Sharp nonexistence results for a linear elliptic inequality involving Hardy and Leray potentials

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Abstract. In this paper we deal with non-negative distributional supersolutions for a class of linear elliptic equations involving inverse-square potentials and logarithmic weights. We prove sharp nonexistence results.

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Introduction

In recent years a great deal work has been made to find necessary and sufficient conditions for the existence of distributional supersolutions to semilinear elliptic equations with inverse-square potentials. We quote for instance [7] (and the references therein), where a problem related to the Hardy and Sobolev inequalities has been studied. In the present paper we are interested in a class of linear elliptic equations.

Let $N \geq 2$ be an integer, $R \in (0,1]$ and let B_R be the ball in $\mathbb{R}^N$ of radius R centered at 0. We focus our attention on non-negative distributional solutions to

$$-\Delta u - \frac{(N-2)^2}{4}|x|^{-2}\, u \geq \alpha |x|^{-2}\, |\log|x||^{-2}\, u \quad \text{in } \mathcal{D}'(B_R \setminus \{0\}), \tag{0.1}$$

where $\alpha \in \mathbb{R}$ is a varying parameter. By a standard definition, a solution to (0.1) is a function $u \in L^1_{\text{loc}}(B_R \setminus \{0\})$ such that

$$-\int_{B_R} u\Delta\varphi\; dx - \frac{(N-2)^2}{4}\int_{B_R} |x|^{-2} u\varphi\; dx \geq \alpha \int_{B_R} |x|^{-2}\, |\log|x||^{-2}\, u\varphi\; dx$$

for any non-negative $\varphi \in C^\infty_c(B_R \setminus \{0\})$. Problem (0.1) is motivated by the inequality

$$\int_{B_1} |\nabla u|^2\; dx - \frac{(N-2)^2}{4}\int_{B_1} |x|^{-2}|u|^2 \geq \frac{1}{4}\int_{B_1} |x|^{-2}\, |\log|x||^{-2}\, |u|^2\; dx\,, \tag{0.2}$$

which holds for any $u \in C^\infty_c(B_1 \setminus \{0\})$ (see for example [2], [5], [8], [12] and Appendix A). Notice that (0.2) improves the Hardy inequality for maps supported by the unit ball if $N \geq 3$. Inequality (0.2) was firstly proved by Leray [14] in the lower dimensional case $N = 2$.

Due to the sharpness of the constants in (0.2), a necessary and sufficient condition for the existence of non-trivial and non-negative solutions to (0.1) is that $\alpha \leq 1/4$ (compare with Theorem B.2 in Appendix B and with Remark 1.5).

In case $\alpha \leq 1/4$ we provide necessary conditions on the parameter α to have the existence of non-trivial solutions satisfying suitable integrability properties.

Theorem 0.1 *Let $R \in (0,1]$ and let $u \geq 0$ be a distributional solution to (0.1). Assume that there exists $\gamma \leq 1$ such that*

$$u \in L^2_{\text{loc}}(B_R; |x|^{-2}\, |\log|x||^{-2\gamma}\; dx)\;, \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2\,.$$

Then $u = 0$ almost everywhere in B_R.

We remark that Theorem 0.1 is sharp, in view of the explicit counter-example in Remark 1.5.

Let us point out some consequences of Theorem 0.1. We use the Hardy-Leray inequality (0.2) to introduce the space $\widetilde{H}^1_0(B_1)$ as the closure of $C^\infty_c(B_1 \setminus \{0\})$ with respect to the scalar product

$$\langle u, v\rangle = \int_{B_1} \nabla u \cdot \nabla v \; dx - \frac{(N-2)^2}{4} \int_{B_1} |x|^{-2} uv \; dx$$

(see for example [9]). It turns out that $\widetilde{H}^1_0(B_1)$ strictly contains the standard Sobolev space $H^1_0(B_1)$, unless $N = 2$.

Take $\gamma = 1$ in Theorem 0.1. Then problem (0.1) has no non-trivial and non-negative solutions $u \in L^2_{\text{loc}}(B_R; |x|^{-2} |\log |x||^{-2} \; dx)$ if $\alpha = 1/4$. Therefore, if in the dual space $\widetilde{H}^1_0(B_R)'$, a function $u \in \widetilde{H}^1_0(B_R)$ solves

$$\begin{cases} -\Delta u - \dfrac{(N-2)^2}{4} |x|^{-2} \, u \geq \dfrac{1}{4} |x|^{-2} |\log |x||^{-2} \; u & \text{in } B_R \\ u \geq 0 \, , \end{cases}$$

then $u = 0$ in B_R.

