Differential and Integral Equations Volume 22, Numbers 11-12 (2009), 1115–1124

MINIMAL DISC-TYPE SURFACES EMBEDDED IN A PERTURBED CYLINDER

Mouhamed Moustapha Fall and Carlo Mercuri
Sissa, Via Beirut 2-4, Trieste, 34014 Italy

(Submitted by: Reza Aftabizadeh)

Abstract. In the present note, we deal with small perturbations of an infinite cylinder in three-dimensional Euclidian space. We find minimal disc-type surfaces embedded in the cylinder and intersecting its boundary perpendicularly. The existence and localization of those minimal discs is a consequence of a non-degeneracy condition for the critical points of a functional related to the oscillations of the cylinder from the flat configuration.

1. Introduction

We define a perturbed cylindrical surface C_ε parallel to the z-axis parametrized by

$$C_\varepsilon(\theta, z) := (1 + \varepsilon h(1, \theta, z))(\cos\theta, \sin\theta, 0) + z e_3, \quad \varepsilon > 0,$$

where $h = h(r, \theta, z)$ is a smooth real-valued function expressed in the cylindrical coordinates $(r, \theta, z) \in (0, \infty) \times (0, 2\pi) \times \mathbb{R}$ and e_3 is the unit vector parallel to the z-direction. It is worth noticing that, if $\varepsilon > 0$ is small, then C_ε is a regular surface of $\mathbb{R}^3$. Hereafter, we define $h(\theta, z) := h(1, \theta, z)$ and we denote by Ω_ε the interior of C_ε. We recall that a minimal surface is a surface with mean curvature, $\mathcal{H}$, vanishing everywhere. In particular, we are interested in finding disc-type minimal surfaces. We will parametrize the z-level disc-type surface D^z_ε by

$$D^z_\varepsilon(r, \theta) := r(1 + \varepsilon h(r, \theta, z))(\cos\theta, \sin\theta, 0) + z e_3, \quad r \in (0, 1), \theta \in (0, 2\pi).$$

The boundary is obviously given by the previous equation with $r = 1$. The aim of the present note is to deform D^z_ε for some z in order to find a surface $\mathcal{D}$ such that, for ε small, $\mathcal{H}(\mathcal{D}) = 0$, whose boundary intersects C_ε perpendicularly. Fixing z and moving along the e_3-direction, we can define a smooth

Accepted for publication: February 2009.
AMS Subject Classifications: 53A10, 53C21, 35R35.

deformation of D^z_ε, denoted by $\mathcal{D}^z_\varepsilon(t)$, simply by

$$F^z_\varepsilon(r,\theta,t) := r(1+\varepsilon h(r,\theta,z+t))(\cos\theta,\sin\theta,0)+(z+t)e_3, \tag{1.1}$$

defined for $(r,\theta,t)\in(0,1]\times(0,2\pi)\times(-t_0,t_0)$. Since

$$F^z_\varepsilon(1,\theta,t)=C_\varepsilon(\theta,z+t),$$

for any fixed t, $F^z_\varepsilon(r,\theta,t)$ parameterizes a disc whose boundary lies on C_ε, therefore $\mathcal{D}^z_\varepsilon(t)$ is an *admissible deformation* of D^z_ε.

Let $\mathcal{C}^{k,\alpha}$ be the standard Hölder spaces $\mathcal{C}^{k,\alpha}(\bar{B})$. We now state our main result.

Theorem 1.1. *Suppose z_0 is a non-degenerate critical point of the functional*

$$\Gamma(z) := \oint h(\theta,z)d\theta. \tag{1.2}$$

Then, an $\varepsilon_0>0$ exists such that for any $\varepsilon\in(0,\varepsilon_0)$ there exist $\delta_\varepsilon>0$, a smooth function $\omega_\varepsilon : B\to\mathbb{R}$ on the unit ball $B\subset\mathbb{R}^2$ and z_ε, satisfying $\|\omega_\varepsilon\|_{\mathcal{C}^{2,\alpha}}<\delta_\varepsilon$ and $|z_0-z_\varepsilon|<\delta_\varepsilon$, such that the set $D^{z_0}_\varepsilon$ can be smoothly deformed, through (1.1)*, to a disc-type minimal surface $\mathcal{D}^{z_\varepsilon}_\varepsilon(\omega_\varepsilon(B))\subset\Omega_\varepsilon$ intersecting C_ε perpendicularly.*

We have to point out that in [18] (Section 3) the existence of closed geodesics for C_ε have been studied with a variant of the Ljapunov-Schmidt reduction scheme. Even in this case the functional Γ defined in (1.2) determines the location.

