Foliations of small tubes in Riemannian manifolds by capillary minimal discs

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Abstract

Letting Γ be an embedded curve in a Riemannian manifold $\mathcal{M}$, we prove the existence of minimal disc-type surfaces centered at Γ inside surface of revolution of $\mathcal{M}$ around Γ, having small radius, and intersecting it with constant angles. In particular we obtain that small tubular neighborhoods can be foliated by minimal discs.

1 Introduction

Minimal surfaces are surfaces with mean curvature vanishing everywhere. These include, but are not limited to, surfaces of minimal area subjected to various constraints.
In this paper we are interested in minimal surfaces which intersect a given hyper-surface with a constant angle. Such surfaces, called *capillary minimal surfaces*, are critical points of an energy functional under some constraints, see (3) below.
Capillary surfaces correspond to the physical problem of describing of an incompressible liquid in a container in the absence of gravity. A great deal of work has been devoted to capillarity phenomena from the point of view of existence, uniqueness and topological properties of solutions mainly in the non-parametric case and in the more general situation of presence of gravity (see the book of R. Finn, [7], for an account of the subject). Some answers to these questions have been obtained by many authors. We refer for example to the papers [9], [10], [4] [12], [16], [18] [17], [20],[21] and the references therein.
Here we prove existence results of capillary surfaces with prescribed topology in Riemannian manifolds. Roughly speaking, we first show the existence of a class of capilary (minimal) disc-type surfaces embedded in a Riemannian surface of revolution (see below). In particular, shrinking enough the thickness of the surface of revolution, this class constitutes a foliation. Secondly we have existence of minimal disc-type surfaces embedded in a geodesic tube of a curve which intersect perpendicularly the boundary of the tube.
The method we use is perturbative in nature and the main idea goes back to R. Ye, [23], subsequently employed with success by many authors to obtain existence of (large) constant mean curvature hyper-surfaces. For example, one can see [5], [6] [13], [14] and [15]. Before stating our results, some preliminaries are required.

A surface of revolution is a surface created by rotating a parametric curve $[a, b] \ni s \to (\kappa(s)\,,\,\phi(s)) \in \mathbb{R}^2$ lying on some plane around a straight line (the axis of rotation) in the same plane.
The resulting surface $\mathscr{C}^1$ therefore always has azimuthal symmetry. Examples of surfaces of revolution include cylinder (excluding the ends), hyperboloid, paraboloid, sphere, torus, etc.
In more generality one can obtain surfaces of revolution in $\mathbb{R}^{n+1}$, $n \geq 2$ using the standard parametrization

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$$S(s,z) = (\kappa(s)\,,\,\phi(s)\,\Theta(z))\,,$$

where $z \mapsto \Theta(z) \in S^{n-1}$, $\phi(s) \neq 0 \quad \forall s \in [a,b]$.
Assuming that the rotating curve is parameterized by arc length namely

$$(\phi'(s))^2 + (\kappa'(s))^2 = 1,$$

clearly the disc $\mathscr{D}_{s,1}$ centered at $(\kappa(s)\,,\,0)$ (on the axis of rotation) with radius $\phi(s)$ parameterized by

$$B_1^n \ni x \mapsto (\kappa(s)\,,\,\phi(s)\,x)\,,$$

has zero mean curvature and intersects the above surface of revolution with a constant angle equal to $\arccos \phi'(s)$, where B_1^n stands for the unit ball of $\mathbb{R}^n$ centered at the origin, namely $\mathscr{D}_{s,1}$ is a capillary surface.
Motivated by capillarity problems, for questions of stability, see [7], it is not restrictive to assume that the angle of contact is in $(0,\pi)$, namely $\phi'(s) \in (-1,1)$ or equivalently

$$\kappa'(s) \neq 0.$$

We shall extend these definitions of surface of revolution in a Riemannian setting.

Let $(\mathcal{M}^{n+1}, g)$ be Riemannian manifold, and Γ an embedded curve parameterized by a map $\gamma : [0,1] \to \mathcal{M}$. We consider a local parallel orthogonal frame $E_1, \cdots, E_n$ of $N\Gamma$ along Γ. This determines a coordinate system by

$$[0,1] \times \mathbb{R}^n \ni (x_0, y) \mapsto f(x_0, y) := \exp_{\gamma(x_0)}(y^i E_i) \,\in\, \mathcal{M}.$$

For a small parameter $\rho > 0$, consider the *Riemannian surface of revolution* $\mathscr{C}^\rho$ around Γ in $\mathcal{M}$ parameterized by

$$(s,z) \longrightarrow f(\rho\, S(s,z)) = f(\rho\,\kappa(s)\,,\,\rho\,\phi(s)\Theta(z)) = \exp_{\gamma(\rho\,\kappa(s))}(\rho\,\phi(s)\Theta^i(z)\,E_i),$$

where $z \mapsto \Theta(z) \in S^{n-1}$, and call its interior $\Omega_\rho := \operatorname{int}\mathscr{C}^\rho$ which is nothing but a tubular neighborhood for Γ if ρ is small enough. Here we are assuming always that $\phi(s) \neq 0$ and that $(\phi'(s))^2 + (\kappa'(s))^2 = 1$.

For any $s \in [a,b]$, we consider the following set

$$D_{s,\rho} := f(\rho\,\kappa(s)\,,\,\rho\,\phi(s)\,B_1^n),$$

it is clear that $\partial D_{s,\rho} \subset \mathscr{C}^\rho$ and we have that the mean curvature $H_{D_{s,\rho}}$ of $D_{s,\rho}$, see § 4.1, satisfies

$$H_{D_{s,\rho}} = \mathcal{O}(\rho) \qquad \text{in } D_{s,\rho} \tag{1}$$

while the angle between the unit outer normals (see also § 4.2) can be expanded as

$$\langle N_{D_{s,\rho}}, N_{\mathscr{C}^\rho}\rangle = \phi'(s) + \mathcal{O}(\rho) \qquad \text{on } \partial D_{s,\rho}. \tag{2}$$

Our aim is to perturb $D_{s,\rho}$ to a capillary minimal submanifold, $\mathscr{D}_{s,\rho}$, of Ω_ρ centered on Γ with contact angle $\arccos \phi'(s)$ along $\partial\mathscr{D}_{s,\rho} \subset \mathscr{C}^\rho$, as it happens in $\mathbb{R}^{n+1}$.

Theorem 1.1 *Suppose we are in the situation described above. Let $[a',b'] \subset [a,b]$ be such that $\phi(s)\phi''(s) > 0$ for every $s \in [a',b']$. Then there exists $\rho_0 > 0$ such that for any $s \in [a',b']$ and $\rho \in (0,\rho_0)$, there exists an embedded minimal disc $\mathscr{D}_{s,\rho} \subset \Omega_\rho$, intersecting $\mathscr{C}^\rho$ by an angle equal to $\phi'(s)$ along its boundary. Moreover $\mathscr{D}_{s,\rho}$ is a normal graph over the set $D_{s,\rho}$ for which the norm (in the $\mathcal{C}^{2,\alpha}$-topology) of this function defining the graph tends to zero uniformly as ρ tends to zero.*
Furthermore there exists a tubular neighborhood O_ρ of $\gamma([a',b'])$ foliated by such minimal discs for which each leaf intersects ∂O_ρ transversally along its boundary.

Remark 1.2
- *When we parameterize in particular $\mathscr{C}^\rho$ with $\kappa(s) = s$, and if we require the capillary discs to be perpendicular to $\mathscr{C}^\rho$, we obtain the conditions $\phi' = 0$ and $\phi'' \neq 0$. This means that non-degenerate extrema of the width ϕ determine the location of such surfaces.*
- *An example is the hyperboloid, $\phi(s) = \cosh s$ and $\kappa(s) = \sinh s$. Here one may see $\mathcal{M}$ as a Lorentzian manifold modeled on the Minkowski space $\mathbb{R}^n_1$. Letting $q \in \mathcal{M}$ and E_0 a unit time-like vector of $T_q\mathcal{M}$ and $\gamma(x_0) = \exp_q(x_0 E_0)$ so one can see $\mathscr{D}_{0,\rho}$ as a space-like minimal disc in the geodesic sphere of radius ρ.*

An interesting particular case which is not covered by Theorem 1.1 is when $\phi \equiv 1$ and $\kappa =$Id, namely when we deal with geodesic tubes. In this situation (recall that in this case the angle of contact is $\frac{\pi}{2}$) it is the geometry of the manifold to determine the position of the discs. More precisely, we have that $\mathscr{C}^\rho$ is the geodesic tube of radius $\rho > 0$ around Γ,

$$\mathscr{C}^\rho = \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q, \Gamma) = \rho\},$$

and its interior is nothing but

$$\Omega_\rho := \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q, \Gamma) < \rho\}.$$

In this case due to invariance by translations along the axis of rotation, we reduced our problem of finding minimal surfaces to a finite-dimensional one. Namely we have obtained the following

Theorem 1.3 *There exists a smooth function $\psi_\rho : [a, b] \to \mathbb{R}$ such that, for ρ small, if s_0 is a critical point of ψ_ρ the set $D_{s_0,\rho}$ can be smoothly perturbed to an embedded minimal hyper-surface $\mathscr{D}_{s_0,\rho} \subset \Omega_\rho$ intersecting $\mathscr{C}^\rho$ perpendicularly along its boundary. Furthermore, for any integer k, there exists a constant c_k (independant on ρ) such that*

$$\|\psi_\rho - \sum_{i,j}^{n} \langle R_p(E_j, E_i)E_j, E_i\rangle\|_{\mathcal{C}^k[a,b]} \leq c_k \rho^2,$$

where R_p is the Riemann tensor of $\mathcal{M}$ at $p = \gamma(\rho\, s)$.

Some remarks are due: let $\Gamma \ni p \to \Psi(p) = \sum_{i,j}^n \langle R_p(E_j, E_i)E_j, E_i\rangle$ any strict maxima or minima of Ψ imply the existence of minimal surfaces. In particular suppose at some point $p_0 = \gamma(\rho\, s_0)$ interior to Γ, there hold

$$d\Psi(p_0)[\dot\gamma(\rho\, s_0)] = 0 \qquad \text{and} \qquad \left|d^2\Psi(p_0)[\dot\gamma(\rho\, s_0), \dot\gamma(\rho\, s_0)]\right| > c,$$

for some constant c independent on ρ. By the implicit function theorem, there exits a curve $(0, \rho_0) \ni \rho \mapsto s_\rho$ with $s_\rho \to s_0$ such that s_ρ is a critical point of ψ_ρ. Hence for every $\rho \in (0, \rho_0)$, there exits an embedded minimal disc $\mathscr{D}_{s_\rho,\rho}$, centered at $\gamma(\rho\, s_\rho)$, contained in Ω_ρ that intersects $\partial\Omega_\rho$ perpendicularly along its boundary.

Remark 1.4
- *We have that*

$$\Psi(p) = \sum_{i,j}^{n} \langle R_p(E_j, E_i)E_j, E_i\rangle = \boldsymbol{S}(p) + 2\, Ric_p(\dot\gamma(\rho\, s), \dot\gamma(\rho\, s)),$$

where

$$\boldsymbol{S}(p) = \sum_{\alpha,\beta=0}^{n} \langle R_p(E_\alpha, E_\beta)E_\alpha, E_\beta\rangle$$

is the scalar curvature of $\mathcal{M}$ at $p = \gamma(\rho\, s)$, $E_0 = \dot\gamma(\rho\, s)$ and Ric_p is the Ricci tensor of $\mathcal{M}$ at p. From Theorem 1.3, we have that if $s \mapsto Ric_p(\dot\gamma(\rho\, s), \dot\gamma(\rho\, s))$ is locally constant along Γ then stable critical points the scalar curvature yields existence of minimal discs.

- *Recall that if $(\mathcal{M}_1^{m_1}, g_1)$ and $(\mathcal{M}_2^{m_2}, g_2)$ are two manifolds, the Riemann tensor R of the (Riemannian) Cartesian product $\mathcal{M}^{m_1+m_2} := (\mathcal{M}_1 \times \mathcal{M}_2, g_1 \oplus g_2)$ decomposes as $R = R^1 \oplus R^2$ since the connection ∇ is given by $\nabla_{X_1+X_2}(Y_1+Y_2) = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2$ for any X_1, Y_1 (resp. X_2, Y_2) vector fields of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$), where ∇^i is the connection of $\mathcal{M}_i$. Clearly for any $p_2 \in \mathcal{M}_2$, the set $(\mathcal{M}_1)(p_2) := \{(p_1, p_2) \in \mathcal{M} \quad : \quad p_1 \in \mathcal{M}_1\}$ is a submanifold of $\mathcal{M}$, diffeomorphic to $\mathcal{M}_1$.*
  In particular if $m_1 = 1$, $R^1 = 0$, by Theorem 1.3 we obtain that stable critical points of the mapping $\boldsymbol{S}\Big|_{(\mathcal{M}_1)(p_2)}$ yield existence of minimal discs inside (small) geodesic tubes around the curve $(\mathcal{M}_1)(p_2)$, where as before $\boldsymbol{S}$ is the scalar curvature of $\mathcal{M}$.

