$ for $\tau$ small, and therefore

$$
\begin{equation*}
\left(w^{\tau}, \psi\right)_{L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)}=O\left(\tau^{\frac{\tilde{\tilde{j}}}{2}} \quad \text { as } \tau \rightarrow 0^{+}\right. \tag{118}
\end{equation*}
$$

for every $\psi \in \mathcal{A}_{0}$. From Lemma 4.2 , for every sequence $\tau_{n} \rightarrow 0^{+}$, there exist a subsequence $\left\{\tau_{n_{k}}\right\}_{k \in \mathbb{N}}$ and an eigenfunction $\widetilde{\psi} \in \mathcal{A}_{0}$

$$
\begin{equation*}
\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \widetilde{\psi}^{2}(\theta) d S=1 \quad \text { and } \quad w^{\tau_{n_{k}}} \rightarrow \widetilde{\psi} \quad \text { in } L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right) \tag{119}
\end{equation*}
$$

From (118) and (119), we infer that

$$
0=\lim _{k \rightarrow+\infty}\left(w^{\tau_{n_{k}}}, \widetilde{\psi}\right)_{L^{2}\left(\mathbb{S}^{N-1}\right)}=\|\widetilde{\psi}\|_{L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)}^{2}=1
$$

thus reaching a contradiction.
We can now completely describe the behavior of solutions to (38) near the singularity, hence proving Theorem 4.1.
Proof of Theorem 4.1. Identity (82) follows from part (i) of Lemma 4.2, thus there exists $k_{0} \in \mathbb{N}, k_{0} \geqslant 1$, such that $\gamma=\lim _{r \rightarrow 0^{+}} \mathcal{N}(r)=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}$. Let us denote as $m$ the multiplicity of $\mu_{j_{0}}(\lambda)$ so that, for some $j_{0} \in \mathbb{N}, j_{0} \geqslant 1, j_{0} \leqslant k_{0} \leqslant j_{0}+m-1, \mu_{j_{0}}(\lambda)=$ $\mu_{j_{0}+1}(\lambda)=\cdots=\mu_{j_{0}+m-1}(\lambda)$ and let $\left\{\psi_{i}\right\}_{i=j_{0}}^{j_{0}+m-1}$ be an $L^{2}\left(\mathbb{S}_{+}^{N} ; \theta_{1}^{1-2 s}\right)$-orthonormal basis for the eigenspace associated to $\mu_{k_{0}}(\lambda)$.

Let $\left\{\tau_{n}\right\}_{n \in \mathbb{N}} \subset(0,+\infty)$ such that $\lim _{n \rightarrow+\infty} \tau_{n}=0$. Then, from part (ii) of Lemma 4.2 and Lemmas 4.3 and 4.5 , there exist a subsequence $\left\{\tau_{n_{k}}\right\}_{k \in \mathbb{N}}$ and $m$ real numbers $\beta_{j_{0}}, \ldots, \beta_{j_{0}+m-1} \in \mathbb{R}$ such that $\left(\beta_{j_{0}}, \beta_{j_{0}+1}, \ldots, \beta_{j_{0}+m-1}\right) \neq(0,0, \ldots, 0)$ and

$$
\begin{align*}
& \tau_{n_{k}}^{-\gamma} w\left(0, \tau_{n_{k}} x\right) \rightarrow|x|^{\gamma} \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i} \psi_{i}\left(0, \frac{x}{|x|}\right) \quad \text { in } C_{\mathrm{loc}}^{1, \alpha}\left(B_{1}^{\prime} \backslash\{0\}\right) \quad \text { as } k \rightarrow+\infty,  \tag{120}\\
& \tau_{n_{k}}^{-\gamma} w\left(\tau_{n_{k}} \theta\right) \rightarrow \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i} \psi_{i}(\theta) \quad \text { in } C^{0, \alpha}\left(\mathbb{S}_{+}^{N}\right) \quad \text { as } k \rightarrow+\infty,  \tag{121}\\
& \tau_{n_{k}}^{-\gamma} w\left(0, \tau_{n_{k}} \theta^{\prime}\right) \rightarrow \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i} \psi_{i}\left(0, \theta^{\prime}\right) \quad \text { in } C^{1, \alpha}\left(\mathbb{S}^{N-1}\right) \quad \text { as } k \rightarrow+\infty, \tag{122}
\end{align*}
$$