A brief comment on the method is in order. Observe that

$$\mathcal{H}(D^z_\varepsilon)=\mathcal{O}(\varepsilon),\quad \text{in } D^z_\varepsilon, \tag{1.3}$$

and

$$\langle N_{D^z_\varepsilon},N_{C_\varepsilon}\rangle=\mathcal{O}(\varepsilon),\quad \text{on } \partial D^z_\varepsilon, \tag{1.4}$$

where $N_{D^z_\varepsilon}$ and N_{C_ε} stand for the unit outer normals respectively to D^z_ε and C_ε. Hereafter, we use the symbol $\mathcal{O}(\varepsilon^\alpha)$ instead of the Landau symbol $O(\varepsilon^\alpha)$ to indicate that it is also a function defined on B, possibly depending on z (see the notation below). On the other hand we search for some D satisfying

$$\mathcal{H}(D)=0,\quad \text{in } D, \tag{1.5}$$

and

$$\langle N_D,N_{C_\varepsilon}\rangle=0,\quad \text{on } \partial D, \tag{1.6}$$

which correspond to the Euler-Lagrange equation associated to the relative area functional, let us call it $\mathcal{E} : D\to \text{Area}(D\cap\Omega_\varepsilon)$, restricted to the class of *admissible (and orientable, smooth) surfaces* $\mathcal{S}$, i.e., those $\mathcal{S}$ such that $\mathcal{S} \subset \Omega_\varepsilon$ and $\partial\mathcal{S} \subset C_\varepsilon$. In order to solve (1.5) and (1.6) we localize the problem considering $z \in [a,b]$, and assuming z_0 is a non-degenerate critical point of $\oint h(\theta,z)d\theta$ and $z_0 \in (a,b)$. Because of (1.3) and (1.4) the set $Z := \{D_\varepsilon^z \,:\, z \in [a,b]\}$ provides a class of approximate solutions. Consider the map F defined in (1.1). Because of (1.1), any admissible set $\mathcal{D}_\varepsilon^z$ near D_ε^z can be parametrized by a function $\omega : B \to \mathbb{R}$ such that $\mathcal{D}_\varepsilon^z(\omega) = F_\varepsilon^z(r,\theta,\omega)$. Define $\mathcal{H}(z,\varepsilon,\omega) := \mathcal{H}(\mathcal{D}_\varepsilon^z(\omega))$ and $\mathcal{B}(z,\varepsilon,\omega) := \langle N_{\mathcal{D}_\varepsilon^z(\omega)}, N_{C_\varepsilon}\rangle$. The idea in solving (1.5)-(1.6) is then to determine ω such that

$$\mathcal{H}(z,\varepsilon,\omega) = 0 \quad \text{in } \mathcal{D}_\varepsilon^z(\omega)$$

$$\mathcal{B}(z,\varepsilon,\omega) = 0 \quad \text{on } \partial\mathcal{D}_\varepsilon^z(\omega).$$

Denoting $L, \bar{L}$ and $Q, \bar{Q}$ respectively, terms linear and quadratic in ω and its derivatives (see the notation below), the previous problem transforms to

$$\begin{cases} \Delta\omega = \varepsilon L(\omega) + Q(\omega) + \mathcal{O}(\varepsilon^2), \quad \text{on } B \\ \frac{\partial\omega}{\partial\eta} = \varepsilon h_z(\theta,z) + \varepsilon\bar{L}(\omega) + \bar{Q}(\omega) + \mathcal{O}(\varepsilon^2), \quad \text{on } \partial B. \end{cases} \tag{1.7}$$

The problem (1.7) can be formulated in a weak sense, expressed as

$$(\Phi_{\varepsilon,z}(\omega),\omega') = 0, \quad \forall\omega' \in H^1(B),$$

and splits in a system of two equations, namely the auxiliary+bifurcation equations system (see for example [1]). Indeed we can project $\Phi_{\varepsilon,z}(\omega)$ to the space of average-zero functions and its orthogonal complement $\langle 1\rangle \oplus \langle 1\rangle^\perp$. Because of the Poincaré -Wirtinger inequality, $\mathcal{L}_{z,\varepsilon}$ defined by $(\mathcal{L}_{z,\varepsilon}\omega,\omega') := \int_B \nabla\omega\nabla\omega'$ is invertible in $\langle 1\rangle^\perp$. The auxiliary equation is then solved by a fixed-point argument. Finally, the bifurcation equation reads as a simple equation in $\mathbb{R}$ and can be solved in a small neighborhood of z_0, using an elementary argument based on the contraction lemma.