- *As a simple byproduct of our analysis, we find that if Γ is a closed curve, we have at least 2 (equal to the Lusternik-Schnierelman category of Γ, see [3]) solutions (without any assumptions on the curvature of $\mathcal{M}$).*

- *We believe that this result might be generalized to higher codimensions namely if N^ℓ, $1 < \ell < n$, is an ℓ-dimensional submanifold of $\mathcal{M}^{n+1}$ and considering the following surface of revolution with axis of rotation $\mathbb{R}^\ell$*
$$S(s,z) = (\kappa^1(s), \dots, \kappa^\ell(s), \phi(s)\,\Theta(z)),$$
  where $z \mapsto \Theta(z) \in S^{n-\ell}$, one could obtain $(n-\ell+1)$-dimensional minimal disc-type submanifolds of $\mathcal{M}$ centered on N^ℓ.

Let us describe the proof of the theorems above. We first recall, see [17], that Capillary hyper-surfaces with constant contact angle arccos $\phi'(s)$ are stationary for the energy functional

$$\mathcal{E}(D) = \mathrm{Area}(D \cap \Omega_\rho) - \phi'(s)\,\mathrm{Area}(\Omega'_\rho), \tag{3}$$

among (orientable smooth) surfaces $D \subset \Omega_\rho$ with $\partial D \subset \partial\Omega_\rho$ and $\Omega'_\rho \subset \partial\Omega_\rho$ is the part (on one side of D) for which the angle is measured. Moreover the Euler-Lagrange equations is nothing but

$$\begin{aligned} H_D \quad &= \quad 0 \qquad \text{in } D, \\ \langle N_D, N_{\partial\Omega_\rho}\rangle \quad &= \quad \phi'(s) \qquad \text{on } \partial D. \end{aligned} \tag{4}$$

Here H_D is the mean curvature of D while N_D and $N_{\partial\Omega_\rho}$ are outer unit normals of D and $\partial\Omega_\rho$ respectively. Since we look for stationary surfaces with a given profile for this energy functional, clearly by (1)-(2) a manifold of approximate solutions is given by $Z_\rho := \{D_{s,\rho} \quad : \quad s \in [a,b]\}$.
For any given hyper-surface $D_{s,\rho} \in Z_\rho$, we parametrize (locally) a neighborhood of $D_{s,\rho}$ (in the manifold in $\mathcal{M}$) by a mapping $F^s : \mathbb{R} \times B_1^n \to \mathcal{M}$ for which $F^s(t, \partial B_1^n) \subset \partial\Omega_\rho$, for every t, while the direction $F^s_*(\partial_t)$ is nearly normal to $D_{s,\rho}$, and moreover $D_{s,\rho} = F^s(0, B_1^n)$, see (8). This allows to parametrize any set $\mathscr{D}$ nearby $D_{s,\rho}$ satisfying $\partial\mathscr{D} \subset \partial\Omega_\rho$ by a function $w : B_1^n \to \mathbb{R}$ such that $\mathscr{D}(w) = F^s(w, B_1^n)$. We call $\mathcal{H}(s,\rho,w)$ the mean curvature of $\mathscr{D}(w)$ and $\mathcal{B}(s,\rho,w)$ the angle between the normals $N_{\partial\mathscr{D}(w)}$ and $N_{\mathscr{C}^\rho}$ of $\partial\mathscr{D}(w)$ and $\partial\Omega_\rho$ respectively.
One of the main features in this work in the (technical) Sections 4.1, § 4.2 is to calculate $\mathcal{H}(s,\rho,w)$ as a nonlinear elliptic partial differential operator, depending on ρ and s acting on w coupled with the mixed boundary operator which we denote by $\mathcal{B}(s,\rho,w)$. In these calculations it is important to gather various different types of error terms, some of which depend linearly and some nonlinearly on w, and some of which are inhomogeneous terms vanishing to some order in ρ. It turns out to be helpful to rescale the local coordinates y by $\varepsilon(s) = \rho\phi(s)$ which is the radius of the discs. The final expression, Proposition 4.5, for the mean curvature of $\mathscr{D}(w)$ then is

$$-\frac{\phi}{\kappa'}\,\mathcal{H}(s,\rho,w) = -\mathbb{L}_{\rho,s}(w) + \mathcal{O}(\rho^2) + \rho\,Q(w) \qquad \text{in } \mathscr{D}(w),$$

where $\mathbb{L}_{\rho,s}$ is the linearized mean curvature operator about $\mathscr{D}(0) = D_{s,\rho}$:

$$\mathbb{L}_{\rho,s}(w) = -\Delta\, w + \rho L_s(w) \qquad \text{in } \mathscr{D}(w);$$

also the angle between the normals satisfies (see Proposition 4.6)

$$\rho^{-1}\,\left(\mathcal{B}(s,\rho,w)-\phi'(s)\right)=\mathbb{B}_{\rho,s}(w)+\mathcal{O}(1)+\rho\,\bar{Q}(w),$$

where

$$\mathbb{B}_{\rho,s}(w)=\left((\kappa'(s))^2\,\frac{\partial w}{\partial\eta}+\phi\phi'' w\right)+\rho\,\bar{L}_s(w)\qquad\text{on }\partial\mathscr{D}(w).$$

Here L_s (resp. $\bar{L}_s$) is a second order (resp. first order) differential operator and $Q(w)$, $\bar{Q}(w)$ are quadratic in w, see also the end of Section 2 for more precise definitions.

It turns out that the problem of finding w such that $\mathscr{D}(w)$ solves (4) namely

$$\begin{aligned}\mathcal{H}(s,\rho,w)&=0\qquad\text{in }\mathscr{D}(w),\\ \mathcal{B}(s,\rho,w)&=\phi'(s)\qquad\text{on }\partial\mathscr{D}(w),\end{aligned}$$

can be transformed to a fixed point problem for which the solvability is based on the invertibility of $\mathbb{L}_{\rho,s}$ on a suitable space of functions w such that $\mathbb{B}_{\rho,s}(w)=0$. If $\phi\phi''>0$, the operator $\mathbb{L}_{\rho,s}$ (resp. $-\mathbb{L}_{\rho,s}$) is invertible by means of usual Sobolev inequalities. Hence after suitable adjustment of the disc $\mathscr{D}(w)$, we readily prove the first theorem. This program is carried out in § 5.1.

Now in the situation where $\phi\equiv 1$ and κ =Id, it is clear that the linearized mean curvature $\mathbb{L}_{\rho,s}$ about any $D\in Z_\rho$ may have small (possibly zero) eigenvalues on the space of functions for which $\mathbb{B}_{\rho,s}(w)=\frac{\partial w}{\partial\eta}+\rho\,\bar{L}_s(w)=0$. This is related to the invariance by translations along the axis of rotation in the "flat" case. Hence $\mathbb{L}_{\rho,s}$ may not be invertible on such space. However restricting again ourselves on space of function orthogonal to the constant function 1, we can perturb Z_ρ to a manifold $\mathcal{Z}_\rho$ (constituted by sets having constant and small mean curvatures, see § 5.3.1) which turns out to be a *natural constraint* for $\mathcal{E}$ namely critical point of $\mathcal{E}\Big|_{\mathcal{Z}_\rho}$ is also stationary for $\mathcal{E}$. For that we use an argument from Kapouleas in [11] which was successfully employed in [15]. We will follow the argument of the latter, we refer to § 5.3.1.
It is worth noticing that this method is also closely related to variational-perturbative methods introduced by Ambrosetti and Badiale in [1] and subsequently used with success to get existence and multiplicity results for a wide class of variational problems in some perturbative setting we refer to the book by Ambrosetti Malchiodi [2] for more details and related applications.

Remark 1.5

- *If $\phi''\equiv 0$, namely when $\phi(s)=cs+d$, S parameterizes the cone we cannot conclude. However we notice that the proof of Theorem 1.3 highlights that near a point $\gamma(\rho\,s_0)$ for which $\phi''(s_0)=0$, there will be capillary surfaces with constant and small mean curvatures, see Remark 5.1.*
- *An interesting question is also to perturb the set*

$$\exp_{\gamma(\rho\,\kappa(s))}(\rho\,\phi(s)S^{n-1})$$

to a closed minimal submanifold of $\mathscr{C}^\rho$. One can see also the work by S.Secchi [19].

- *Another problem can be set as follows. Let $\mathcal{U}$ be a smooth bounded domain of $\mathcal{M}$, and $\Gamma\hookrightarrow\partial\mathcal{U}$ be a smooth curve. We let $\tilde{y}=(y^1,\dots,y^n)$ and $N_{\partial\mathcal{U}}$ be a unit interior normal field along $\partial\mathcal{U}$. Choosing an oriented orthogonal frame $(E_1\,\dots,E_{n-1})$ along Γ in $\partial\mathcal{U}$, one obtains a coordinate system by letting, for any $y=(\tilde{y},y^n)=(y^1,\dots,y^{n-1},y^n)$,*

$$F(x_0,\tilde{y},y^n):=\exp^{\mathcal{M}}_{\exp^{\partial\mathcal{U}}_{\gamma(x_0)}(y^iE_i)}(y^n\,N_{\partial\mathcal{U}}).$$

Now consider the set

$$F(\rho\,\kappa(s),\rho\,\phi(s)B^n_+),$$

where $B^n_+=\{x=(x^1,\dots,x^{n-1},x^n)\in\mathbb{R}^n\quad:\quad|x|=1,\quad x^n>0\}$. One may be tempted to perturb the set above into capillary minimal surfaces that meet the "half"-surface of revolution

$$F(\rho\,\kappa(s),\rho\,\phi(s)S^{n-1}_+)$$

by an angle equal to $\arccos \phi'$*. In this case, as we believe, a result like Theorem 1.1 would carry over. On the other hand one would need maybe to impose some conditions on the principal curvature of* $\partial\mathcal{U}$ *along* Γ *in order to obtain a variant of Theorem 1.3.*

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2 Preliminaries and notations

We consider $(\phi,\,\kappa)\,:\,[a,b]\to\mathbb{R}^2$ smooth with $\kappa'(s)$, $\phi(s)\neq 0$ for every $s\in[a,b]$ moreover we assume that s is the arc length of the rotating curve $s\mapsto(\phi(s),\kappa(s))$ precisely

$$(\phi'(s))^2+(\kappa'(s))^2=1 \qquad \forall s\in[a,b].$$

We also assume that x_0 is the arc length of γ, and we will let $E_0:=\gamma'$. We choose a parallel (local) orthonormal frame $E_1,\cdots,E_n$ of $N\Gamma$ along Γ. This determines a coordinate system by defining

$$f(x_0,y):=\exp_{\gamma(x_0)}(y^iE_i) \qquad \text{for } y=(y^1,\cdots,y^n)$$

which therefore defines coordinates vector fields :

$$Y_0:=f_*(\partial_{x_0}), \qquad Y_i:=f_*(\partial_{y_i}).$$

We will adopt the convention that the indices $i,j,k,\dots\in\{1,\dots,n\}$ while $\alpha,\beta,\dots\in\{0,\dots,n\}$ with $Y_\alpha=Y_0$ when $\alpha=0$.
By construction, $\nabla_{X_i}Y_0\Big|_\Gamma\in T\Gamma$ so that we can define

$$\langle\nabla_{X_i}Y_0,Y_0\rangle\Big|_\Gamma=-\Gamma_0^0(E_i).$$

There also holds

$$\nabla_{Y_i}Y_j(y)=O(|y|)_\gamma Y_\gamma. \tag{5}$$

If $q=f(x_0,y)\in\mathcal{M}$ near the point $p=f(x_0,0)\in\Gamma$, we can expand the metric $g_{\alpha\beta}(q)=\langle Y_\alpha,Y_\beta\rangle$ in y, (we refer to [13] Proposition 2.1 for the proof).