and
(123) $\tau_{n_{k}}^{1-\gamma} \nabla_{x} w\left(0, \tau_{n_{k}} \theta^{\prime}\right) \rightarrow \sum_{i=j_{0}}^{j_{0}+m-1} \beta_{i}\left(\gamma \psi_{i}\left(0, \theta^{\prime}\right) \theta^{\prime}+\nabla_{\mathbb{S}^{N-1}} \psi_{i}\left(0, \theta^{\prime}\right)\right) \quad$ in $C^{0, \alpha}\left(\mathbb{S}^{N-1}\right)$ as $k \rightarrow+\infty$ for some $\alpha \in(0,1)$.

We now prove that the $\beta_{i}$ 's depend neither on the sequence $\left\{\tau_{n}\right\}_{n \in \mathbb{N}}$ nor on its subsequence $\left\{\tau_{n_{k}}\right\}_{k \in \mathbb{N}}$.

Defining $\varphi_{i}$ and $\zeta_{i}$ as in (108) and (111), from (121) it follows that, for any $i=j_{0}, \ldots, j_{0}+m-1$,

$$
\begin{equation*}
\tau_{n_{k}}^{-\gamma} \varphi_{i}\left(\tau_{n_{k}}\right)=\int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \frac{w\left(\tau_{n_{k}} \theta\right)}{\tau_{n_{k}}^{\gamma}} \psi_{i}(\theta) d S \rightarrow \sum_{j=j_{0}}^{j_{0}+m-1} \beta_{j} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} \psi_{j}(\theta) \psi_{i}(\theta) d S=\beta_{i} \tag{124}
\end{equation*}
$$

as $k \rightarrow+\infty$. As deduced in the proof of Lemma 4.5, for any $i=j_{0}, \ldots, j_{0}+m-1$ and $\tau \in(0, R]$ there holds

$$
\begin{align*}
\varphi_{i}(\tau) & =\tau^{\sigma_{i}^{+}}\left(c_{1}^{i}+\int_{\tau}^{R} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t\right)+\tau^{\sigma_{i}^{-}}\left(\int_{0}^{\tau} \frac{t^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t\right)  \tag{125}\\
& =\tau^{\sigma_{i}^{+}}\left(c_{1}^{i}+\int_{\tau}^{R} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t+O\left(\tau^{\tilde{\delta}}\right)\right) \quad \text { as } \tau \rightarrow 0^{+}
\end{align*}
$$

for some $c_{1}^{i} \in \mathbb{R}$, where $\sigma_{i}^{ \pm}$are defined in (113). Choosing $\tau=R$ in the first line of (125), we obtain

$$
c_{1}^{i}=R^{-\sigma_{i}^{+}} \varphi_{i}(R)-R^{\sigma_{i}^{-}-\sigma_{i}^{+}} \int_{0}^{R} \frac{s^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(s) d s
$$

Hence (125) yields

$$
\tau^{-\gamma} \varphi_{i}(\tau) \rightarrow R^{-\sigma_{i}^{+}} \varphi_{i}(R)-R^{\sigma_{i}^{-}-\sigma_{i}^{+}} \int_{0}^{R} \frac{t^{-\sigma_{i}^{-}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t+\int_{0}^{R} \frac{t^{-\sigma_{i}^{+}+1}}{\sigma_{i}^{+}-\sigma_{i}^{-}} \zeta_{i}(t) d t \quad \text { as } \tau \rightarrow 0^{+}
$$