Next take $\gamma = 0$ and $\alpha \geq -3/4$. From Theorem 0.1 it follows that problem (0.1) has no non-trivial and non-negative solutions $u \in L^2_{\text{loc}}(B_R; |x|^{-2} \; dx)$. In particular, if $N \geq 3$ and if $u \in H^1_0(B_R) \hookrightarrow L^2(B_R; |x|^{-2} \; dx)$ is a weak solution to

$$\begin{cases} -\Delta u - \dfrac{(N-2)^2}{4} |x|^{-2} \, u \geq -\dfrac{3}{4} |x|^{-2} |\log |x||^{-2} \, u & \text{in } B_R \\ u \geq 0 \, , \end{cases}$$

then $u = 0$ in B_R. Thus Theorem 0.1 improves some of the nonexistence results in [1] and in [13].

The case of boundary singularities has been little studied. In Section 2 we prove sharp nonexistence results for inequalities in cone-like domains in $\mathbb{R}^N$, $N \geq 1$, having a vertex at 0. A special case concerns linear problems in half-balls. For $R > 0$ we let $B^+_R = B_R \cap \mathbb{R}^N_+$, where $\mathbb{R}^N_+$ is any half-space. Notice that $B^+_R = (0, R)$ or $B^+_R = (-R, 0)$ if $N = 1$. A necessary and sufficient condition for the existence of non-negative and non-trivial distributional solutions to

$$-\Delta u - \frac{N^2}{4} |x|^{-2} \, u \geq \alpha |x|^{-2} |\log |x||^{-2} \; u \quad \text{in } \mathcal{D}'(B^+_R) \tag{0.3}$$

is that $\alpha \le 1/4$ (see Theorem B.3 and Remark 2.3), and the following result holds.

Theorem 0.2 *Let $R \in (0,1]$, $N \ge 1$ and let $u \ge 0$ be a distributional solution to (0.3). Assume that there exists $\gamma \le 1$ such that*

$$u \in L^2(B_R^+; |x|^{-2} \left|\log |x|\right|^{-2\gamma} \ dx) \ , \quad \alpha \ge \frac{1}{4} - (1-\gamma)^2 \, .$$

Then $u = 0$ almost everywhere in B_R^+.

The key step in our proofs consists in studying the ordinary differential inequality

$$\begin{cases} -\psi'' \ge \alpha s^{-2} \psi & \text{in } \mathcal{D}'(a,\infty) \\ \psi \ge 0 \, , \end{cases} \tag{0.4}$$

where $a > 0$. In our crucial Theorem 1.3 we prove a nonexistence result for (0.4), under suitable weighted integrability assumptions on ψ. Secondly, thanks to an "averaged Emden-Fowler transform", we show that distributional solutions to problems of the form (0.1) and (0.3) give rise to solutions of (0.4), see Section 1.2 and 2 respectively. Our main existence results readily follow from Theorem 1.3. A similar idea, but with a different functional change, was already used in [6] to obtain nonexistence results for a large class of superlinear problems.

In Appendix A we give a simple proof of the Hardy-Leray inequality for maps with support in cone-like domains that includes (0.2) and that motivates our interest in problem (0.3).

Appendix B deals in particular with the case $\alpha > 1/4$. The nonexistence Theorems B.2 and B.3 follow from an Allegretto-Piepenbrink type result (Lemma B.1).

In the last appendix we point out some related results and some consequences of our main theorems.

Notation

We denote by $\mathbb{R}_+$ the half real line $(0,\infty)$. For $a > 0$ we put $I_a = (a,\infty)$.

We denote by $|\Omega|$ the Lebesgue measure of the domain $\Omega \subset \mathbb{R}^N$. Let $q \in [1,+\infty)$ and let ω be a non-negative measurable function on Ω. The weighted Lebesgue space $L^q(\Omega;\omega(x)\,dx)$ is the space of measurable maps u in Ω with finite norm $\left(\int_\Omega |u|^q \omega(x)\,dx\right)^{1/q}$. For $\omega \equiv 1$ we simply write $L^q(\Omega)$. We embed $L^q(\Omega;\omega(x)\,dx)$ into $L^q(\mathbb{R}^N;\omega(x)\,dx)$ via null extension.

1 Proof of Theorem 0.1

The proof consists of two steps. In the first one we prove a nonexistence result for a class of linear ordinary differential inequalities that might have some interest in itself.