Lemma 2.1 *In the above coordinates* (x_0,y)*, for any* $i,j=1,...,n$*, we have*

$$\begin{aligned}
g_{ij}(q) &= \delta_{ij}+\tfrac{1}{3}\langle R_p(Y,E_i)Y,E_j\rangle+\tfrac{1}{6}\langle\nabla_Y R_p(Y,E_i)Y,E_j\rangle+\,O_p(|y|^4);\\
g_{0j}(q) &= \tfrac{2}{3}\langle R_p(Y,E_0)Y,E_j\rangle+O_p(|y|^3);\\
g_{00}(q) &= 1-2\Gamma_0^0(Y)+\,\langle R_p(Y,E_0)Y,E_0\rangle+\,O_p(|y|^3),
\end{aligned}$$

where $Y:=y^iE_i$*.*

Notation for error terms: Any expression of the form $L(\omega)$ (resp. $\bar{L}(\omega)$) denotes a linear combination of the function ω together with its derivatives with respect to the vector fields Y_i up to order 2 (resp. order 1). The coefficients of L or $\bar{L}$ might depend on ρ and s but, for all $k\in\mathbb{N}$, there exists a constant $c>0$ independent of $\rho\in(0,1)$ and $s\in[a,b]$ such that

$$\|L(\omega)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^n})}\le c\,\|\omega\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^n})},$$

$$\|\bar{L}_s(\omega)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^n})}\le c\,\|\omega\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_+^n})}.$$

Similarly, any expression of the form $Q(\omega)$ (resp $\bar{Q}(\omega)$) denotes a nonlinear operator in the function ω together with its derivatives with respect to the vector fields Y_i up to order 2 (resp. 1). The coefficients of the Taylor expansion of $Q^a(\omega)$ in powers of ω and its partial derivatives might depend on ρ and s and, given $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\rho \in (0,1)$ and $s \in [a,b]$ such that

$$\|Q(\omega_1) - Q(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^n})} \leq c \left(\|\omega_1\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^n})} + \|\omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^n})} \right) \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^n})},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^n})} \leq 1$, $i = 1, 2$. Also

$$\|\bar{Q}(\omega_1) - \bar{Q}(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{B_1^n})} \leq c \left(\|\omega_1\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_1^n})} + \|\omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_1^n})} \right) \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{B_1^n})},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{B_1^n})} \leq 1$. We also agree that any term denoted by $\mathcal{O}(r^d)$ (with $r \in \mathbb{R}$ may depend on s) is a smooth function on B_1^n that might depend on s but satisfies

$$\| \frac{\mathcal{O}(r^d)}{|r|^d} \|_{\mathcal{C}^{k,\alpha}(\overline{B_1^n})} \leq c$$

for a constant c independent of s.

3 On the surface of revolution around Γ

We start by fixing the following notations which will be useful later.

Notations:

Through the following of this paper,

$$\varepsilon(s) = \rho\, \phi(s) \qquad \text{and} \qquad \varepsilon_1(s) = \rho\, \kappa(s) \qquad \text{for every } s \in [a,b].$$

In terms of cylindrical coordinates, letting $\Theta(z) : \mathbb{R}^{n-1} \to S^{n-1}$, the surface of revolution $\mathscr{C}^\varepsilon$ around Γ can be parameterized by

$$\mathcal{C}^\varepsilon(s,z) := f(\varepsilon_1(s), \varepsilon(s)\Theta(z)) = \exp_{\gamma(\varepsilon_1(s))}(\varepsilon(s)\Theta^i(z)\, E_i).$$

The tangent plane is spanned by the vector fields

$$\begin{aligned} Z_0^c &= \mathcal{C}^\rho(\partial_{x_0}) = \varepsilon_1' Y_0 + \varepsilon'\, \Upsilon, \\ Z_j^c &= \mathcal{C}^\rho(\partial_{z^j}) = \varepsilon\, \Upsilon_j, \qquad j = 1, \cdots, n, \end{aligned}$$

where

$$\Upsilon = \Theta^i Y_i.$$

We recall also from [13]

Lemma 3.1 *Let $q = \mathcal{C}^\rho(s,z) \in \mathscr{C}^\rho$, there hold*

$$\begin{aligned} \langle \Upsilon, \Upsilon \rangle_q &= 1, \\ \langle \Upsilon, Y_0 \rangle_q &= 0, \\ \langle \Upsilon, \Upsilon_j \rangle_q &= 0. \end{aligned}$$

Lemma 3.2 *In the notations above, the first fundamental form of $\mathscr{C}^\rho$ has the following expansions*

$$\begin{aligned}
\langle Z_0^c, Z_0^c\rangle &= \frac{\varepsilon^2}{\phi^2} - 2\varepsilon|\varepsilon_1'|^2\Gamma_0^0(\Theta) + \varepsilon^2|\varepsilon_1'|^2\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^5),\\
\langle Z_l^c, Z_k^c\rangle &= \varepsilon^2\langle\Theta_l,\Theta_k\rangle + \mathcal{O}(\varepsilon^4),\\
\langle Z_0^c, Z_k^c\rangle &= \mathcal{O}(\varepsilon^4).
\end{aligned}$$

Proof. Recalling that

$$|\varepsilon_1'|^2 + |\varepsilon'|^2 = \frac{\varepsilon^2}{\phi^2}$$

hence we obtain, using also the Lemmas 3.1, 2.1, that

$$\begin{aligned}
\langle Z_0^c, Z_0^c\rangle &= |\varepsilon_1'|^2\left(1 - 2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^3)\right) + |\varepsilon'|^2\\
&= \frac{\varepsilon^2}{\phi^2} - 2\varepsilon|\varepsilon_1'|^2\Gamma_0^0(\Theta) + \varepsilon^2|\varepsilon_1'|^2\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^5)
\end{aligned}$$

The other expansions are easy consequences of the Lemmas 3.1 2.1. ■

3.1 The unit normal field to the surface of revolution

Call

$$M := \varepsilon' X_0 - \varepsilon_1'\Upsilon$$

and set

$$\tilde N_c(s,z) = M + \alpha_0 Z_0^c + \alpha_k Z_k^c.$$

Note that this vector filed is normal (not necessary unitary) to the surface whenever we can determined α_k so that $\langle\tilde N_c, Z_k^c\rangle = \langle\tilde N_c, Z_0^c\rangle = 0$ for all $k = 1,\dots,n$. This therefore leads to solving a linear system.

Observe that

$$\begin{aligned}
\langle M, Z_0^c\rangle &= \varepsilon_1'\varepsilon'\langle Y_0,Y_0\rangle - \varepsilon_1'\varepsilon'\langle\Upsilon,\Upsilon\rangle\\
&= \varepsilon_1'\varepsilon'\left(1 - 2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta,E_0)\Theta,E_0\rangle - 1 + \mathcal{O}(\varepsilon^3)\right),
\end{aligned}$$

hence

$$\langle M, Z_0^c\rangle = -2\varepsilon_1'\varepsilon'\varepsilon\Gamma_0^0(\Theta) + \varepsilon_1'\varepsilon'\varepsilon^2\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^5).$$

Also we have

$$\langle M, Z_k^c\rangle = \varepsilon\varepsilon'\langle X_0,Y_k\rangle - \varepsilon_1'\varepsilon\langle\Upsilon,Y_k\rangle = \mathcal{O}(\varepsilon^4).$$

If we use Lemma 3.2, we have

$$\begin{aligned}
\alpha_0\langle Z_0^c, Z_0^c\rangle &= -\alpha_k\langle Z_k^c, Z_0^c\rangle - \langle M, Z_0^c\rangle\\
&= \alpha_k\mathcal{O}(\varepsilon^4) + 2\varepsilon\varepsilon_1'\varepsilon'\Gamma_0^0(\Theta) - \varepsilon^2\varepsilon_1'\varepsilon'\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^5)
\end{aligned}$$

so

$$(6)\quad \alpha_0 = \frac{\varepsilon'\varepsilon_1'}{\varepsilon^2}\phi^2\left(2\varepsilon\Gamma_0^0(\Theta) - |\varepsilon_1'|^2\langle R(\Theta,E_0)\Theta,E_0\rangle\right) + 4\phi^2\frac{|\varepsilon_1'|^3}{\varepsilon^2}\varepsilon'\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) + \mathcal{O}(\varepsilon^3) + \alpha_k\mathcal{O}(\varepsilon^4).$$

Since

$$\alpha_k\langle Z_k^c, Z_l^c\rangle + \alpha_0\langle Z_0^c, Z_l^c\rangle = -\langle M, Z_l^c\rangle$$

and using (6)

$$\alpha_k\langle Z_k^c, Z_l^c\rangle + \alpha_k\mathcal{O}(\varepsilon^6) + \mathcal{O}(\varepsilon^5) = \mathcal{O}(\varepsilon^4)$$

we get

$$\alpha_k\left(\varepsilon^2\langle\Theta_l,\Theta_k\rangle + \mathcal{O}(\varepsilon^4)\right) = \mathcal{O}(\varepsilon^4),$$

therefore

$$\alpha_k = \mathcal{O}(\varepsilon^2).$$

Recalling that $\varepsilon = \rho\,\phi$ while $\varepsilon_1 = \rho\,\kappa$ we define $\bar\alpha_0$ by the relation

$$\alpha_0 = \phi'\bar\alpha_0 + \mathcal{O}(\varepsilon^3).$$

Namely

$$\bar\alpha_0 = 2\varepsilon_1'\phi\Gamma_0^0(\Theta) - \varepsilon'\varepsilon_1'\phi\langle R(\Theta,E_0)\Theta,E_0\rangle + 4\varepsilon_1'\kappa'\varepsilon\Gamma_0^0(\Theta)\Gamma_0^0(\Theta).$$

Now let us compute the norm of this normal vector field. Since

$$\tilde N_c(s,z) := M + \alpha_0 Z_0^c + \alpha_k Z_k^c$$

we have by construction

$$\langle\tilde N_c,\tilde N_c\rangle = \langle M,M\rangle + a_0^2\langle Z_0^c,Z_0^c\rangle + \alpha_k\alpha_l\langle Z_k^c,Z_l^c\rangle + 2\alpha_0\langle M,Z_0^c\rangle + 2\alpha_k\langle M,Z_k^c\rangle + 2\alpha_k\alpha_0\langle Z_k^c,Z_0^c\rangle.$$

Notice that

$$\begin{aligned}\alpha_0\langle Z_0^c,Z_0^c\rangle &= -\langle M,Z_0^c\rangle - \alpha_k\langle Z_k^c,Z_0^c\rangle\\ &= -\langle M,Z_0^c\rangle + \mathcal{O}(\varepsilon^6)\end{aligned}$$

and

$$\alpha_0^2\langle Z_0^c,Z_0^c\rangle = -\alpha_0\langle M,Z_0^c\rangle + \mathcal{O}(\varepsilon^6),$$

hence

$$\langle\tilde N_c,\tilde N_c\rangle = \langle M,M\rangle + \alpha_0\langle M,Z_0^c\rangle + \mathcal{O}(\varepsilon^6).$$

Now observe that

$$\begin{aligned}\langle M,M\rangle &= |\varepsilon'|^2\langle X_0,X_0\rangle + |\varepsilon_1'|^2\langle\Upsilon,\Upsilon\rangle\\ &= \frac{\varepsilon^2}{\phi^2} + |\varepsilon'|^2\left(-2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^3)\right)\end{aligned}$$

and

$$\alpha_0\langle M,Z_0^c\rangle = -2\varepsilon\varepsilon_1'\varepsilon'\Gamma_0^0(\Theta)\alpha_0 + \mathcal{O}(\varepsilon^4)\alpha_0 = -4\frac{\varepsilon^2}{\rho^2}|\varepsilon'|^2|\varepsilon_1'|^2\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) + \mathcal{O}(\varepsilon^5).$$

So we have

$$\tfrac{\phi^2}{\varepsilon^2}\langle\tilde N_c,\tilde N_c\rangle \;=\; 1 + \tfrac{|\varepsilon'|^2}{\varepsilon^2}\phi^2\left(-2\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2\langle R(\Theta,E_0)\Theta,E_0\rangle\right) - 4\tfrac{\phi^4}{\varepsilon^2}|\varepsilon'|^2|\varepsilon_1'|^2\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) + \mathcal{O}(\varepsilon^3)$$

Finally we conclude that

$$\tfrac{\varepsilon}{\phi}|\tilde N_c|^{-1} \;=\; 1 + \tfrac{|\varepsilon'|^2}{\varepsilon}\phi^2\Gamma_0^0(\Theta) + \left(3|\tfrac{\varepsilon'}{\varepsilon}|^4\phi^4\varepsilon^2 + 2|\tfrac{\varepsilon_1'}{\varepsilon}|^2|\varepsilon'|^2\phi^4\right)\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) - \frac{|\varepsilon'|^2}{2}\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^3)$$

and, setting

$$H_c(\Theta,\Theta) := \left(3\frac{|\varepsilon'|^2}{\varepsilon^2} + 2\frac{|\varepsilon_1'|^2}{\varepsilon^2}\right)\phi^2\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) - \frac{1}{2}\langle R(\Theta,E_0)\Theta,E_0\rangle,$$

we can simply write

$$\frac{\varepsilon}{\phi}|\tilde N_c|^{-1} = 1 + |\varepsilon'|^2\phi^2\left[\frac{1}{\varepsilon}\Gamma_0^0(\Theta) + H_c(\Theta,\Theta)\right] + \mathcal{O}(\varepsilon^3).$$