A great deal of work has been devoted to minimal surfaces from the point of view of existence, uniqueness and topological properties of the solutions. We refer to the papers [2], [7], [8], [9], [11], [15], [16], [17], [19], [20] and the references therein. The method we used is perturbative in nature and the main idea goes back to Ye [21] and was employed by many authors in the study of constant mean curvature hyper-surfaces, see for example [4], [5], [6], [10], [12], [13] and [14].

Notation for error terms. Any expression of the form $L(\omega)$ (respectively $\bar{L}(\omega)$) denotes a linear combination of the function ω together with its derivatives with respect to the function ω up to order 2 (respectively order 1). The coefficients of L or $\bar{L}$ might depend on ε and z but, for all $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\varepsilon \in (0,1)$ and $z \in [a,b]$ such that

$$\|L(\omega)\|_{\mathcal{C}^{k,\alpha}} \le c\,\|\omega\|_{\mathcal{C}^{k+2,\alpha}},$$

and similarly for $\bar{L}$. Furthermore, any expression of the form $Q(\omega)$ (respectively $\bar{Q}(\omega)$) denotes a nonlinear operator in the function ω together with its derivatives with respect to ω up to order 2 (respectively 1).

Given $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\varepsilon \in (0,1)$ and $z \in [a,b]$ such that

$$\|Q(\omega_1) - Q(\omega_2)\|_{\mathcal{C}^{k,\alpha}} \le c\,(\|\omega_1\|_{\mathcal{C}^{k+2,\alpha}} + \|\omega_2\|_{\mathcal{C}^{k+2,\alpha}})\ \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+2,\alpha}},$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}} \le 1$, $i = 1,2$, and similarly for $\bar{Q}$. We also agree that any term denoted by $\mathcal{O}(r^d)$ (with $r \in \mathbb{R}$ possibly depending on z) is a smooth function on B that might depend on z but satisfies

$$\|\frac{\mathcal{O}(r^d)}{|r|^d}\|_{\mathcal{C}^{k,\alpha}} \le c,$$

for a constant c independent of z.

2. Expansions of the geometrical quantities for $\mathcal{D}^z_\varepsilon(\omega)$

For the sake of convenience we define hereafter

$$n := (\cos\theta, \sin\theta, 0),$$

and for every fixed z we use the following parametrization for $\mathcal{D}^z_\varepsilon(\omega)$:

$$X(r,\theta) := r(1 + \varepsilon h(r,\theta,z+\omega(r,\theta)))n + (z + \omega(r,\theta))e_3.$$

We can expand the last equation in terms of ω, yielding

$$X = X\big|_{\omega=0} + \frac{\partial X}{\partial \omega}\Big|_{\omega=0}\omega + \varepsilon Q(\omega).$$

Then we have

$$X(r,\theta) = D^z_\varepsilon(r,\theta) + (e_3 + r\varepsilon h_z(r,\theta,z)n)\omega + \varepsilon Q(\omega),$$

or, equivalently,

$$X(r,\theta) = r(1 + \varepsilon h(r,\theta,z))n + ze_3 + \omega(r,\theta)e_3 + \varepsilon L(\omega) + Q(\omega).$$

In order to compute the mean curvature of $\mathcal{D}^z_\varepsilon(\omega)$, some preliminary computations are in order. We will determine the meaningful geometrical quantities at a first order of approximation in ε. We refer, for example, to the book [3] for notation and terminology.