1.1 Nonexistence results for problem (0.4)

We start by fixing some terminologies. Let $\mathcal{D}^{1,2}(\mathbb{R}_+)$ be the Hilbert space obtained via the Hardy inequality

$$(1.1) \qquad \int_0^\infty |v'|^2 \, ds \ge \frac{1}{4} \int_0^\infty s^{-2}|v|^2 \, ds \;, \qquad v \in C_c^\infty(\mathbb{R}_+)$$

as the completion of $C_c^\infty(\mathbb{R}_+)$ with respect to the scalar product

$$\langle v, w \rangle = \int_0^\infty v'w' \, ds \, .$$

Notice that $\mathcal{D}^{1,2}(\mathbb{R}_+) \hookrightarrow L^2(\mathbb{R}_+; s^{-2} \, ds)$ with a continuous embedding and moreover $\mathcal{D}^{1,2}(I_a) \subset C^0(\mathbb{R}_+)$ by Sobolev embedding theorem. By Hölder inequality, the space $L^2(\mathbb{R}_+; s^2 \, ds)$ is continuously embedded into the dual space $\mathcal{D}^{1,2}(\mathbb{R}_+)'$.

Finally, for any $a > 0$ we put $I_a = (a, \infty)$ and

$$\mathcal{D}^{1,2}(I_a) = \{ v \in \mathcal{D}^{1,2}(\mathbb{R}_+) \mid v(a) = 0 \ \} \, .$$

We need two technical lemmata.

Lemma 1.1 *Let $f \in L^2(I_a; s^2 \, ds)$ and $v \in C^2(\mathbb{R}_+) \cap L^2(I_a; s^{-2} \, ds)$ be a function satisfying $v(a) = 0$ and*

$$(1.2) \qquad -v'' \le f \quad \text{in } I_a \, .$$

Put $v^+ := \max\{v, 0\}$. Then $v^+ \in \mathcal{D}^{1,2}(I_a)$ and

$$(1.3) \qquad \int_a^\infty |(v^+)'|^2 \, ds \le \int_a^\infty f v^+ \, ds \, .$$

Proof. We first show that $(v^+)' \in L^2(\mathbb{R})$ and that (1.3) holds. Let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying

$$0 \le \eta \le 1\ , \quad \eta(s) \equiv 1 \text{ for } |s| \le 1\ , \quad \eta(s) \equiv 0 \text{ for } s \ge 2$$

and put $\eta_h(s) = \eta(s/h)$. Then $\eta_h v^+ \in \mathcal{D}^{1,2}(I_a)$ and $\eta_h v^+ \ge 0$. Multiply (1.2) by $\eta_h v^+$ and integrate by parts to get

$$\int_a^\infty \eta_h |(v^+)'|^2\ ds - \frac{1}{2}\int_a^\infty \eta_h'' |v^+|^2\ ds \le \int_a^\infty \eta_h f v^+\ ds\,. \tag{1.4}$$

Notice that for some constant c depending only on η it results

$$\left| \int_a^\infty \eta_h'' |v^+|^2\ ds \right| \le c \int_h^{2h} s^{-2} |v^+|^2\ ds \to 0$$

as $h \to \infty$, since $v^+ \in L^2(I_a; s^{-2}\ ds)$. Moreover,

$$\int_a^\infty \eta_h f v^+\ ds \to \int_a^\infty f v^+\ ds$$

by Lebesgue theorem, as $fv^+ \in L^1(I_a)$ by Hölder inequality. In conclusion, from (1.4) we infer that

$$\int_a^h |v_+'|^2\ ds \le \int_a^\infty f v^+\ ds + o(1) \tag{1.5}$$

since $\eta_h \equiv 1$ on (a, h). By Fatou's Lemma we get that $(v^+)' \in L^2(I_a)$ and (1.3) readily follows from (1.5). To prove that $v^+ \in \mathcal{D}^{1,2}(I_a)$, it is enough to notice that $\eta_h v^+ \to v^+$ in $\mathcal{D}^{1,2}(I_a)$. Indeed,

$$\int_a^\infty |1 - \eta_h|^2 |(v^+)'|^2 \le \int_h^\infty |(v^+)'|^2\ ds = o(1)$$
$$\int_a^\infty |\eta_h'|^2 |v^+|^2\ ds \le c \int_h^\infty s^{-2} |v^+|^2\ ds = o(1)$$

as $(v^+)' \in L^2(I_a)$ and $v^+ \in L^2(I_a; s^{-2}\ ds)$. □

Through the paper we let (ρ_n) to be a standard mollifier sequence in $\mathbb{R}$, such that the support of ρ_n is contained in the interval $(-\frac{1}{n}, \frac{1}{n})$.