We collect all these in the following

Proposition 3.3 *There exists an interior (non unit) normal vector field of $\mathscr{C}^\rho$ which has the following expansions*

$$\tilde{N}_{\mathscr{C}^\rho}(s,z) = -\varepsilon_1'\Upsilon + \varepsilon' Y_0 + (\phi'\bar{\alpha}_0 + \mathcal{O}(\varepsilon^3))Z_0^c + \alpha_k Z_k^c,$$

where

$$\bar{\alpha}_0 = 2\varepsilon_1'\phi\Gamma_0^0(\Theta) - \varepsilon'\varepsilon_1'\phi\langle R(\Theta,E_0)\Theta,E_0\rangle + 4\varepsilon_1'\kappa'\varepsilon\Gamma_0^0(\Theta)\Gamma_0^0(\Theta);$$
$$\alpha_k = \mathcal{O}(\varepsilon^2).$$

Moreover

$$\rho\left|\tilde{N}_{\mathscr{C}^\rho}\right|^{-1} = 1 + |\phi'|^2\left(\varepsilon\Gamma_0^0(\Theta) + \varepsilon^2 H_c(\Theta,\Theta)\right) + \mathcal{O}(\varepsilon^3),$$

where

$$H_c(\Theta,\Theta) = \left(|\phi'|^2 + 2\phi^4\right)\Gamma_0^0(\Theta)\Gamma_0^0(\Theta) - \frac{1}{2}\langle R(\Theta,E_0)\Theta,E_0\rangle + \mathcal{O}(\varepsilon^3).$$

4 Discs centered on Γ with boundary on $\mathscr{C}^\rho$

For $\delta > 0$, B_δ^n will denote the ball of $\mathbb{R}^n$ with radius δ centered at the origin. For any s, we consider the disc $\mathscr{D}^{s,\varepsilon}$ of radius ε centered at $\gamma(\varepsilon(s))$ given by

$$\mathscr{D}^{s,\varepsilon} := f(\varepsilon_1(s)\,,\,\varepsilon(s)\,B_1^n),$$

parameterized by

$$B_1^n \ni x \mapsto \mathcal{D}^{s,\varepsilon}(x) = f(\varepsilon_1,\,\varepsilon\,x).$$

Notations

$$\bar{\varepsilon}(s,t) := \varepsilon(s+\varepsilon(s)t) \qquad \bar{\varepsilon}_1(s,t) := \varepsilon_1(s+\varepsilon(s)t);$$

$$\bar{\varepsilon}'(s,t) := \partial_t\varepsilon(s+\varepsilon(s)t) = \varepsilon(s)\varepsilon'(s+\varepsilon(s)t) \qquad \bar{\varepsilon}_1'(s,t) := \partial_t\varepsilon_1(s+\varepsilon(s)t) = \varepsilon(s)\varepsilon_1'(s+\varepsilon(s)t).$$

Notice that

$$\bar{\varepsilon}'(s,t) = (\varepsilon' + \varepsilon\varepsilon''\,t + \varepsilon^3\mathcal{O}(t^2))\varepsilon, \qquad \bar{\varepsilon}_1'(s,t) = (\varepsilon_1' + \varepsilon\varepsilon_1''\,t + \varepsilon^3\mathcal{O}(t^2))\varepsilon. \tag{7}$$

A parametrization of the neighborhood of the disc (in $\mathcal{M}$) centered at $p = \gamma(\varepsilon_1(s)) \in \Gamma$ with radius ε (ρ small) can be defined by

$$F^s(t,x) := f(\bar{\varepsilon}_1(s,t)\,,\,\bar{\varepsilon}(s,t)\,x) \qquad \forall x \in B_1^n,\ |t| \ll 1 \tag{8}$$

note that by construction,

$$F^s(t\,,\,\partial B_1^n) \subset \mathscr{C}^\rho, \qquad \forall |t| \ll 1$$

and more precisely, for every $|t| \ll 1$

$$F^s(t,x) = \mathcal{C}^\rho(s+\varepsilon\,t, \varepsilon(s+\varepsilon\,t)\,x) \qquad \forall x \in \partial B_1^n = S^{n-1}.$$

We consider the following vector fields induced by F^s

$$\begin{aligned}\varepsilon\,T_0 \quad &:= \quad F_*(\partial_t) \quad = \quad \bar{\varepsilon}_1'\,Y_0(t) + \bar{\varepsilon}'\,X(t),\\ \bar{\varepsilon}\,T_j \quad &:= \quad F_*(\partial_{y^j}) \quad = \quad \bar{\varepsilon}\,Y_j(t).\end{aligned}$$

Here $X = x^i E_i$.

Lemma 4.1 *At $q = F^s(t,x)$, we have*

$$\begin{aligned}
\langle Y_i(t), Y_j(t)\rangle_q &= \delta_{ij} + \tfrac{\varepsilon^2}{3}\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\langle \nabla_X R_p(X,E_i)X,E_j\rangle + \mathcal{O}_p(\varepsilon^4) + \varepsilon^3\mathcal{O}(t) + \varepsilon^4\mathcal{O}_p(t^2);\\
\langle Y_0(t), Y_j(t)\rangle_q &= \tfrac{2\varepsilon^2}{3}\langle R_p(X,E_0)X,E_j\rangle + \mathcal{O}_p(\varepsilon^3) + \varepsilon^3\mathcal{O}_p(t) + \varepsilon^4\mathcal{O}_p(t^2);\\
\langle Y_0(t), Y_0(t)\rangle_q &= 1 - 2\bar\varepsilon\Gamma_0^0(X) + 2\varepsilon^2 U_0^0(X)t + \varepsilon^2\langle R_p(X,E_0)X,E_0\rangle + \mathcal{O}_p(\varepsilon^3) + \varepsilon^3\mathcal{O}_p(t) + \varepsilon^4\mathcal{O}_p(t^2),
\end{aligned}$$

where $p = \gamma(\varepsilon_1(s)) \in \Gamma$ and

$$\varepsilon U_0^0(X) = \varepsilon_1'\Gamma_{00}^0 - \varepsilon'\Gamma_0^0(X).$$

Proof. There holds

$$\begin{aligned}
\frac{d}{dt}\langle Y_0(t), Y_0(t)\rangle_q\Big|_{t=0} &= 2\langle \nabla_{\varepsilon T_0} Y_0, Y_0\rangle\Big|_{t=0}\\
&= 2\bar\varepsilon_1'\langle \nabla_{Y_0}Y_0, Y_0\rangle\Big|_{t=0} + 2\bar\varepsilon'\langle \nabla_X Y_0, Y_0\rangle\Big|_{t=0}\\
&= 2\varepsilon(\varepsilon_1'\Gamma_{00}^0 - \varepsilon'\Gamma_0^0(X)) + \mathcal{O}_p(\varepsilon^3).
\end{aligned}$$

we have used (7). Hence the last expansion follows. On the other hand one has

$$\begin{aligned}
\frac{d}{dt}\langle Y_0(t), Y_i(t)\rangle_q\Big|_{t=0} &= \langle \nabla_{\varepsilon T_0} Y_0, Y_i\rangle\Big|_{t=0} + \langle Y_0, \nabla_{\varepsilon T_0} Y_i\rangle\Big|_{t=0}\\
&= \bar\varepsilon_1'\langle \nabla_{Y_0}Y_0, Y_i\rangle\Big|_{t=0} + 2\bar\varepsilon'\langle \nabla_X Y_0, Y_i\rangle\Big|_{t=0} + \bar\varepsilon_1'\langle Y_0, \nabla_{Y_0}Y_i\rangle\Big|_{t=0} + 2\bar\varepsilon'\langle Y_0, \nabla_X Y_i\rangle\Big|_{t=0}
\end{aligned}$$

Since by construction $\langle \nabla_{E_0}E_0, E_i\rangle + \langle E_0, \nabla_{E_0}E_i\rangle = 0$ and also $\langle \nabla_X E_0, E_i\rangle + \langle E_0, \nabla_X E_i\rangle = 0$ on Γ, we infer that

$$\frac{d}{dt}\langle Y_0(t), Y_i(t)\rangle_q\Big|_{t=0} = \mathcal{O}_p(\varepsilon^3).$$

In the same vain, the first expansions follows similarly. ■

Using the above lemma, and (7) we get

Lemma 4.2 *The following expansions hold*

$$\begin{aligned}
\langle T_i, T_j\rangle &= \delta_{ij} + \tfrac{\varepsilon^2}{3}\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\langle \nabla_X R_p(X,E_i)X,E_j\rangle + \mathcal{O}_p(\varepsilon^4) + \varepsilon^3\mathcal{O}(t) + \varepsilon^4\mathcal{O}_p(t^2);\\
\langle T_0, T_j\rangle &= \varepsilon' x^i + \mathcal{O}_p(\varepsilon^3) + \varepsilon^4\mathcal{O}_p(t) + \varepsilon^5\mathcal{O}_p(t^2);\\
\langle T_0, T_0\rangle &= \tfrac{\varepsilon^2}{\phi^2} + |\varepsilon_1'|^2\left(-2\bar\varepsilon\Gamma_0^0(X) + 2\varepsilon^2 U_0^0(X)t\right) + \mathcal{O}_p(\varepsilon^5) + \varepsilon^5\mathcal{O}_p(t) + \varepsilon^6\mathcal{O}_p(t^2),
\end{aligned}$$

where $p = \gamma(\varepsilon_1(s)) \in \Gamma$.

Observe that all disc-type surfaces nearby $\mathscr{D}_{s,\rho}$ with boundary contained in $\mathscr{C}^\rho$ can be parameterized by

$$G^s(x) := F^s(w(x)\,,\,x), \tag{9}$$

for some smooth function $w : B_1^n \to \mathbb{R}$. We will call $\mathscr{D}^{s,\varepsilon(s)}(w) = G^s(B_1^n)$.

4.1 Mean curvature of Perturbed disc $\mathscr{D}(w)$

It is not difficult to see that the tangent plane of $\mathscr{D}(w) = \mathscr{D}^{s,\varepsilon(s)}(w)$ is spanned by the vector fields

$$Z_j \;=\; G^s_*(\partial_{x^j}) \;=\; \varepsilon\, w_{x^j} T_0 + \bar\varepsilon(s,w)\, T_j.$$

From Lemma 4.2, it is clear that at the point $q = G^s(x) = F^s(w(x)\,,\,x)$ there hold

(10)

$$
\begin{aligned}
\langle T_i, T_j\rangle_q &= \delta_{ij} + \tfrac{\varepsilon^2}{3}\,\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\,\langle \nabla_X R_p(X,E_i)X,E_j\rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^4 Q(w);\\
\langle T_0, T_j\rangle_q &= \varepsilon' x^j + \mathcal{O}(\varepsilon^3) + \varepsilon^4 L(w) + \varepsilon^5 Q(w);\\
\langle T_0, T_0\rangle_q &= \tfrac{\varepsilon^2}{\phi^2} + |\varepsilon_1'|^2\left(-2\bar\varepsilon\Gamma_0^0(X) + 2\varepsilon^2 U_0^0(X) w\right) + \mathcal{O}_p(\varepsilon^5) + \varepsilon^5 L(w) + \varepsilon^6 Q(w).
\end{aligned}
$$

Observing that $\bar\varepsilon(s,w) = \varepsilon + \varepsilon\varepsilon'\,w + \varepsilon^3 Q(w)$ and using (10), we get the first fundamental form $h_{ij} := \langle Z_i, Z_j\rangle$,

(11)

$$
\begin{aligned}
\varepsilon^{-2} h_{ij} &= (1 + 2\varepsilon'\,w)\,\delta_{ij} + \varepsilon'(w_{x^i}x^j + w_{x^j}x^i) + \tfrac{\varepsilon^2}{3}\,\langle R_p(X,E_i)X,E_j\rangle + \tfrac{\varepsilon^3}{6}\,\langle \nabla_X R_p(X,E_i)X,E_j\rangle\\
&+ \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).
\end{aligned}
$$

4.1.1 The normal vector field

Considering the vector field

$$\tilde N_{\mathscr D} = Y_0 + a_k Z_k.$$

Observe that it is normal (not necessary unitary) to the disc whenever we can find a_k such that $\langle \tilde N_{\mathscr D}, Z_k\rangle = 0$ for any $k = 1,\dots,n$. Namely a_k satisfies

$$\mathscr{D}^{s,\varepsilon(s)}(w)\,a_k\,h_{ik} = -\langle Y_0, Z_i\rangle. \tag{12}$$

Since

$$\langle Y_0, Z_i\rangle = \bar\varepsilon_1' w_{x^i}\langle Y_0, Y_0\rangle + \bar\varepsilon' w_{x^i}\langle Y_0, X\rangle + \bar\varepsilon\langle Y_0, Y_i\rangle$$

then from (11) and (7), we get the formula

$$\varepsilon^2 a_k = \varepsilon\varepsilon_1' w_{x^k}\langle Y_0, Y_0\rangle + \varepsilon\langle Y_0, Y_k\rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^4 L(w) + \varepsilon^3 Q(w). \tag{13}$$