and therefore from (124) we deduce that

$$
\begin{aligned}
\beta_{i}= & R^{-\gamma} \int_{\mathbb{S}_{+}^{N}} \theta_{1}^{1-2 s} w(R \theta) \psi_{i}(\theta) d S \\
& -R^{-2 \gamma-N+2 s} \int_{0}^{R} \frac{\kappa_{s} \rho^{\gamma+N-1}}{2 \gamma+N-2 s}\left(\int_{\mathbb{S}^{N-1}}\left(h\left(\rho \theta^{\prime}\right) w\left(0, \rho \theta^{\prime}\right)+f\left(\rho \theta^{\prime}, w\left(0, \rho \theta^{\prime}\right)\right)\right) \psi_{i}\left(0, \theta^{\prime}\right) d S^{\prime}\right) d \rho \\
& +\int_{0}^{R} \frac{\kappa_{s} \rho^{2 s-\gamma-1}}{2 \gamma+N-2 s}\left(\int_{\mathbb{S}^{N-1}}\left(h\left(\rho \theta^{\prime}\right) w\left(0, \rho \theta^{\prime}\right)+f\left(\rho \theta^{\prime}, w\left(0, \rho \theta^{\prime}\right)\right)\right) \psi_{i}\left(0, \theta^{\prime}\right) d S^{\prime}\right) d \rho
\end{aligned}
$$

In particular the $\beta_{i}$ 's depend neither on the sequence $\left\{\tau_{n}\right\}_{n \in \mathbb{N}}$ nor on its subsequence $\left\{\tau_{n_{k}}\right\}_{k \in \mathbb{N}}$, thus implying that the convergences in (120), (121), (122), and (123) actually hold as $\tau \rightarrow 0^{+}$and proving the theorem.

## 5. Proof of Theorem 1.1

Let $\mathcal{D}^{s, 2}(\Omega)$ denote the completion on $C_{\mathrm{c}}^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)}$. Simple density arguments show that (7) is equivalent to

$$
\begin{equation*}
(u, \varphi)_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)}=\int_{\Omega}\left(\frac{\lambda}{|x|^{2 s}} u(x)+h(x) u(x)+f(x, u(x))\right) \varphi(x) d x, \text { for all } \varphi \in \mathcal{D}^{s, 2}(\Omega) \tag{126}
\end{equation*}
$$

Since $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$, we can let $\mathcal{H}(u) \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)$ such that

$$
\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s} \nabla \mathcal{H}(u) \cdot \nabla \varphi d t d x=0, \quad \text { for all } \varphi \in C_{\mathrm{c}}^{\infty}\left(\mathbb{R}_{+}^{N+1}\right)
$$

and $\mathcal{H}(u)=u$ on $\mathbb{R}^{N}$ identified with $\partial \mathbb{R}_{+}^{N+1}$, i.e. $\mathcal{H}(u)$ weakly satisfies

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla \mathcal{H}(u)\right)=0, & \text { in } \mathbb{R}_{+}^{N+1}, \\ \mathcal{H}(u)=u, & \text { on } \partial \mathbb{R}_{+}^{N+1}=\{0\} \times \mathbb{R}^{N}\end{cases}
$$

From [4] we have that

$$
\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s} \nabla \mathcal{H}(u) \cdot \nabla \widetilde{\varphi} d t d x=\kappa_{s}(u, \widetilde{\varphi})_{\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)} \quad \text { for all } \widetilde{\varphi} \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)
$$

where

$$
\kappa_{s}=\frac{\Gamma(1-s)}{2^{2 s-1} \Gamma(s)}
$$

i.e.