Lemma 1.2 *Let $a > 0$ and $\psi \in L^2(I_a; s^{-2}\ ds)$. Then $\rho_n \star \psi \in L^2(I_a; s^{-2}\ ds)$ and*

$$\rho_n \star \psi \to \psi \quad \text{in } L^2(I_a; s^{-2}\ ds), \tag{1.6}$$

$$g_n := \rho_n \star (s^{-2}\psi) - s^{-2}(\rho_n \star \psi) \to 0 \quad \text{in } L^2(I_a; s^2\ ds)\,. \tag{1.7}$$

Proof. We start by noticing that $\rho_n \star \psi \to \psi$ almost everywhere. Then we use Hölder inequality to get

$$\begin{aligned} s^{-2}|(\rho_n \star \psi)(s)|^2 &= s^{-2}\left|\int \rho_n(s-t)^{1/2}\rho_n(s-t)^{1/2}\psi(t)\ dt\right|^2 \\ &\leq s^{-2}\left(\frac{1}{n}+s\right)^2 \int \rho_n(s-t)t^{-2}|\psi(t)|^2\ dt \\ &\leq \left(1+\frac{1}{na}\right)^2 |(\rho_n \star (s^{-2}\psi^2))(s)| \end{aligned}$$

for any $s > a > 0$. Since $s^{-2}\psi^2 \in L^1(I_a)$ then $\rho_n \star (s^{-2}\psi^2) \to s^{-2}\psi^2$ in $L^1(I_a)$. Thus $s^{-1}(\rho_n \star \psi) \to s^{-1}\psi$ in $L^2(I_a)$ by the (generalized) Lebesgue Theorem, and (1.6) follows.

To prove (1.7) we first argue as before to check that

$$s^2\left|\int \rho_n(s-t)t^{-2}\psi(t)\ dt\right|^2 \leq \left(1-\frac{1}{na}\right)^{-2} |(\rho_n \star (s^{-2}\psi^2))(s)|$$

for any $s > a > 0$. Thus $\rho_n \star (s^{-2}\psi)$ converges to $s^{-2}\psi$ in $L^2(I_a; s^2\ ds)$ by Lebesgue's Theorem. In addition, $s^{-2}(\rho_n \star \psi) \to s^{-2}\psi$ in $L^2(I_a; s^2\ ds)$ by (1.6). Thus $g_n \to 0$ in $L^2(I_a; s^2\ ds)$ and the Lemma is completely proved. $\square$

The following result for solutions to (0.4) is a crucial step in the proofs of our main theorems.

Theorem 1.3 *Let $a > 0$ and let ψ be a distributional solution to (0.4). Assume that there exists $\gamma \leq 1$ such that*

$$\psi \in L^2(I_a; s^{-2\gamma}\ ds)\ , \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2\,.$$

Then $\psi = 0$ almost everywhere in I_a.

Proof. We start by noticing that $L^2(I_a; s^{-2\gamma}\, ds) \hookrightarrow L^2(I_a; s^{-2}\, ds)$ with a continuous immersion for any $\gamma < 1$. In addition, we point out that we can assume

$$\alpha = \frac{1}{4} - (1-\gamma)^2 \,. \tag{1.8}$$

Let ρ_n be a standard sequence of mollifiers, and let

$$\psi_n = \rho_n \star \psi \;, \quad g_n = \rho_n \star (s^{-2}\psi) - s^{-2}(\rho_n \star \psi)\,.$$

Then $\psi_n \to \psi$ in $L^2(I_a; s^{-2\gamma}\, ds)$ and almost everywhere, and $g_n \to 0$ in $L^2(I_a; s^2\, ds)$ by Lemma 1.2. Moreover, $\psi_n \in C^\infty(\overline{I}_a)$ is a non-negative solution to

$$-\psi_n'' \ge \alpha s^{-2}\psi_n + \alpha g_n \quad \text{in } \mathcal{D}'(I_a). \tag{1.9}$$

We assume by contradiction that $\psi \ne 0$. We let $s_0 \in I_a$ such that $\varepsilon_n := \psi_n(s_0) \to \psi(s_0) > 0$. Up to a scaling and after replacing g_n with $s_0^2 g_n$, we may assume that $s_0 = 1$. We will show that

$$\varepsilon_n := \psi_n(1) \to \psi(1) > 0 \tag{1.10}$$

leads to a contradiction. We fix a parameter

$$\delta > \frac{1}{2} - \gamma \ge -\frac{1}{2} \tag{1.11}$$

and for n large we put

$$\varphi_{\delta,n}(s) := \varepsilon_n\, s^{-\delta} \in L^2(I_1; s^{-2\gamma}\, ds)\,.$$