2.1. **The first fundamental form.** We compute the following quantities:

$$X_r = (1+\varepsilon(h+rh_r))n + \omega_r e_3 + \varepsilon L(\omega) + Q(\omega), \tag{2.1}$$

$$E := |X_r|^2 = 1 + 2\varepsilon(h+rh_r) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega), \tag{2.2}$$

$$X_\theta := r\varepsilon h_\theta n + r(1+\varepsilon h)n_\theta + \omega_\theta e_3 + \varepsilon L(\omega) + Q(\omega), \tag{2.3}$$

$$G := |X_\theta|^2 = r^2(1+2\varepsilon h) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega), \tag{2.4}$$

$$F := \langle X_\theta, X_r\rangle = r\varepsilon h_\theta + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{2.5}$$

Finally, we use the first fundamental form, E, F, G in terms of the basis X_r, X_θ, to determine the following quantity for later use:

$$EG - F^2 = r^2(1+4\varepsilon h) + 2r^3\varepsilon h_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{2.6}$$

2.2. **The normal.** We are going to expand the Gauss map for $\mathcal{D}^z_\varepsilon(\omega)$:

$$N := \frac{X_r \wedge X_\theta}{|X_r \wedge X_\theta|} = \frac{X_r \wedge X_\theta}{(EG-F^2)^{1/2}}.$$

Using (2.1) and (2.3) we compute

$$\bar{N} := X_r \wedge X_\theta = r(1+2\varepsilon h)e_3 - \omega_\theta n_\theta + r^2\varepsilon h_r e_3 - r\omega_r n + O(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega),$$

and

$$|\bar{N}|^{-1} = \frac{1}{(EG-F^2)^{1/2}} = \frac{1}{r}(1-2\varepsilon h) - \frac{r}{2}\varepsilon h_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

Therefore, we have

$$N = \frac{\bar{N}}{|\bar{N}|} = e_3 - \frac{1}{r}\omega_\theta n_\theta - \omega_r n + \frac{r}{2}\varepsilon h_r e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{2.7}$$

Finally, we can compute the derivatives N_r, N_θ:

$$N_r = (\frac{1}{r^2}\omega_\theta - \frac{1}{r}\omega_{\theta r})n_\theta - \omega_{rr}n + (\frac{\varepsilon}{2}h_r + \frac{r}{2}\varepsilon h_{rr})e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega), \tag{2.8}$$

$$N_\theta = -(\frac{1}{r}\omega_{\theta\theta} + \omega_r)n_\theta + (\frac{1}{r}\omega_\theta - \omega_{r\theta})n + \frac{r}{2}\varepsilon h_{r\theta}e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{2.9}$$

2.3. **The second fundamental form.** As a preliminary step to determining the mean curvature for$\mathcal{D}^z_\varepsilon(\omega)$ we expand the second fundamental form given by the following e, f, g quantities. Using (2.1), (2.3),(2.8) and (2.9) we can determine the second fundamental form:

$$-g := \langle N_\theta, X_\theta\rangle = -r(\frac{1}{r}\omega_{\theta\theta} + \omega_r) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega),$$

so that

$$g = \omega_{\theta\theta} + r\omega_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{2.10}$$

$$-e := \langle N_r, X_r \rangle = -\omega_{rr} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

$$e = \omega_{rr} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega) \tag{2.11}$$

$$-f := \langle N_\theta, X_r \rangle = \frac{1}{r}\omega_\theta - \omega_{r\theta} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{2.12}$$

2.4. **The mean curvature.** Using (2.11),(2.4), (2.10), (2.2), (2.12) and (2.5), we have

$$eG = r^2\omega_{rr} + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

$$gE = \frac{1}{r}\omega_{\theta\theta} + \omega_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega)$$

$$-fF = \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

By equation (2.6) one has that

$$(EG - F^2)^{-1} = r^{-2}(1 - 2\varepsilon h + r\varepsilon h_r) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

We now estimate the mean curvature. Through the usual formula

$$\mathcal{H} = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2},$$

we finally compute $\mathcal{H}(z, \varepsilon, \omega) := \mathcal{H}(\mathcal{D}_\varepsilon^z(\omega))$,

$$\mathcal{H}(z, \varepsilon, \omega) = \frac{1}{2}(\omega_{rr} + \frac{\omega_r}{r} + \frac{\omega_{\theta\theta}}{r^2}) + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{2.13}$$