$$
-\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial \mathcal{H}(u)}{\partial t}=\kappa_{s}(-\Delta)^{s} u(x),
$$

in a weak sense. Therefore $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ weakly solves (1) in $\Omega$ in the sense of (7) if and only if its extension $w=\mathcal{H}(u)$ satisfies

$$
\begin{cases}\operatorname{div}\left(t^{1-2 s} \nabla w\right)=0, & \text { in } \mathbb{R}_{+}^{N+1}  \tag{127}\\ w=u, & \text { on } \mathbb{R}^{N} \\ -\lim _{t \rightarrow 0^{+}} t^{1-2 s} \frac{\partial w}{\partial t}(t, x)=\kappa_{s}\left(\frac{\lambda}{|x|^{2 s}} w+h w+f(x, w)\right), & \text { on } \Omega\end{cases}
$$

in a weak sense, i.e. if for all $\widetilde{\varphi} \in \mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right)$ such that $x \mapsto \widetilde{\varphi}(0, x) \in \mathcal{D}^{s, 2}(\Omega)$ we have

$$
\begin{equation*}
\int_{\mathbb{R}_{+}^{N+1}} t^{1-2 s} \nabla w \cdot \nabla \widetilde{\varphi} d t d x=\kappa_{s} \int_{\Omega}\left(\frac{\lambda}{|x|^{2 s}} w+h w+f(x, w)\right) \widetilde{\varphi} d x \tag{128}
\end{equation*}
$$

Since $\mathcal{D}^{1,2}\left(\mathbb{R}_{+}^{N+1} ; t^{1-2 s}\right) \hookrightarrow H^{1}\left(B_{R}^{+} ; t^{1-2 s}\right)$ for all $R>0$, the result follows from Theorem 4.1.

Proof of Theorem 1.2. The proof follows from the proof of Theorem 1.1, observing that, if $\lambda=0$, since no singularity occurs in the coefficients of the equation, the convergences in (92) and (93) hold in $C^{0, \alpha}\left(\overline{B_{r}^{+}}\right)$by virtue of Lemma 3.3.

Proof of Theorem 1.3. The proof follows directly from Theorem 1.1. Indeed, let $u$ be a solution to (1) satisfying $u(x)=o\left(|x|^{n}\right)=o(1)|x|^{n}$ as $|x| \rightarrow 0$ for all $n \in \mathbb{N}$; if, by contradiction, $u \not \equiv 0$ in $\Omega$, convergence (13) in Theorem 1.1 would hold, thus contradicting the assumption that $u(x)=o\left(|x|^{n}\right)$ if $n>-\frac{N-2 s}{2}+\sqrt{\left(\frac{2 s-N}{2}\right)^{2}+\mu_{k_{0}}(\lambda)}$.

## 6. Proof of Theorem 1.4

Let $u \in \mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ be a weak solution to (14) in $\Omega$ such that $u \equiv 0$ on a set $E \subset \Omega$ with $\mathcal{L}(E)>0$, where $\mathcal{L}$ denotes the $N$-dimensional Lebesgue measure. By Lebesgue's density Theorem, for a.e. point $x \in E$ there holds

$$
\lim _{r \rightarrow 0^{+}} \frac{\mathcal{L}(E \cap B(x, r))}{\mathcal{L}(B(x, r))}=1 \quad \text { and } \quad \lim _{r \rightarrow 0^{+}} \frac{\mathcal{L}\left(\left(\mathbb{R}^{N} \backslash E\right) \cap B(x, r)\right)}{\mathcal{L}(B(x, r))}=0
$$

where $B(x, r)=\left\{y \in \mathbb{R}^{N}:|y-x|<r\right\}$, i.e. a.e. point of $E$ is a density point of $E$. Let $x_{0}$ be a density point of $E$; then for all $\varepsilon>0$ there exists $r_{0}=r_{0}(\varepsilon) \in(0,1)$ such that, for all $r \in\left(0, r_{0}\right)$,

$$
\begin{equation*}
\frac{\mathcal{L}\left(\left(\mathbb{R}^{N} \backslash E\right) \cap B\left(x_{0}, r\right)\right)}{\mathcal{L}\left(B\left(x_{0}, r\right)\right)}<\varepsilon \tag{129}
\end{equation*}
$$