Clearly $\varphi_{\delta,n} \in C^\infty(\mathbb{R}_+)$ and one easily verifies that $(\varphi_{\delta,n})_n$ is a bounded sequence in $L^2(I_1; s^{-2\gamma}\, ds)$ by (1.10) and (1.11). Finally we define

$$v_{\delta,n} = \varphi_{\delta,n} - \psi_n = \varepsilon_n\, s^{-\delta} - \psi_n \,,$$

so that $v_{\delta,n} \in L^2(I_1; s^{-2\gamma}\, ds)$ and $v_{\delta,n}(1) = 0$. In addition $v_{\delta,n}$ solves

$$-v_{\delta,n}'' \le \alpha s^{-2}\, v_{\delta,n} - c_\delta\, \varepsilon_n\, s^{-2-\delta} - \alpha g_n \quad \text{in } I_1, \tag{1.12}$$

where $c_\delta := \delta(\delta+1) + \alpha = \delta(\delta+1) + 1/4 - (1-\gamma)^2$. Notice that $c_\delta > 0$ and that all the terms in the right hand side of (1.12) belong to $L^2(I_1; s^2\, ds)$, by (1.11). Thus Lemma 1.1 gives $v_{\delta,n}^+ \in \mathcal{D}^{1,2}(I_1)$ and

$$\int_1^\infty |(v_{\delta,n}^+)'|^2\, ds \le \alpha \int_1^\infty s^{-2}|v_{\delta,n}^+|^2\, ds - c_\delta \varepsilon_n \int_1^\infty s^{-2-\delta} v_{\delta,n}^+\, ds + o(1)\,,$$

since $v_{\delta,n}^+$ is bounded in $L^2(I_1; s^{-2}\, ds)$ and $g_n \to 0$ in $L^2(I_1; s^2\, ds)$. By (1.8) and Hardy's inequality (1.1), we conclude that

$$(1-\gamma)^2 \int_1^\infty s^{-2}|v_{\delta,n}^+|^2 + c_\delta\, \varepsilon_n \int_1^\infty s^{-2-\delta} v_{\delta,n}^+ \, ds = o(1)\,.$$

Thus, for any fixed δ we get that $v_{\delta,n}^+ \to 0$ almost everywhere in I_1 as $n \to \infty$, since $\varepsilon_n c_\delta$ is bounded away from 0 by (1.10). Finally we notice that

$$\psi_n = \varphi_{\delta,n} - v_{\delta,n} \geq \varepsilon_n s^{-\delta} - v_{\delta,n}^+\,.$$

Since $\psi_n \to \psi$ and $v_{\delta,n}^+ \to 0$ almost everywhere in I_1, and since $\varepsilon_n \to \psi(1) > 0$, we infer that $\psi \geq \psi(1)s^{-\delta}$ in I_1. This conclusion clearly contradicts the assumption $\psi \in L^2(I_1; s^{-2\gamma}\, ds)$, since $\delta > 1/2 - \gamma$ was arbitrarily chosen. Thus (1.10) cannot hold and the proof is complete. □

Remark 1.4 *If $\alpha > 1/4$ then every non-negative solution $\psi \in L^1_{\rm loc}(I_a)$ to problem (0.4) vanishes. This is an immediate consequence of Lemma B.1 in Appendix B and the sharpness of the constant $1/4$ in the Hardy inequality (1.1).*

1.2 Conclusion of the proof

We will show that any non-negative distributional solution u to problem (0.1) gives rise to a function ψ solving (0.4), and such that $\psi = 0$ if and only if $u = 0$. To this aim, we introduce the Emden-Fowler transform $u \mapsto Tu$ by letting

$$u(x) = |x|^{\frac{2-N}{2}}\, (Tu)\left(|\log|x||\,,\frac{x}{|x|}\right)\,. \tag{1.13}$$