And also since $\bar\varepsilon = \varepsilon + \varepsilon^2 L(w)$, $\bar\varepsilon_1 = \varepsilon_1 + \varepsilon^2 L(w)$, we get

$$\varepsilon^2\,a_k = -\varepsilon\varepsilon_1'(1 - 2\varepsilon\Gamma_0^0(X))w_{x^k} - \frac{2\varepsilon^3}{3}\,\langle R(X,E_0)X,E_k\rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^4 L(w) + \varepsilon^3 Q(w)$$

and thus

$$a_k = -\frac{\varepsilon_1'}{\varepsilon}(1 - 2\varepsilon\Gamma_0^0(X))w_{x^i} - \frac{2\varepsilon}{3}\,\langle R(X,E_0)X,E_i\rangle + \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon Q(w).$$

Moreover using also (12) we have

$$
\begin{aligned}
\langle \tilde N_{\mathscr D}, \tilde N_{\mathscr D}\rangle &= \langle Y_0, Y_0\rangle - a_k a_l\,h_{kl}\\
&= \langle Y_0, Y_0\rangle - a_k(\mathcal{O}(\varepsilon^3) + \varepsilon^2 L(w) + \varepsilon^4 Q(w))\\
&= \langle Y_0, Y_0\rangle - \big(\mathcal{O}(\varepsilon) + L(w) + \varepsilon^2 Q(w)\big)\big(\mathcal{O}(\varepsilon^3) + \varepsilon^2 L(w) + \varepsilon^2 Q(w)\big)\\
&= \langle Y_0, Y_0\rangle + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).
\end{aligned}
$$

Hence

$$\left|\langle \tilde N_{\mathscr D}, \tilde N_{\mathscr D}\rangle\right|^{-1} = |\langle Y_0, Y_0\rangle|^{-1} + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^2 Q(w) \tag{14}$$

Therefore

$$\left|\langle \tilde N_{\mathscr D}, \tilde N_{\mathscr D}\rangle\right|^{-1} = 1 + \bar\varepsilon\Gamma_0^0(X) + \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon^2 Q(w). \tag{15}$$

We then conclude that the unit normal has the following expansions:

$$
\begin{aligned}
N_{\mathscr{D}} &= \left(1+\varepsilon\Gamma_0^0(X)+\mathcal{O}(\varepsilon^2)+\varepsilon^2 L(w)+\varepsilon^2 Q(w)\right) Y_0 \\
&+ \left(-\tfrac{\varepsilon_1'}{\varepsilon}(1+\varepsilon\Gamma_0^0(X))w_{x^k} - \tfrac{2\varepsilon}{3}\langle R(X,E_0)X,E_k\rangle + \mathcal{O}(\varepsilon^2)+\varepsilon^2 L(w)+\varepsilon Q(w)\right) Z_k.
\end{aligned}
\tag{16}
$$

Sometimes we will simply need to write $N_{\mathscr{D}}$ in the more compact form

$$N_{\mathscr{D}} = Y_0 + \left(\mathcal{O}(\varepsilon^2)+\varepsilon L(w)+\varepsilon^2 Q(w)\right)_\alpha Y_\alpha.$$

4.1.2 The Second Fundamental Form

Observe that in the scaled variables $y=\varepsilon\, x$, since the functions $\mathcal{O}(\varepsilon^m)$, $L(w)$ and $Q(w)$ are depending on x whereas the vector fields Y_α depend on $y=\varepsilon\, x$, we have for any

$$E_i(\mathcal{O}(\varepsilon^m)) = \mathcal{O}(\varepsilon^{m-1}), \qquad E_i(\varepsilon^m L(w)) = \varepsilon^{m-1}L(w), \qquad E_i(\varepsilon^m Q(w)) = \varepsilon^{m-1}Q(w).$$

Having this in mind, we state the following

Lemma 4.3 *There holds*

$$\langle T_0, \nabla_{Z_i} N_{\mathscr{D}}\rangle = \mathcal{O}(\varepsilon^3)+\varepsilon L(w)+\varepsilon^2 Q(w).$$

PROOF. Using (16) and recall that $T_0 = \varepsilon_1' Y_0 + \varepsilon' X + \varepsilon L(w)_\alpha Y_\alpha$, we have

$$
\begin{aligned}
\langle T_0, \nabla_{Z_i} N_{\mathscr{D}}\rangle\Big|_{w=0} &= \langle T_0, \nabla_{\varepsilon Y_i}(1+\varepsilon\Gamma_0^0(X))Y_0\rangle + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon\Gamma_0^0(E_i)\langle T_0, Y_0\rangle + \varepsilon(1+\varepsilon\Gamma_0^0(X))\langle T_0, \nabla_{Y_i}Y_0\rangle + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon\varepsilon_1'\Gamma_0^0(E_i)\langle Y_0, Y_0\rangle + \varepsilon\varepsilon_1'(1+\varepsilon\Gamma_0^0(X))\langle Y_0, \nabla_{Y_i}Y_0\rangle + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon\varepsilon_1'\Gamma_0^0(E_i) - \varepsilon\varepsilon_1'\Gamma_0^0(E_i) + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Hence we get the result. ■

Let us now estimate the second fundamental form of $\mathscr{D}(w)$.

Lemma 4.4

$$
\begin{aligned}
\langle \nabla_{Z_i} Z_j, N_{\mathscr{D}}\rangle &= \varepsilon\left(1-\varepsilon\Gamma_0^0(E_l)x^l\right) w_{x^i x^j} + \varepsilon^2\langle \nabla_{Y^i} Y_j, Y_0\rangle + \mathcal{O}(\varepsilon^4) \\
&- \varepsilon^2\left(w_{x^j}\Gamma_0^0(E_i) + w_{x^i}\Gamma_0^0(E_j)\right) + \varepsilon^3 L(w) + \varepsilon^3 Q(w).
\end{aligned}
$$

PROOF. we have

$$\langle \nabla_{Z_i} Z_j, N_{\mathscr{D}}\rangle = \varepsilon\langle \nabla_{Z_i}(w_{x^j}T_0), N\rangle + \langle \nabla_{Z_i}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle.$$

We first estimate $\langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle$.
Observe that

$$\frac{\partial}{\partial x^i}\langle w_{x^j}T_0, N_{\mathscr{D}}\rangle = \langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle + \langle w_{x^j}T_0, \nabla_{Z_i}N_{\mathscr{D}}\rangle,$$

which implies that

$$\langle \nabla_{Z_i}(w_{x^j}T_0), N_{\mathscr{D}}\rangle = \frac{\partial}{\partial x^i}\langle w_{x^j}T_0, N_{\mathscr{D}}\rangle - w_{x^j}\langle T_0, \nabla_{Z_i}N_{\mathscr{D}}\rangle.$$

The formula (12) shows that

$$\langle Y_0, \tilde{N}_{\mathscr{D}}\rangle = \langle Y_0, Y_0\rangle + a_k\langle Z_k, Y_0\rangle = \langle Y_0, Y_0\rangle - a_k a_l\langle Z_k, Z_l\rangle = \left|\tilde{N}_{\mathscr{D}}\right|^2$$

and then

$$\langle Y_0, N_{\mathscr{D}} \rangle = \left| \tilde{N}_{\mathscr{D}} \right| .$$

From the fact that $\langle Y_0, X\rangle = 0$ when $w = 0$ and that

$$\langle Z_k, X\rangle = \varepsilon x^k + \mathcal{O}(\varepsilon^4) + \varepsilon^2 L(w) + \varepsilon^5 Q(w)$$

we obtain $a_k\langle Z_k, X\rangle = \mathcal{O}(\varepsilon^2) + \varepsilon L(w) + \varepsilon^2 Q(w)$, from which the following hold

$$\begin{aligned}
\langle \varepsilon T_0, N_{\mathscr{D}}\rangle &= \bar{\varepsilon}_1' \langle Y_0, N_{\mathscr{D}}\rangle + \bar{\varepsilon}'\langle X, N_{\mathscr{D}}\rangle \\
&= \bar{\varepsilon}_1' \left| \tilde{N}_{\mathscr{D}} \right| + \bar{\varepsilon}'(\varepsilon^2 + \varepsilon L(w) + \varepsilon^2 Q(w)) \\
&= \varepsilon\varepsilon_1'(1 - \varepsilon\Gamma_0^0(X)) + \mathcal{O}(\varepsilon^4) + \varepsilon^3 L(w) + \varepsilon^4 Q(w).
\end{aligned}$$

From this, we deduce that

$$\frac{\partial}{\partial x^i}\langle w_{x^j} T_0, N_{\mathscr{D}}\rangle = \varepsilon_1' \left(1 - \varepsilon\Gamma_0^0(X)\right) w_{x^i x^j} - \varepsilon\varepsilon_1'\Gamma_0^0(E_i) w_{x^j} + \varepsilon^3 L(w) + \varepsilon^2 Q(w).$$

We conclude using also Lemma 4.3 that

$$\langle \nabla_{Z_i}(w_{x^j} T_0), N_{\mathscr{D}}\rangle = \varepsilon_1' \left(1 - \varepsilon\Gamma_0^0(X)\right) w_{x^i x^j} - \varepsilon\varepsilon_1'\Gamma_0^0(E_i) w_{x^j} + \varepsilon^3 L(w) + \varepsilon^2 Q(w). \tag{17}$$

It remains the term $\langle \nabla_{Z_i}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle = \varepsilon w_{x^i}\langle \nabla_{T_0}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle + \bar{\varepsilon}\langle \nabla_{T_i}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle$. Since

$$\bar{\varepsilon}(s, w) = \varepsilon(s) + \varepsilon L(w) + \varepsilon^3 Q(w), \tag{18}$$

we can write

$$\langle \nabla_{T_i}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle \;=\; \varepsilon\langle \nabla_{T_i} T_j, N\rangle + \langle \nabla_{T_i}((\varepsilon^2 L + \varepsilon^3 Q) T_j), N_{\mathscr{D}}\rangle.$$

Recalling that $\nabla_{Y_i} Y_j = (\mathcal{O}(\varepsilon) + \varepsilon L + \varepsilon^2 Q)_\alpha Y_\alpha$ also $\langle T_i, N_{\mathscr{D}}\rangle = \varepsilon^2 + \varepsilon L + \varepsilon^2 Q$ thus

$$\langle \nabla_{T_i}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle \;=\; \varepsilon\langle \nabla_{Y_i} Y_j, Y_0\rangle\Big|_{w=0} + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^3 Q(w).$$

Moreover (7) and (18) yield

$$\langle \nabla_{T_0}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle\Big|_{w=0} \;=\; \varepsilon\varepsilon\varepsilon_1'\langle \nabla_{Y_0} Y_j, N_{\mathscr{D}}\rangle + \varepsilon\varepsilon\varepsilon'\langle \nabla_X Y_j, N_{\mathscr{D}}\rangle = -\varepsilon\varepsilon\varepsilon_1'\Gamma_0^0 + \mathcal{O}(\varepsilon^4).$$

This implies that

$$\langle \nabla_{T_0}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle = -\varepsilon\varepsilon\varepsilon_1'\Gamma_0^0 + \mathcal{O}(\varepsilon^4) + \varepsilon^2 L(w) + \varepsilon^3 Q(w).$$

Finally, collecting these and using (18) it turns out that

$$\langle \nabla_{Z_i}(\bar{\varepsilon}T_j), N_{\mathscr{D}}\rangle = \varepsilon\varepsilon\langle \nabla_{Y_i} Y_j, Y_0\rangle\Big|_{w=0} + \mathcal{O}(\varepsilon^4) - \varepsilon\varepsilon\varepsilon_1'\Gamma_0^0(E_j) w_{x^i} + \varepsilon^4 L(w) + \varepsilon^3 Q(w). \tag{19}$$

The result follows from (17) and (19). ■

We need also to expand more precisely $\langle \nabla_{Y_i} Y_j, Y_0\rangle\Big|_{w=0}$. By construction it vanish on Γ and

$$Y_l\langle \nabla_{Y_i} Y_j, Y_0\rangle = \langle \nabla_{Y_l}\nabla_{Y_i} Y_j, Y_0\rangle + \langle \nabla_{Y_i} Y_j, \nabla_{Y_l} Y_0\rangle$$

Furthermore by (5) and since (see for instance [8] Lemma 9.20)

$$\nabla_{Y_l}\nabla_{Y_i} Y_j\Big|_{\gamma(x^0)} = -\frac{1}{3}\left(R(E_l, E_i)E_j + R(E_l, E_j)E_i\right),$$

it follows that

$$\langle \nabla_{Y_i} Y_j, Y_0\rangle = -\frac{\varepsilon}{3}(\langle R(X,E_i)E_j,E_0\rangle + \langle R(X,E_j)E_i,E_0\rangle) + \mathcal{O}(\varepsilon^2)$$