3. EXPANSION OF THE ORTHOGONALITY CONDITION ON THE BOUNDARY

In what follows $h(\theta, z)$ stands for $h(1, \theta, z)$. In order to express the orthogonality condition we compute the normal to $C_\varepsilon(\theta, z)$ with parametrization

$$Y(z, \theta) := (1 + \epsilon h(\theta, z))n + ze_3.$$

The basis of the tangent space at each point is given by

$$Y_z = \varepsilon h_z n + e_3,$$

and

$$Y_\theta = \varepsilon h_\theta n + (1 + \varepsilon h)n_\theta.$$

This yields

$$\bar{N} := Y_z \wedge Y_\theta = \varepsilon h_z e_3 + \varepsilon h_\theta n_\theta - (1 + \varepsilon h)n + \mathcal{O}(\varepsilon^2),$$

hence,

$$|\bar{N}| = 1 + \varepsilon h + \mathcal{O}(\varepsilon^2)$$

and finally

$$N(\theta,z) := \frac{\bar N}{|\bar N|} = \varepsilon h_z(\theta,z)e_3 + \varepsilon h_\theta(\theta,z)n_\theta - (1+\varepsilon h(\theta,z))n + \mathcal{O}(\varepsilon^2).$$

This yields

$$\begin{aligned} N(\theta,z+\omega) &:= \frac{\bar N}{|\bar N|} \\ &= \varepsilon h_z(\theta,z+\omega)e_3 + \varepsilon h_\theta(\theta,z+\omega)n_\theta - (1+\varepsilon h(\theta,z+\omega))n + \mathcal{O}(\varepsilon^2). \end{aligned}$$

From (2.7), we expand the normal of $\mathcal{D}^z_\varepsilon(\omega)$ in $r=1$:

$$N_{\mathcal{D}^z_\varepsilon(\omega)} = e_3 - \omega_\theta n_\theta - \omega_r n + \frac{\varepsilon}{2} h_r e_3 + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

Therefore, we get

$$\langle N_{\mathcal{D}^z_\varepsilon(\omega)}, N(\theta,z+\omega)\rangle = \varepsilon h_z(z+\omega) + \omega_r + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega).$$

Finally, since

$$\varepsilon h_z(\theta,z+\omega) = \varepsilon h_z(\theta,z) + \varepsilon L(\omega) + Q(\omega),$$

the last equation becomes

$$\mathcal{B}(z,\varepsilon,\omega) := \omega_r + \varepsilon h_z + \mathcal{O}(\varepsilon^2) + \varepsilon L(\omega) + Q(\omega). \tag{3.1}$$

4. Proof of Theorem 1.1

Collecting equations (2.13) and (3.1) we get the following nonlinear system of PDE's:

$$\begin{cases} -\Delta\omega = \varepsilon L(\omega) + Q(\omega) + \mathcal{O}(\varepsilon^2), & \text{on } B \\ \frac{\partial\omega}{\partial\eta} = \varepsilon h_z(\theta,z) + \varepsilon\bar L(\omega) + \bar Q(\omega) + \bar{\mathcal{O}}(\varepsilon^2), & \text{on } \partial B, \end{cases} \tag{4.1}$$

where $\frac{\partial\omega}{\partial\eta}$ stands for the unit outer normal derivative of ω and B is the unit ball in $\mathbb{R}^2$.

We shall introduce the following operators: for any $\omega,\omega' \in H^1(B)$, let

$$(\mathcal{L}(\omega),\omega') := \int_B \nabla\omega\nabla\omega';$$

for $\omega \in \mathcal{C}^{2,\alpha}$, let

$$\begin{aligned} (\mathcal{F}_{\varepsilon,z}(\omega),\omega') &:= \int_B (Q(\omega) + \varepsilon L(\omega) + \mathcal{O}(\varepsilon^2))\omega' \\ &+ \oint_{\partial B} (\varepsilon h_z + \varepsilon\bar L(\omega) + \bar Q(\omega) + \bar{\mathcal{O}}(\varepsilon^2))\omega', \quad \forall \omega' \in H^1(B). \end{aligned}$$

With this notation, (4.1) is solved provided

$$\mathcal{L}(\omega) = \mathcal{F}_{\varepsilon,z}(\omega). \tag{4.2}$$

With an abuse of notation, by the Riesz representation theorem, we may assume that, for any $\omega \in \mathcal{C}^{2,\alpha}$, $\mathcal{F}_{\varepsilon,z}(\omega) \in \mathcal{C}^{0,\alpha}$. Moreover it satisfies

$$\|\mathcal{F}_{\varepsilon,z}(\omega)\|_{\mathcal{C}^{0,\alpha}} \leq c\varepsilon\,(1 + \|\omega\|_{\mathcal{C}^{2,\alpha}}) + O\,(\|\omega\|_{\mathcal{C}^{2,\alpha}})\;\|\omega\|_{\mathcal{C}^{2,\alpha}}; \tag{4.3}$$