By change of variable formula, for any $R' \in (0,R)$ it results

$$\int_{B_{R'}} |x|^{-2}\,|\log|x||^{-2\gamma}\,|u|^2\, dx = \int_{|\log R'|}^\infty \int_{\mathbb{S}^{N-1}} s^{-2\gamma}|Tu|^2\, ds d\sigma\,, \tag{1.14}$$

so that $Tu \in L^2(I_a \times \mathbb{S}^{N-1}; s^{-2\gamma}\, dsd\sigma)$ for any $a > a_R := |\log R|$. Now, for an arbitrary $\varphi \in C_c^\infty(I_{a_R})$ we define the radially symmetric function $\tilde\varphi \in C_c^\infty(B_R)$ by setting

$$\tilde\varphi(x) = |x|^{\frac{2-N}{2}}\varphi(|\log|x||)\,,$$

so that $\varphi = T\tilde{\varphi}$. By direct computations we get

$$\int_{B_R} u(\Delta\tilde{\varphi} + \frac{(N-2)^2}{4}|x|^{-2}\tilde{\varphi})\, dx = \int_{a_R}^{\infty} \varphi'' \int_{\mathbb{S}^{N-1}} Tu\, d\sigma ds \tag{1.15}$$

$$\int_{B_R} |x|^{-2}\,|\log|x||^{-2}\, u\tilde{\varphi}\, dx = \int_{a_R}^{\infty} s^{-2}\varphi \int_{\mathbb{S}^{N-1}} Tu\, d\sigma ds\,. \tag{1.16}$$

Thus we are led to introduce the function ψ defined in I_{a_R} by setting

$$\psi(s) = \int_{\mathbb{S}^{N-1}} (Tu)(s,\sigma)\, d\sigma\,.$$

We notice that $\psi \in L^2(I_a; s^{-2\gamma}\, ds)$ for any $a > a_R$, since

$$\int_a^{\infty} s^{-2\gamma}|\psi|^2\, ds \le \left|\mathbb{S}^{N-1}\right| \int_a^{\infty}\int_{\mathbb{S}^{N-1}} s^{-2\gamma}|Tu|^2\, dsd\sigma$$

by Hölder inequality. Moreover, from (1.15) and (1.16) it immediately follows that $\psi \ge 0$ is a distributional solution to

$$-\psi'' \ge \alpha s^{-2}\psi \quad \text{in } \mathcal{D}'(I_{a_R}).$$

By Theorem 1.3 we infer that $\psi = 0$ in I_{a_R}, and hence $u = 0$ in B_R. The proof of Theorem 0.1 is complete. □

Remark 1.5 *The assumption on the integrability of u in Theorem 0.1 are sharp. If $\alpha > 1/4$ use the results in Appendix B. For $\alpha \le 1/4$ put $\delta_\alpha := (\sqrt{1-4\alpha}-1)/2$ and notice that the function $u_\alpha : B_1 \to \mathbb{R}$ defined by*

$$u_\alpha(x) = |x|^{\frac{2-N}{2}}|\log|x||^{-\delta_\alpha}$$

solves

$$-\Delta u_\alpha - \frac{(N-2)^2}{4}|x|^{-2}\, u_\alpha = \alpha|x|^{-2}\,|\log|x||^{-2}\, u_\alpha \quad \textit{in } \mathcal{D}'(B_1 \setminus \{0\}).$$

Moreover, if $\gamma \le 1$ then

$$u_\alpha \in L^2_{\mathrm{loc}}(B_1; |x|^{-2}\,|\log|x||^{-2\gamma}\, dx) \quad \textit{if and only if} \quad \alpha < \frac{1}{4} - (1-\gamma)^2\,.$$

2 Cone-like domains

Let $N \ge 2$. To any Lipschitz domain $\Sigma \subset \mathbb{S}^{N-1}$ we associate the *cone*

$$\mathcal{C}_\Sigma := \left\{ r\sigma \in \mathbb{R}^N \mid \sigma \in \Sigma \ , \ r > 0 \ \right\} .$$

For any given $R > 0$ we introduce also the *cone-like* domain

$$\mathcal{C}_\Sigma^R := \mathcal{C}_\Sigma \cap B_R = \left\{ r\sigma \in \mathbb{R}^N \mid r \in (0, R) \ , \ \sigma \in \Sigma \ \right\} .$$

Notice that $\mathcal{C}_{\mathbb{S}^{N-1}} = \mathbb{R}^N \setminus \{0\}$ and $\mathcal{C}^R_{\mathbb{S}^{N-1}} = B_R \setminus \{0\}$. If Σ is an half-sphere $\mathbb{S}^{N-1}_+$ then $\mathcal{C}_{\mathbb{S}^{N-1}_+}$ is an half-space $\mathbb{R}^N_+$ and $\mathcal{C}^R_{\mathbb{S}^{N-1}_+}$ is an half-ball B^+_R, as in Theorem 0.2.