We conclude that from Lemma 4.4 that the Second fundamental form $\mathrm{II}_{ij} = \langle \nabla_{Z_i} Z_j, N^{\mathscr{D}}\rangle$ of the perturbed disc $\mathscr{D}^{s,\varepsilon(s)}(w)$ centered at the point $\gamma(\varepsilon(s))$ with radius $\varepsilon(s)$ is given by

$$\begin{aligned} \mathrm{II}_{ij} \;\; &= \;\; \varepsilon\varepsilon_1' \left(1-\varepsilon\Gamma_0^0(E_l)x^l\right)\, w_{x^j x^i} - \tfrac{\varepsilon^3}{3}\left(\langle R(X,E_i)E_j,E_0\rangle + \langle R(X,E_j)E_i,E_0\rangle\right) \\ &\quad + \mathcal{O}(\varepsilon^4) - \varepsilon^2\varepsilon_1'\left(w_{x^j}\Gamma_0^0(E_i) + w_{x^i}\Gamma_0^0(E_j)\right) + \varepsilon^4 L(w) + \varepsilon^3 Q(w). \end{aligned} \tag{20}$$

We recall that if E_α is an orthogonal basis of $T_p\mathcal{M}$, then

$$\mathrm{Ric}_p(X,Y) = -\langle R_p(X,E_\alpha)Y,E_\alpha\rangle \qquad \forall X,Y \in T_p\mathcal{M}.$$

Finally we obtain

$$\begin{aligned} \mathrm{II}_{ij}\, h^{ij} \;\; &= \;\; \tfrac{\varepsilon_1'}{\varepsilon}\left(1-\varepsilon\Gamma_0^0(X)\right)\,\Delta w - \tfrac{2\varepsilon}{3}\mathrm{Ric}_p(X,E_0) \\ &\quad + \mathcal{O}(\varepsilon^2) - 2\varepsilon_1'\,\Gamma_0^0(\nabla w) + \varepsilon^2 L(w) + \varepsilon Q(w), \end{aligned}$$

where $X = x^l E_l$.

Proposition 4.5 *In the above notations, the mean curvature $\mathcal{H}(s,\rho,w)$ of $\mathscr{D}_{s,\rho}(w)$ has the following expansions*

$$\frac{\phi}{\kappa'}\,\mathcal{H}(s,\rho,w) \;\; = \;\; \Delta w - \frac{2\rho}{3}\frac{\phi^2}{\kappa'}\,Ric_p(X,E_0) + \mathcal{O}(\rho^2) + \rho L(w) + \rho Q(w).$$

In particular if Γ *is a geodesic,* $\Gamma_0^0 = 0$ *then*

$$\frac{\phi}{\kappa'}\,\mathcal{H}(s,\rho,w) \;\; = \;\; \Delta w - \frac{2\rho}{3}\frac{\phi^2}{\kappa'}\,Ric_p(X,E_0) + \mathcal{O}(\rho^2) + \rho^2 L(w) + \rho Q(w).$$

4.2 Angle between the normals

By construction, at $q = G^s(x)$ we have $F^s(w(x),x) = \mathcal{C}^\varepsilon(s+\varepsilon w(x), \varepsilon(s+\varepsilon w(x)))$, for every $x \in \partial B_1^n$. Recall from § 3.1 and 4.1.1 that

$$\tilde N_{\mathscr{C}^\rho}(s,z) = -\varepsilon_1'(s)X + \varepsilon'(s)Y_0 + \alpha_0 Z_0^c + \alpha_k Z_k^c,$$

where $\alpha_0 = \phi'\bar\alpha_0 + \mathcal{O}(\varepsilon^3)$ and $\alpha_k = \mathcal{O}(\varepsilon^2)$ also

$$\tilde N_{\mathscr{D}} = Y_0 + a_k Z_k.$$

One easily verifies that

$$\begin{aligned} \langle \tilde N_{\mathscr{D}}, \rho^{-1}\tilde N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q \;\; &= \;\; -\kappa'\langle X,Y_0\rangle_q + \phi'\langle Y_0,Y_0\rangle_q + \frac{\alpha_0}{\rho}\langle Z_0^c,Y_0\rangle_q + \frac{\alpha_k}{\rho}\langle Z_k^c,Y_0\rangle_q \\ &\quad - \;\; \kappa' a_k\langle X,Z_k\rangle_q + \phi' a_k\langle Z_k,Y_0\rangle_q + \frac{a_k}{\rho}\alpha_0\langle Z_0^c,Z_k\rangle_q + \frac{\alpha_k}{\rho}a_l\langle Z_k,Z_l^c\rangle_q. \end{aligned}$$

We have to expand

$$\kappa'(s+\varepsilon w) = \kappa'(s) + \varepsilon\kappa''(s)w + \varepsilon^2 Q(w) \qquad \phi'(s+\varepsilon w) = \phi'(s) + \varepsilon\phi''(s)w + \varepsilon^2 Q(w).$$

We will also need the following result which uses just the expansions of the metric Lemma 4.1

$$\begin{aligned} \langle Z_0^c, Z_k\rangle \;\; &= \;\; \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon^3 Q(w), \\ \langle Z_l^c, Z_k\rangle \;\; &= \;\; \mathcal{O}(\varepsilon^2) + \varepsilon^2 L(w) + \varepsilon^3 Q(w), \\ \langle X, Z_k\rangle_q \;\; &= \;\; (\varepsilon + \varepsilon'\varepsilon w)x^k + \varepsilon^3 L(w) + \varepsilon^5 Q(w). \end{aligned} \tag{21}$$

We use the fact that $a_k\langle Z_l, Z_k\rangle = -\langle Y_0, Z_k\rangle$ to have

$$\begin{aligned}\rho^{-1}\langle \tilde N_{\mathscr D}, \tilde N_{\mathscr C^\varepsilon}(s+\varepsilon w)\rangle_q &= (\phi' + \varepsilon w \phi'')\langle Y_0, Y_0\rangle_q + (\kappa' + \varepsilon\,\kappa'' w)\alpha_0 \langle Y_0, Y_0\rangle_q + \frac{\alpha_k}{\rho}\langle Z_k^c, Y_0\rangle_q \\ &- (\kappa' + \varepsilon\kappa'' w) a_k \langle X, Z_k\rangle_q - \phi' a_k a_l \langle Z_k, Z_l\rangle_q + \frac{a_k}{\rho}\alpha_0\langle Z_0^c, Z_k\rangle_q + \frac{\alpha_k}{\rho} a_l \langle Z_k, Z_l^c\rangle_q \\ &+ \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

Now from (21) we get

$$-\phi' a_k a_l \langle Z_k, Z_l\rangle_q + \frac{\alpha_0}{\rho} a_k \langle Z_0^c, Z_k\rangle_q + \frac{\alpha_k}{\rho} a_l \langle Z_k, Z_l^c\rangle_q = \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w)$$

and also since

$$\langle Z_k^c, Y_0\rangle_q = \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^4 Q(w),$$

one has

$$\begin{aligned}\rho^{-1}\langle \tilde N_{\mathscr D}, \tilde N_{\mathscr C^\varepsilon}(s+\varepsilon w)\rangle_q &= (\phi' + \varepsilon w \phi'')\langle Y_0, Y_0\rangle_q + (\kappa' + \varepsilon w \kappa'')\bar\alpha_0 \langle Y_0, Y_0\rangle_q \\ &- (\kappa' + \varepsilon\kappa'' w) a_k \langle X, Z_k\rangle_q + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

From (21) and recalling the formula for a_k in (13) we get

$$\begin{aligned}a_k\langle X, Z_k\rangle_q &= -\varepsilon_1' \frac{\partial w}{\partial \eta}\langle Y_0, Y_0\rangle_q - (1+\varepsilon' w)\langle Y_0, X\rangle_q + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w) \\ &= -\varepsilon_1' \frac{\partial w}{\partial \eta}\langle Y_0, Y_0\rangle_q + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w)\end{aligned}$$

and then we deduce that

$$\begin{aligned}\rho^{-1}\langle \tilde N_{\mathscr D}, \tilde N_{\mathscr C^\varepsilon}(s+\varepsilon w)\rangle_q &= \phi'\left(1+\kappa'\bar\alpha_0\right)\langle Y_0, Y_0\rangle_q + (\varepsilon w\phi'' + \kappa'\varepsilon_1')\frac{\partial w}{\partial \eta}\langle Y_0, Y_0\rangle_q + 2\varepsilon^2\kappa''\kappa' w \Gamma_0^0(X) \\ &+ \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

Using (14), we have that

$$\begin{aligned}\rho^{-1}\langle N_{\mathscr D}, \tilde N_{\mathscr C^\varepsilon}(s+\varepsilon w)\rangle_q &= \phi'\left(1+\kappa'\bar\alpha_0\right)|Y_0|_q + (\varepsilon w\phi'' + \kappa'\varepsilon_1')\frac{\partial w}{\partial \eta}|Y_0|_q + 2\varepsilon^2\kappa''\kappa' w \Gamma_0^0(X) \\ &+ \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).\end{aligned}$$

Since $\alpha_0 = \phi'(s+\varepsilon w)\bar\alpha_0 + \mathcal{O}(\varepsilon^3)$, one has

$$\alpha_0 = (\phi'(s) + \phi''(s)\varepsilon(s) w)\bar\alpha_0 + \varepsilon^2 Q(w)\bar\alpha_0$$

so that

$$\alpha_0 = \phi'(s)\bar\alpha_0 - 2\phi''(s)\varepsilon_1'\varepsilon w \Gamma_0^0(X) + \varepsilon^3 L(w) + \varepsilon^2 Q(w).$$

Moreover notice that

$$\Gamma_0^0(X)\Big|_q = \Gamma_0^0(X) + \varepsilon^2 L(w) + \varepsilon^3 Q(w)$$

and also

$$|\phi'(s+\varepsilon w)|^2 = (\phi' + 2\varepsilon w\phi'')\phi' + \varepsilon^2 Q(w),$$

we have that

$$\rho\left|\tilde N_{\mathscr C^\rho}(s+\varepsilon w)\right|^{-1} = 1 + (\phi' + 2\varepsilon\phi'' w + \varepsilon' w)\varepsilon\phi'\Gamma_0^0(X) + |\phi'(s)|^2\varepsilon^2 H_c(X,X) + \mathcal{O}(\varepsilon^3) + \varepsilon^3 L(w) + \varepsilon^3 Q(w)$$

from which we deduce that

$$\begin{aligned}
\rho\left|\tilde N_{\mathscr{C}^\rho}(s+\varepsilon w)\right|_q^{-1}|Y_0|_q &= 1-(\kappa')^2\varepsilon\Gamma_0^0(X)+\varepsilon^2\left(3-(\phi')^2+(\phi')^4+2(\phi')^2\phi^4\right)\Gamma_0^0(X)\Gamma_0^0(X)\\
&+ \frac{2-(\phi')^2}{2}\langle R_p(X,E_0)X,E_0\rangle+\varepsilon w\left(-\varepsilon'+2\varepsilon\phi\phi''+\varepsilon'\phi'\right)\Gamma_0^0(X)+2\varepsilon^2U_0^0(X)w\\
&+ \mathcal{O}(\varepsilon^3)+\varepsilon^3L(w)+\varepsilon^2Q(w).
\end{aligned}$$

Consequently we may expand the angle as

$$\begin{aligned}
\langle N_{\mathscr{D}},N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= \rho\phi'\left(1+\kappa'\bar\alpha_0\right)|Y_0|_q\left|\tilde N_{\mathscr{C}^\rho}(s+\varepsilon w)\right|_q^{-1}+2\varepsilon^2\kappa''\kappa' w\Gamma_0^0(X)\\
&+ \left(1-(\kappa')^2\varepsilon\Gamma_0^0(X)\right)\left(\varepsilon w\phi''+\kappa'\varepsilon_1'\frac{\partial w}{\partial\eta}\right)\\
&+ \mathcal{O}(\varepsilon^3)+\varepsilon^3L(w)+\varepsilon^2Q(w).\\
\langle N_{\mathscr{D}(w)},N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q &= \phi'(s)\Big(1+(\kappa')^2\varepsilon\Gamma_0^0(X)\Big)+\Big(1-(\kappa')^2\varepsilon\Gamma_0^0(X)\Big)\left(\varepsilon w\phi''+\kappa'\varepsilon_1'\frac{\partial w}{\partial\eta}\right)\\
&+ \phi'\varepsilon^2\Big(3-2(\kappa')^4-(\phi')^2+(\phi')^4+2(\phi')^2\phi^4\Big)\Gamma_0^0(X)\Gamma_0^0(X)\\
&+ \tfrac{2-(\phi')^2}{2}\phi'\varepsilon^2\langle R_p(X,E_0)X,E_0\rangle+\varepsilon\phi' w\Big(-\varepsilon'+2\varepsilon\kappa''\kappa'+2\varepsilon\phi\phi''+\varepsilon'\phi'\Big)\Gamma_0^0(X)\\
&+ 2\varepsilon^2U_0^0(X)w+\mathcal{O}(\varepsilon^3)+\varepsilon^3L(w)+\varepsilon^2Q(w).
\end{aligned}$$