$$\|\mathcal{F}_{\varepsilon,z}(\omega_1) - \mathcal{F}_{\varepsilon,z}(\omega_2)\|_{\mathcal{C}^{0,\alpha}} \leq c\varepsilon\|\omega_1 - \omega_2\|_{\mathcal{C}^{2,\alpha}} \tag{4.4}$$
$$+O(\|\omega_1\|_{\mathcal{C}^{2,\alpha}} + \|\omega_2\|_{\mathcal{C}^{2,\alpha}})\|\omega_1 - \omega_2\|_{\mathcal{C}^{2,\alpha}}.$$

Now call

$$H := \Big\{\omega \in H^1(B) : \int_B \omega = 0\Big\},$$

and

$$\mathcal{P}f := f - \frac{1}{\pi}\int_B f, \qquad \forall f \in L^2.$$

From now on, we will assume that $\omega \in \mathcal{P}\mathcal{C}^{2,\alpha}$. Therefore, we have to solve the system

$$\begin{cases} \mathcal{L}(\omega) = \mathcal{P}\mathcal{F}_{\varepsilon,z}(\omega) & \text{auxiliary equation} \\ (Id - \mathcal{P})\mathcal{F}_{\varepsilon,z}(\omega) = 0. & \text{bifurcation equation .} \end{cases} \tag{4.5}$$

With a slight abuse of terminology we refer to the system (4.5) as the system of auxiliary+bifurcation equations.

Notice that the auxiliary equation is equivalent to the following fixed-point problem:

$$\omega = \mathcal{L}^{-1}\mathcal{P}\mathcal{F}_{\varepsilon,z}(\omega), \quad \text{in } \mathcal{P}\mathcal{C}^{2,\alpha}. \tag{4.6}$$

By the Poincaré-Wirtinger inequality, $\mathcal{L}$ is coercive in H. Furthermore, by elliptic regularity theory, $\mathcal{L} : \mathcal{P}\mathcal{C}^{2,\alpha} \to \mathcal{P}\mathcal{C}^{0,\alpha}$ is an isomorphism. Hence, by (4.3) and (4.4), we have that (4.6) can be readily solved in a small ball in $\mathcal{P}\mathcal{C}^{2,\alpha}$. More precisely, there exists a positive constant c such that, for any z and $\varepsilon > 0$ small, there exists a unique $\omega_{\varepsilon,z}$ with $\|\omega_{\varepsilon,z}\|_{\mathcal{C}^{2,\alpha}(\bar{B})} \leq c\varepsilon$ solving (4.6). Moreover, reducing ε if necessary, we may assume that $\mathcal{D}^z_\varepsilon(\omega_{\varepsilon,z})$ is embedded in Ω_ε.

By construction we have also that $\omega_{\varepsilon,z}$ smoothly depends on z; furthermore, for $k \geq 0$ we have

$$\|\frac{\partial^k \omega_{\varepsilon,z}}{\partial z^k}\|_{\mathcal{C}^{2,\alpha}} \leq c_k\varepsilon, \quad z \in [a,b], \tag{4.7}$$

where c_k is independent of z and small ε. The bifurcation equation becomes $(Id - \mathcal{P})\,\mathcal{F}_{\varepsilon,z}(\omega_{\varepsilon,z}) = 0$ which is equivalent to

$$\int_B (Q(\omega_{\varepsilon,z}) + \varepsilon L(\omega_{\varepsilon,z}) + \mathcal{O}(\varepsilon^2)) + \oint_{\partial B} (\varepsilon h_z + \varepsilon \bar{L}(\omega_{\varepsilon,z}) + \bar{Q}(\omega_{\varepsilon,z}) + \bar{\mathcal{O}}(\varepsilon^2)) = 0.$$

That is,

$$\oint_{\partial B} h_z(\theta, z) d\theta = g_\varepsilon(z),$$

where g_ε is a smooth function in z such that, because of (4.7), $\|g_\varepsilon\|_{\mathcal{C}^k[a,b]} \leq c'_k \varepsilon$. Define $f_\varepsilon(z) := \oint h_z(\theta, z) d\theta - g_\varepsilon(z)$. If z_0 is a non-degenerate critical point (strict maxima or minima) for $\oint h(\theta, z) d\theta$, then $f'_\varepsilon(z_0)$ is invertible if ε is sufficiently small. Consequently, as above, the bifurcation equation is equivalent to the fixed-point problem

$$z - z_0 = [f'_\varepsilon(z_0)]^{-1}(f_\varepsilon(z_0) + O(|z - z_0|^2)),$$

which has a unique solution z_ε such that $|z_\varepsilon - z_0| < c\varepsilon$. This concludes the proof.