Assume by contradiction that $u \not \equiv 0$ in $\Omega$. Theorem 1.2 implies that

$$
\begin{equation*}
r^{-\gamma} u\left(x_{0}+r\left(x-x_{0}\right)\right) \rightarrow\left|x-x_{0}\right|^{\gamma} \psi\left(0, \frac{x-x_{0}}{\left|x-x_{0}\right|}\right) \quad \text { as } r \rightarrow 0^{+} \tag{130}
\end{equation*}
$$

in $C^{1, \alpha}\left(B_{1}\left(x_{0}, 1\right)\right)$, where $\gamma=-\frac{N-2 s}{2}+\sqrt{\left(\frac{N-2 s}{2}\right)^{2}+\mu_{k_{0}}}$ and $\mu_{k_{0}}=\mu_{k_{0}}(0) \geqslant 0$ is an eigenvalue of problem (10) with $\lambda=0$ and $\psi$ is a corresponding eigenfunction. Since $u \equiv 0$ in $E$, by (129) we have

$$
\begin{aligned}
& \int_{B\left(x_{0}, r\right)} u^{2}(x) d x=\int_{\left(\mathbb{R}^{N} \backslash E\right) \cap B\left(x_{0}, r\right)} u^{2}(x) d x \\
& \leqslant\left(\int_{\left(\mathbb{R}^{N} \backslash E\right) \cap B\left(x_{0}, r\right)}|u(x)|^{2^{*}(s)} d x\right)^{2 / 2^{*}(s)}\left|\mathcal{L}\left(\left(\mathbb{R}^{N} \backslash E\right) \cap B\left(x_{0}, r\right)\right)\right|^{\frac{2}{}^{2^{*}(s)-2}}{ }^{2^{*}(s)} \\
&<\varepsilon^{\frac{2^{*}(s)-2}{2^{*}(s)}}\left|\mathcal{L}\left(B\left(x_{0}, r\right)\right)\right|^{2^{2^{*}(s)-2}}{ }^{2^{*(s)}} \\
&\left(\int_{\left(\mathbb{R}^{N} \backslash E\right) \cap B\left(x_{0}, r\right)}|u(x)|^{2^{*}(s)} d x\right)^{2 / 2^{*}(s)}
\end{aligned}
$$

for all $r \in\left(0, r_{0}\right)$, i.e. letting $u^{r}(x):=r^{-\gamma} u\left(x_{0}+r\left(x-x_{0}\right)\right)$,

$$
\int_{B\left(x_{0}, 1\right)}\left|u^{r}(x)\right|^{2} d x<\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2^{*}(s)-2}{2^{*(s)}}} \varepsilon^{\frac{2^{*}(s)-2}{2^{*}(s)}}\left(\int_{B\left(x_{0}, 1\right)}\left|u^{r}(x)\right|^{2^{*}(s)} d x\right)^{2 / 2^{*}(s)}
$$

for all $r \in\left(0, r_{0}\right)$. Letting $r \rightarrow 0^{+}$, from (130) we have that

$$
\begin{aligned}
& \int_{B\left(x_{0}, 1\right)}\left|x-x_{0}\right|^{2 \gamma} \psi^{2}\left(0, \frac{x-x_{0}}{\left|x-x_{0}\right|}\right) d x \\
& \qquad \leqslant\left(\frac{\omega_{N-1}}{N}\right)^{\frac{2^{*}(s)-2}{2^{*}(s)}} \varepsilon^{\frac{2^{*}(s)-2}{2^{*(s)}}}\left(\int_{B\left(x_{0}, 1\right)}\left|x-x_{0}\right|^{2^{*}(s) \gamma} \psi^{2^{*}(s)}\left(0, \frac{x-x_{0}}{\left|x-x_{0}\right|}\right) d x\right)^{2 / 2^{*}(s)}
\end{aligned}
$$

which yields a contradiction as $\varepsilon \rightarrow 0^{+}$.
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