We define

$$\mathcal{B}(s,\rho,w):=\langle N_{\mathscr{D}(w)},N_{\mathscr{C}^\varepsilon}(s+\varepsilon w)\rangle_q.$$

Now we conclude this section by collecting all these in the following

Proposition 4.6 *In the above notations, there holds*

$$\begin{aligned}
\mathcal{B}(s,\rho,w) &= \phi'(s)\Big(1+(\kappa')^2\rho\phi\Gamma_0^0(X)\Big)+\rho\left((\kappa')^2\frac{\partial w}{\partial\eta}+\phi\phi''\,w\right)\\
&+ \mathcal{O}(\rho^2)+\rho^2\bar L(w)+\rho^2\bar Q(w),
\end{aligned}$$

while if $\phi'(s)=0$, one has

$$\mathcal{B}(s,\rho,w)=\rho(\kappa')^2\frac{\partial w}{\partial\eta}+\mathcal{O}(\rho^3)+\rho^3\bar L(w)+\rho^2\bar Q(w).$$

In particular if Γ is a geodesic, we get precisely

$$\begin{aligned}
\mathcal{B}(s,\rho,w) &= \phi'(s)\left(1+\frac{1+(\kappa')^2}{2}\rho^2\phi^2\langle R_p(X,E_0)X,E_0\rangle\right)+\rho\left((\kappa')^2\frac{\partial w}{\partial\eta}+\phi\phi''\,w\right)\\
&+ \mathcal{O}(\rho^3)+\rho^3\bar L(w)+\rho^2\bar Q(w).
\end{aligned}$$

5 Existence of capillary minimal submanifolds

5.1 Case where $\phi(s_0)\phi''(s_0)>0$

We may assume that $\phi(s)\phi''(s)>0$ for all $s\in I_{s_0}(\delta):=[s_0-\delta,s_0+\delta]$ for some $\delta>0$ small. We define the following operator by

$$(\mathcal{L}_s w,v):=\int_{B_1^n}\nabla w\nabla v\,dx+\frac{\phi\phi''}{(\kappa')^2}\oint_{\partial B_1^n}wv\,d\sigma.$$

It is clear that from the inequality (see [22], Theorem A.9)

$$\int_{B_1^n} w^2\,dx \;\le\; C(n)\left(\int_{B_1^n}|\nabla w|^2\,dx+\oint_{\partial B_1^n} w^2\,d\sigma\right),\qquad \forall w\in H^1, \tag{22}$$

the operator $\mathcal{L}_s$ is coercive if ρ is small. We call $w_1^{s,\rho}$ the unique solution to the equation

$$(\mathcal{L}_s w_1, v) := -\phi\oint_{\partial B_1^n}\Gamma_0^0(X)v\,d\sigma.$$

Namely it solves the problem

$$\begin{array}{rclr} -\Delta w_1 & = & 0 & \text{in } B_1^n,\\ \dfrac{\partial w_1}{\partial\eta}+\dfrac{\phi\phi''}{(\kappa')^2}\,w_1 & = & -\phi\,\Gamma_0^0(X) & \text{on } \partial B_1^n. \end{array}$$

By elliptic regularity theory, there exist a constant $c>0$ (independent of ρ and s) such that

$$\|w_1^{s,\rho}\|_{\mathcal{C}^{2,\alpha}}\le c\qquad \forall s\in I_{s_0}(\delta).$$

Moreover we have that for all $k\ge 0$

$$\|\frac{\partial^k w_1^{s,\rho}}{\partial s^k}\|_{\mathcal{C}^{2,\alpha}}\le c_k\qquad \forall s\in I_{s_0}(\delta),$$

for some constant c_k which does not depend on s nor on ρ small.
Clearly by construction there holds

$$\left\{\begin{array}{rclll} \mathcal{H}(s,\rho,w_1^{s,\rho}) & = & \mathcal{O}(\rho) & \text{in} & \mathscr{D}_{s,\rho}(w_1^{s,\rho}),\\ \mathcal{B}(s,\rho,w_1^{s,\rho}) & = & \phi'(s)+\mathcal{O}(\rho^2) & \text{on} & \partial\mathscr{D}_{s,\rho}(w_1^{s,\rho}) \end{array}\right.$$

We define the space

$$\begin{aligned} \mathcal{C}^{2,\alpha}_{s,\rho} &:= \left\{w\in\mathcal{C}^{2,\alpha}(\overline{B_1^n}) \;:\; \frac{\partial}{\partial v}\mathcal{B}(s,\rho,w_1^{s,\rho}+v)\Big|_{v=0}[w]=0\right\}\\ &= \left\{w\in\mathcal{C}^{2,\alpha}(\overline{B_1^n}) \;:\; \frac{\partial w}{\partial\eta}+\frac{\phi\phi''}{(\kappa')^2}w+\rho\bar{L}_s(w)=0\right\} \end{aligned}$$

We consider the linearized mean curvature operator about $\mathscr{D}^{s,\rho}(w_1^{s,\rho})$ (see Proposition 4.5), $\mathbb{L}_{\rho,s}(w):\mathcal{C}^{2,\alpha}(\overline{B_1^n})\to\mathcal{C}^{0,\alpha}(\overline{B_1^n})$ defined by

$$\mathbb{L}_{\rho,s}(w) := -\frac{\phi}{\kappa'}\frac{\partial}{\partial v}\mathcal{H}(s,\rho,w_1^{s,\rho}+v)\Big|_{v=0}[w] = -\Delta\,w+\rho L_s(w).$$

We define also $\Phi(s,\rho,x)$, $\mathcal{Q}_{s,\rho}(w)\in\mathcal{C}^{0,\alpha}(\overline{B_1^n})$ by duality as

$$(\Phi(s,\rho,x),w') := -\frac{\phi}{\kappa'}\int_{B_1^n}\mathcal{H}(s,\rho,w_1^{s,\rho})w'\,dx+\rho^{-1}\oint_{\partial B_1^n}(\mathcal{B}(s,\rho,w_1^{s,\rho})-\phi'(s))\,w'\,ds$$

and for every $w\in\mathcal{C}^{2,\alpha}$

$$(\mathcal{Q}_{s,\rho}(w),w') := \int_{B_1^n}Q(w)w'\,dx+\oint_{\partial B_1^n}\bar{Q}(w)w'\,ds,\qquad \forall w'\in L^2.$$

Clearly the solvability of the system

$$\left\{\begin{array}{rclll} \mathcal{H}(s,\rho,w_1^{s,\rho}+w) & = & 0 & \text{in} & \mathscr{D}_{s,\rho}(w_1^{s,\rho}+w),\\ \mathcal{B}(s,\rho,w_1^{s,\rho}+w) & = & \phi'(s) & \text{on} & \partial\mathscr{D}_{s,\rho}(w_1^{s,\rho}+w) \end{array}\right. \tag{23}$$

is equivalent to the fixed point problem

$$w = -\left(\mathbb{L}_{\rho,s}\Big|_{\mathcal{C}^{2,\alpha}_{s,\rho}}\right)^{-1} \{\Phi(s,\rho,x) + \rho\,\mathcal{Q}_{s,\rho}(w)\}\,. \tag{24}$$

Furthermore one has

$$\begin{aligned}\|\mathcal{Q}_{s,\rho}(w)\|_{\mathcal{C}^{0,\alpha}} &= O(\|w\|_{\mathcal{C}^{2,\alpha}})\,\|w\|^2_{\mathcal{C}^{2,\alpha}};\\ \|\mathcal{Q}_{s,\rho}(w_1) - \mathcal{Q}_{s,\rho}(w_2)\|_{\mathcal{C}^{0,\alpha}} &= O(\|w_1\|_{\mathcal{C}^{2,\alpha}}, \|w_2\|_{\mathcal{C}^{2,\alpha}})\|w_1 - w_2\|_{\mathcal{C}^{2,\alpha}},\end{aligned}$$

also by construction, there exist a constant $c > 0$ (independent of ρ and s) such that

$$\|\Phi(s,\rho,\cdot)\|_{\mathcal{C}^{0,\alpha}} \le c\rho \qquad \forall s \in I_{s_0}(\delta).$$

By (22) the operator $\mathbb{L}_{\rho,s}$ is coercive on $\mathcal{C}^{2,\alpha}_{s,\rho}$ if ρ is small enough and also by elliptic regularity theory, $\mathbb{L}_{\rho,s}$ is an isomorphism from $\mathcal{C}^{2,\alpha}_{s,\rho}$ into $\mathcal{C}^{0,\alpha}(\overline{B^n_1})$ therefore we can solve the fixed point problem (24) in a ball of $\mathcal{C}^{2,\alpha}_{s,\rho}$ with radius $C\rho$ for some $C > 0$ which does not depend neither on ρ small nor s.
And thus for ρ small and $s \in I_{s_0}(\delta)$ there exists a function $w^{s,\rho} \in \mathcal{C}^{2,\alpha}_{s,\rho}$, with $\|w^{s,\rho}\|_{\mathcal{C}^{2,\alpha}} \le C\rho$ such that

$$\begin{cases} \mathcal{H}(s,\rho,w^{s,\rho}_1 + w^{s,\rho}) = 0 & \text{in} \quad \mathscr{D}_{s,\rho}(w^{s,\rho}_1 + w^{s,\rho}),\\ \mathcal{B}(s,\rho,w^{s,\rho}_1 + w^{s,\rho}) = \phi'(s) & \text{on} \quad \partial\mathscr{D}_{s,\rho}(w^{s,\rho}_1 + w^{s,\rho}).\end{cases}$$

Namely $\mathscr{D}_{s,\rho}(w^{s,\rho}_1 + w^{s,\rho})$ is a capillary submanifold of Ω_ρ with constant contact angle arccos $\phi'(s)$ if ρ is small enough by $\mathcal{C}^{2,\alpha}$ bound up to the boundary of $\tilde{w}^{s,\rho} = w^{s,\rho}_1 + w^{s,\rho}$. Furthermore it follows from the construction that, for all $k \ge 0$

$$\|\frac{\partial^k \tilde{w}^{s,\rho}}{\partial s^k}\|_{\mathcal{C}^{2,\alpha}} \le c_k\rho \qquad \forall s \in I_{s_0}(\delta), \tag{25}$$

for some constant c_k which does not depend on s nor on ρ small.

5.2 Foliation by minimal discs

Call $\tilde{w}^{s,\rho} = w^{s,\rho}_1 + w^{s,\rho}$. From (8), Lemma 4.1 and (25) the mapping

$$I_{s_0}(\delta) \times B^n_1 \ni (s,x) \xrightarrow{\Psi_\rho} F^s(\tilde{w}^{s,\rho}(x), x) = f(\bar{\varepsilon}_1(s,\tilde{w}^{s,\rho}(x))\,,\, \bar{\varepsilon}(s,\tilde{w}^{s,\rho}(x))\,x)$$

has Jacobian determinant which expands as

$$\rho^{2n+2}\left(\left(1 - |x|^2(\phi')^2\right)\,\phi^{2n} + \mathcal{O}_s(\rho)\right)$$

and hence since $(\phi')^2 \in (0,1)$ (see section 1), Ψ_ρ is a local homeomorphism if ρ is small enough. In particular it is a homeomorphism of a neighborhood of $(s_0, 0)$ which implies that there exist $0 < \delta' < \delta$ and $\varrho > 0$ such that

$$\Psi_\rho(s\,,\,B^n_\varrho) \cap \Psi_\rho(s'\,,\,B^n_\varrho) = \emptyset \qquad \forall s \neq s' \in I_{s_0}(\delta'),$$

for every ρ sufficiently small.
In this way the family of discs $\mathscr{D}_{s,\rho\varrho}(\tilde{w}^{s,\rho})$, $s \in I_{s_0}(\delta')$ with radius $\rho\varrho\,\phi(s)$ centered at $\gamma(\rho\,\kappa(s))$ constitutes a foliation of a neighborhood of $\gamma(\rho\,\kappa(s_0))$ for which each leaf $\mathscr{D}_{s,\rho\varrho}(\tilde{w}^{s,\rho})$ is a minimal disc intersecting $\mathscr{C}^{\rho\varrho}$ transversely along its boundary (the angle of contact may not be equal to arccos $\phi'(s)$).