Acknowledgments The authors would like to thank Prof. Antonio Ambrosetti for taking their attention to these questions. They are grateful to Prof. Antonio Ambrosetti and Prof. Andrea Malchiodi at SISSA/ISAS for helpful discussions. They have been supported by M.U.R.S.T within the PRIN 2008 Variational Methods and Nonlinear Differential Equations.

References

[1] A. Ambrosetti and G. Prodi, "A Primer of Nonlinear Analysis," Cambridge Univ. Press, Cambridge Studies in Advanced Mathematics, No.34 (1993).

[2] W. Bürger and E. Kuwert, *Area-minimizing disks with free boundary and prescribed enclosed volume,* Preprint 2005, to appear in J. Reine Angew. Math.

[3] M.P. Do Carmo, "Differential Geometry of Curves and Surfaces," Prentice Hall, Inc. (Englewood Cliffs, New Jersey) 1976.

[4] M.M. Fall, *Embedded disc-type surfaces with large constant mean curvature and free boundaries,* preprint SISSA 2007, to appear in Com. Cont. Math.

[5] M.M. Fall and F. Mahmoudi, *Hypersurfaces with free boundary and large constant mean curvature: concentration along submanifolds,* Ann. Sc. Norm. Super. Pisa Cl. Sci., (5) Vol. VII (2008).

[6] M.M. Fall and C. Mercuri, *Foliations of small tubes in Riemannian manifolds by capillary minimal discs,* Nonlinear Anal., 70 (2009), 4422-4440.

[7] R. Finn, "Equilibrium Capillary Surfaces," Springer-Verlag, New York, 1986

[8] M. Grüter and J. Jost, *On Embedded Minimal Discs in Convex Bodies,* Annales de l'institut Henri Poincaré (C) Analyse non linéaire, 3 (1986), 345–390.

[9] J. Jost, *Existence results for embedded minimal surfaces of controlled, topological type I,* Ann. Sci. Sup. Pisa. Classe di Scienze Ser. 4, 13 (1986), 15–50.

[10] N. Kapouleas, *Compact constant mean curvature surfaces in Euclidean three-space,* J. Differ. Geom., 33 (1991), 683–715.

[11] H.B. Lawson, "Lectures on Minimal Submanifolds," Vol.I. Second edition. Mathematics Lecture Series, 9. Pulish or Perish, Wimington, Del., 1980.

[12] F. Mahmoudi, R. Mazzeo, and F. Pacard, *Constant mean curvature hypersurfaces condensing along a submanifold,* Geom. Funct. Anal., 16 (2006) 924–958.

[13] R. Mazzeo and F. Pacard, *Foliations by constant mean curvature tubes,* Comm. Anal. Geom., 13 (2005), 633–670.

[14] F. Pacard and X. Xu, *Constant mean curvature spheres in Riemannian manifolds,* preprint (2005) to appear in Manuscripta Math.

[15] A. Ros, "The Isoperimetric Problem," Lecture series given during the Caley Mathematics Institute Summer School on the Global Theory of Minimal Surfaces at the MSRI, Berkley, California (2001).

[16] A. Ros and R. Souam, *On stability of capillary surfaces in a ball,* Pacific J. Math., 178 (1997) 345–361.

[17] A. Ros and E. Vergasta, *Satability for hypersurfaces of constant mean curvature with free boundary,* Geom. Dedicata, 56 (1995), 19–33.

[18] S. Secchi, *A note on closed geodesics for a class of non-compact Riemannian manifolds,* Adv. Nonlinear Stud., 1 (2001), 132–142.

[19] M. Struwe, *On a free boundary problem for minimal surfaces,* Inv. Math., 75 (1984), 547–560.

[20] M. Struwe, *The existence of surfaces of constant mean curvature with free boundaries,* Acta Math., 160 (1988), 19–64.

[21] R. Ye, *Foliation by constant mean curvature spheres,* Pacific J. Math., 147 (1991), 381–396.