Assume that Σ is properly contained in $\mathbb{S}^{N-1}$. Then we let $\lambda_1(\Sigma) > 0$ to be the first eigenvalue of the Laplace operator on Σ. If $\Sigma = \mathbb{S}^{N-1}$ we put $\lambda_1(\mathbb{S}^{N-1}) = 0$. It has been noticed in [15], [11], that

$$(2.1) \qquad \mu(\mathcal{C}_\Sigma) := \inf_{\substack{u \in C^\infty_c(\mathcal{C}_\Sigma) \\ u \neq 0}} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |\nabla u|^2 \ dx}{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-2} |u|^2 \ dx} = \frac{(N-2)^2}{4} + \lambda_1(\Sigma) \, .$$

The infimum $\mu(\mathcal{C})$ is the best constant in the Hardy inequality for maps having compact support in $\mathcal{C}_\Sigma$. In particular, for any half-space $\mathbb{R}^N_+$ it holds that

$$\mu(\mathbb{R}^N_+) = \frac{N^2}{4} \, .$$

The aim of this section is to study the elliptic inequality

$$(2.2) \qquad -\Delta u - \mu(\mathcal{C}_\Sigma)|x|^{-2} \, u \ge \alpha |x|^{-2} \left|\log |x|\right|^{-2} \, u \quad \text{in } \mathcal{D}'(\mathcal{C}^R_\Sigma).$$

Notice that (2.2) reduces to (0.1) if $\Sigma = \mathbb{S}^{N-1}$. Problem (2.2) is related to an improved Hardy inequality for maps supported in cone-like domains which will be discussed in Appendix A.

Theorem 2.1 *Let Σ be a Lipschitz domain properly contained in $\mathbb{S}^{N-1}$, $R \in (0, 1]$ and let $u \ge 0$ be a distributional solution to (2.2). Assume that there exists $\gamma \le 1$ such that*

$$u \in L^2(\mathcal{C}^R_\Sigma; |x|^{-2} \left|\log |x|\right|^{-2\gamma} \ dx) \ , \quad \alpha \ge \frac{1}{4} - (1-\gamma)^2 \, .$$

Then $u = 0$ almost everywhere in $\mathcal{C}^R_\Sigma$.

Proof. We introduce the first eigenfunction $\Phi \in C^2(\Sigma) \cap C(\overline{\Sigma})$ of the Laplace-Beltrami operator $-\Delta_\sigma$ in Σ. Thus Φ is positive in Σ and Φ solves

$$\begin{cases} -\Delta_\sigma \Phi = \lambda_1(\Sigma)\Phi & \text{in } \Sigma \\ \Phi = 0 & \text{on } \partial\Sigma. \end{cases} \tag{2.3}$$

Let $u \in L^2(\mathcal{C}^R_\Sigma; |x|^{-2} \left|\log |x|\right|^{-2\gamma}\ dx)$ be as in the statement, and put $a_R = |\log R|$. We let $Tu \in L^2(I_{a_R} \times \Sigma; s^{-2\gamma}\ ds d\sigma)$ be the Emden-Fowler transform, as in (1.13). We further let $\psi \in L^2(I_{a_R}; s^{-2\gamma}\ ds)$ defined as

$$\psi(s) = \int_\Sigma (Tu)(s,\sigma)\Phi(\sigma)\, d\sigma\,.$$

Next, for $\varphi \in C^\infty_c(I_{a_R})$ being an arbitrary non-negative test function, we put

$$\tilde\varphi(x) = |x|^{\frac{2-N}{2}} \varphi(|\log|x||)\Phi\left(\frac{x}{|x|}\right)\,. \tag{2.4}$$

In essence, our aim is to test (2.2) with $\tilde\varphi$ to prove that ψ satisfies (0.4) in I_{a_R}. To be more rigorous, we use a density argument to approximate Φ in $W^{2,2}(\Sigma) \cap H^1_0(\Sigma)$ by a sequence of smooth maps $\Phi_n \in C^\infty_c(\Sigma)$. Then we define $\tilde\varphi_n$ accordingly with (2.4), in such a way that $T\tilde\varphi_n = \varphi\Phi_n$. By direct computation we get

$$\begin{aligned} \int_{\mathcal{C}^R_\Sigma} u(\Delta\tilde\varphi_n + \frac{(N-2)^2}{4}|x|^{-2}\tilde\varphi_n)\ dx \quad &= \quad \int_{a_R}^\infty \int_\Sigma (Tu)\varphi''\Phi_n\ d\sigma ds \\ &+ \quad \int_{a_R}^\infty \int_\Sigma (Tu)\varphi\Delta_\sigma\Phi_n\ d\sigma ds \end{aligned}$$

$$\begin{aligned} \lambda_1(\Sigma)\int_{\mathcal{C}^R_\Sigma} |x|^{-2}u\tilde\varphi_n\ dx &= \lambda_1(\Sigma)\int_{a_R}^\infty \int_\Sigma (Tu)\varphi\Phi_n\ d\sigma ds \\ \int_{\mathcal{C}^R_\Sigma} |x|^{-2} \left|\log|x|\right|^{-2} u\tilde\varphi_n\ dx &= \int_{a_R}^\infty \int_\Sigma s^{-2}(Tu)\varphi\Phi_n\ d\sigma ds \end{aligned}$$