5.3 $\phi \equiv 1$ and $\kappa =$Id, $\mathscr{C}^\rho$ is the geodesic tube around Γ

In this situation,

$$\mathscr{C}^\rho = \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q,\Gamma) = \rho\}$$

and its interior is

$$\Omega_\rho = \{q \in \mathcal{M} \quad : \quad \mathrm{dist}_g(q,\Gamma) < \rho\}.$$

By [17], it is well known that (smooth) minimal surfaces $D \subset \Omega_\rho$ with $\partial D \subset \mathscr{C}^\rho$ are stationary for the area functional relative to $\mathscr{C}^\rho$ which is $D \mapsto \mathrm{Area}(D \cap \Omega_\rho)$ under variations $\Psi_t : D \to \mathcal{M}$ such that $\partial \Psi_t(D) \subset \mathscr{C}^\rho$ moreover the Euler-Lagrange equations are given by

$$\begin{aligned} H_D &= 0 \qquad \text{in } D, \\ \langle N_D, N_{\mathscr{C}^\rho}\rangle &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{26}$$

5.3.1 A finite-dimensional reduction

For every $s \in [a,b]$ and $X = x^i E_i$, we let $w_1^{s,\rho}$ be the solution of the following problem:

$$\begin{aligned} -\Delta w_1 &= -\tfrac{2}{3}\,\mathrm{Ric}_p(X, E_0) && \text{in } B_1^n, \\ \frac{\partial w_1}{\partial \eta} &= 0 && \text{on } \partial B_1^n, \end{aligned}$$

where $p = \gamma(\rho\,\kappa(s))$.
By elliptic regularity theory, there exist a constant $c > 0$ (independent of ρ and s) such that

$$\|w_1^{s,\rho}\|_{\mathcal{C}^{2,\alpha}} \le c \qquad \forall s \in [a,b]. \tag{27}$$

As in § 5.1, we let

$$\begin{aligned} \mathcal{C}^{2,\alpha}_{s,\rho} &:= \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{B_1^n}) \quad : \quad \frac{\partial}{\partial v}\mathcal{B}(s,\rho,\rho w_1^{s,\rho} + v)\Big|_{v=0}[w] = 0 \right\} \\ &= \left\{ w \in \mathcal{C}^{2,\alpha}(\overline{B_1^n}) \quad : \quad \frac{\partial w}{\partial \eta} + \rho \bar{L}(w) = 0 \right\}. \end{aligned}$$

As explained in the first section, the linearized mean curvature operator about $\mathscr{D}^{s,\rho}(\rho\, w_1^{s,\rho})$ restricted on $\mathcal{C}^{2,\alpha}_{s,\rho}$ defined by

$$\mathbb{L}_{\rho,s}(w) := -\frac{\partial}{\partial v}\mathcal{H}(s,\rho,\rho\, w_1^{s,\rho} + v)\Big|_{v=0}[w] = -\Delta\, w + \rho L_s(w).$$

may have small (possibly zero) eigenvalues hence it may not be invertible on $\mathcal{C}^{2,\alpha}_{s,\rho}$. However instead of solving (26), we will prove that there exists a constant $\lambda_{s,\rho} \in \mathbb{R}$ and a function $w^{s,\rho} \in \mathcal{C}^{2,\alpha}_{s,\rho}$ such that

$$\left\{ \begin{aligned} \mathcal{H}(s,\rho,w^{s,\rho}) &= \lambda_{s,\rho} && \text{in} \quad \mathscr{D}_{s,\rho}(w^{s,\rho}), \\ \mathcal{B}(s,\rho,w^{s,\rho}) &= 0 && \text{on} \quad \partial\mathscr{D}_{s,\rho}(w^{s,\rho}). \end{aligned} \right. \tag{28}$$

To achieve this we let P be the L^2 projection on the space of functions $w \in L^2$ which are orthogonal to the constant function 1, $\int_{B_1^n} w\,dx = 0$. Now if ρ is small enough, the Poincare inequality implies together with elliptic regularity theory that the operator $P \circ \mathbb{L}_{s,\rho}$ is an isomorphism from $P\mathcal{C}^{2,\alpha}_{s,\rho}$ into $P\mathcal{C}^{0,\alpha}(\overline{B_1^n})$. Here letting

$$(\Phi(s,\rho,x), w') := -\int_{B_1^n} \mathcal{H}(s,\rho,\rho w_1^{s,\rho}) w'\,dx + \oint_{\partial B_1^n} \mathcal{B}(s,\rho,\rho w_1^{s,\rho}) w'\,ds$$

one has

$$\|\Phi(s,\rho,\cdot)\|_{\mathcal{C}^{0,\alpha}} \le c\rho^2 \qquad \forall s \in [a,b].$$

Consequently for ρ small, our fixed point problem

$$w = \left(P \circ \mathbb{L}_{\rho,s}\Big|_{\mathcal{C}^{2,\alpha}_{s,\rho}}\right)^{-1} \{P \circ \Phi(s,\rho,x) + \rho P \circ \mathcal{Q}_{s,\rho}(w)\}$$

admits a unique solution $w^{s,\rho} \in P\mathcal{C}^{2,\alpha}_{s,\rho}$, in a ball of radius $c\,\rho^2$ of $P\mathcal{C}^{2,\alpha}_{s,\rho}$. More precisely

$$\int_{B_1^n} w^{s,\rho}\,dx = 0 \qquad \text{and} \qquad \|w^{s,\rho}\|_{\mathcal{C}^{2,\alpha}(\overline{B_1^n})} \le c\rho^2 \qquad \forall s \in [a,b]. \tag{29}$$

Furthermore it follows from the construction that, for all $k \geq 0$

$$\|\frac{\partial^k w^{s,\rho}}{\partial s^k}\|_{\mathcal{C}^{2,\alpha}(\overline{B_1^n})} \le c_k\rho^2 \qquad \forall s \in [a,b],$$

for some constant c_k which does not depend on s nor on ρ small. We then conclude that $P \circ \mathcal{H}(s,\rho,\rho\,w_1^{s,\rho} + w^{s,\rho}) = 0$ hence the existence of a real number $\lambda_{s,\rho} \sim \rho^2$ such that (28) is satisfied.

We have to mention that by (29), provided ρ is small, the corresponding disc $\mathscr{D}_{s,\rho} := \mathscr{D}_{s,\rho}(\tilde{w}^{s,\rho})$ with $\tilde{w}^{s,\rho} = \rho\,w_1^{s,\rho} + w^{s,\rho}$ is embedded into Ω_ρ. This defines a one dimensional manifold of sets satisfying (28):

$$\mathcal{Z}_\rho := \{\mathscr{D}_{s,\rho} \subset \Omega_\rho,\ \partial\mathscr{D}_{s,\rho} \subset \partial\Omega_\rho \quad : \quad s \in [a,b]\}.$$

Remark 5.1 *We notice that, in section 5.1, the same argument as above implies that whenever $\phi''(s_0) = 0$, there will be a capillary disc centered at $\gamma(\rho\,s_0)$ with constant and small mean curvature.*

Variational argument:

We will show that in fact problem (26) can be reduced to a finite dimensional one. We now define the *reduced functional* $\varphi_\rho : [a,b] \to \mathbb{R}$ by

$$\varphi_\rho(s) := \mathrm{Area}(\mathscr{D}_{s,\rho})$$

for any $\mathscr{D}_{s,\rho} \in \mathcal{Z}_\rho$. We have to show the following

Lemma 5.2 *There exists ρ_0 small such that for any $\rho \in (0,\rho_0)$ if s is a critical point of φ_ρ then $\lambda_{s,\rho} = 0$.*

Proof. Let $\lambda \in \mathbb{R}$ and let $q = \gamma(\rho(s + \lambda\,t))$. Then provided t is small, it is clear that the hyper-surface $\mathscr{D}_{q,\rho}$ can be written as a normal graph over $\mathscr{D}_{p,\rho}$, $p = \gamma(\rho s)$ by a smooth function $g_{p,\rho,t,\lambda}$. This defines the variation vector field

$$\zeta_{p,\rho,\lambda} = \frac{\partial g_{p,\rho,t,\lambda}}{\partial t}\Big|_{t=0} N_{\mathscr{D}_{p,\rho}}.$$

Letting Z be the parallel transport of λE_0 along geodesics issued from $p = \gamma(\rho\,s)$. Then, we can easily get the estimates:

$$\|\zeta - Z\| \le c\rho|\lambda|.$$

Assume that s is a critical point of φ_ρ then from the first variation of area see [17],

$$\begin{aligned} 0 \quad &= \quad \frac{d\varphi_\rho(\rho(s+\lambda t))}{dt}\Big|_{t=0} = \lambda\rho\varphi'_\rho(s) \\ &= \quad n\int_{\mathscr{D}_{s,\rho}} H_{\mathscr{D}_{s,\rho}}\langle\zeta, N_{\mathscr{D}_{s,\rho}}\rangle\,ds + \oint_{\partial\mathscr{D}_{s,\rho}} \langle\zeta, N^{\mathscr{D}s,\rho}_{\partial\mathscr{D}_{s,\rho}}\rangle, \end{aligned}$$

where $N^{\mathscr{D}s,\rho}_{\partial\mathscr{D}_{s,\rho}} \in T\mathscr{D}_{s,\rho}$ stands for the normal of $\partial\mathscr{D}_{s,\rho}$ in $\mathscr{D}_{s,\rho}$. Therefore by construction one has

$$0 = \lambda_{s,\rho}\int_{\mathscr{D}_{s,\rho}} \langle\zeta, N_{\mathscr{D}_{s,\rho}}\rangle\,ds. \tag{30}$$

Notice that

$$\langle \zeta, N_{\mathscr{D}_{s,\rho}} \rangle - \langle Z, Y_0 \rangle = \langle \zeta - Z, N_{\mathscr{D}_{s,\rho}} \rangle + \langle Z, N_{\mathscr{D}_{s,\rho}} - Y_0 \rangle,$$

so using the fact that $N_{\mathscr{D}_{s,\rho}} = Y_0 + \mathcal{O}(\rho)$, see § 4.1.1, we have

$$\left| \langle \zeta, N_{\mathscr{D}_{s,\rho}} \rangle - \lambda \right| \le c\rho |\lambda|.$$

Inserting this in (30), we get

$$-\lambda\, \lambda_{s,\rho} \operatorname{Area}(\mathscr{D}_{s,\rho}) \le c\rho\, |\lambda_{s,\rho}|\, |\lambda| \operatorname{Area}(\mathscr{D}_{s,\rho})$$

but since $\operatorname{Area}(\mathscr{D}_{s,\rho}) = \operatorname{Area}(\rho B_1^n) + O_s(\rho^{2+n})$ by (11); (27) and (29), it follows that

$$-\lambda\, \lambda_{s,\rho} \le c\rho\, |\lambda_{s,\rho}|\, |\lambda|\, .$$

Therefore taking $\lambda = -\lambda_{s,\rho}$, we see that $|\lambda_{s,\rho}|^2 \le c\rho\, |\lambda_{s,\rho}|^2$ and this implies that $\lambda_{s,\rho=0}$. ■

We shall end the proof of the Theorem 1.3 by giving the expansion of φ_ρ. From (29) the first fundamental form h_{ij} of a disc $\mathscr{D}_{s,\rho}$ expands as

$$\rho^{-2}\, h_{ij} = \delta_{ij} + \frac{\rho^2}{3} \langle R_p(X, E_i)X, E_j \rangle + \frac{\rho^3}{6} \langle \nabla_X R_p(X, E_i)X, E_j \rangle + \mathcal{O}_s(\rho^4),$$

where $p = \gamma(\rho s) \in \Gamma$.
From the formula

$$\sqrt{\det(I + A)} = 1 + \frac{1}{2}\operatorname{tr}(A) + O(|A|^2),$$

we obtain the volume form:

$$\rho^{-n} \sqrt{\det(h)} = 1 - \frac{\rho^2}{6} \langle R_p(X, E_i)X, E_i \rangle + \frac{\rho^3}{12} \langle \nabla_X R_p(X, E_i)X, E_i \rangle + \mathcal{O}_s(\rho^4)$$

and since by oddness $\int_{B_1^n} \langle \nabla_X R_p(X, E_i)X, E_j \rangle\, dx = 0$ we deduce that

$$\varphi_\rho(s) = \operatorname{Area}(\mathscr{D}_{s,\rho}) = \operatorname{Area}(B_\rho^n) \left(1 - \frac{\rho^2}{6n} \sum_{i,j=1}^{n} \langle R_p(E_j, E_i)E_j, E_i \rangle + O_s(\rho^4) \right).$$

Thus setting

$$\psi_\rho(s) := \frac{6n}{\rho^2} \left(1 - \frac{\varphi_\rho(s)}{\operatorname{Area}(B_\rho^n)} \right) = \sum_{i,j=1}^{n} \langle R_p(E_j, E_i)E_j, E_i \rangle + O_s(\rho^4)$$

we get the result.

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