Since $\tilde\varphi_n \in C^\infty_c(\mathcal{C}^R_\Sigma)$ is an admissible test function for (2.2), using also (2.1) we get

$$\begin{aligned} -\int_{a_R}^\infty \int_\Sigma (Tu)\varphi''\Phi_n\ d\sigma ds \quad &\ge \quad \alpha \int_{a_R}^\infty \int_\Sigma s^{-2}(Tu)\varphi\Phi_n\ d\sigma ds \\ &- \quad \int_{a_R}^\infty \int_\Sigma (Tu)\varphi(\Delta_\sigma\Phi_n + \lambda_1(\Sigma)\Phi_n)\ d\sigma ds\,. \end{aligned}$$

Since $\Phi_n \to \Phi$ and $\Delta_\sigma \Phi_n + \lambda_1(\Sigma)\Phi_n \to 0$ in $L^2(\Sigma)$, we conclude that

$$-\int_{a_R}^{\infty} \varphi'' \psi ds \geq \alpha \int_{a_R}^{\infty} s^{-2} \varphi \psi ds \, .$$

By the arbitrariness of φ, we can conclude that ψ is a distributional solution to (0.4). Theorem 1.3 applies to give $\psi \equiv 0$, that is, $u \equiv 0$ in $\mathcal{C}_\Sigma^R$. □

The next result extends Theorem 2.1 to cover the case $N = 1$. Notice that $\mathbb{R}_+ = (0, \infty)$ is a cone and $(0,1)$ is a cone-like domain in $\mathbb{R}$.

Theorem 2.2 *Let $R \in (0,1]$ and let $u \geq 0$ be a distributional solution to*

$$-u'' - \frac{1}{4}\, t^{-2} u \geq \alpha t^{-2} |\log t|^{-2}\, u \quad \textit{in } \mathcal{D}'(0,R).$$

Assume that there exists $\gamma \leq 1$ such that

$$u \in L^2((0,R); t^{-2}|\log t|^{-2\gamma} \ dt) \ , \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2 \, .$$

Then $u = 0$ almost everywhere in $(0,R)$.

Proof. Write $u(t) = t^{1/2}\psi\left(|\log t|\right) = t^{1/2}\psi(s)$ for a function $\psi \in L^2(I_{a_R}; s^{-2\gamma} \ ds)$ and then notice that ψ is a distributional solution to

$$-\psi'' \geq \alpha s^{-2} \psi \quad \text{in } \mathcal{D}'(I_{a_R}).$$

The conclusion readily follows from Theorem 1.3. □

Remark 2.3 *If $\alpha > 1/4$ then every non-negative solution $u \in L^1_{\text{loc}}(\mathcal{C}_\Sigma^R)$ to problem (2.2) vanishes by Theorem B.3.*

In case $\alpha \leq 1/4$ the assumptions on α and on the integrability of u in Theorems 2.1, 2.2 are sharp. Fix $\alpha \leq 1/4$, let $\delta_\alpha := (\sqrt{1-4\alpha} - 1)/2$, and define the function

$$u_\alpha(r\sigma) = r^{\frac{2-N}{2}} |\log r|^{-\delta_\alpha} \Phi(\sigma) \, .$$

Here Φ solves (2.3) if $N \geq 2$. If $N = 1$ we agree that $\sigma = 1$ and $\Phi \equiv 1$. By direct computations one has that u_α solves (2.2). Moreover, if $\gamma \leq 1$ and $R \in (0,1)$ then $u_\alpha \in L^2(\mathcal{C}_\Sigma^R; |x|^{-2} \left|\log |x|\right|^{-2\gamma} \ dx)$ if and only if $\alpha < \frac{1}{4} - (1-\gamma)^2$.

Remark 2.4 *Nonexistence results for linear inequalities involving the differential operator $-\Delta - \mu(\mathcal{C}_\sigma)|x|^{-2}$ were already obtained in [11].*

A Hardy-Leray inequalities on cone-like domains

In this appendix we give a simple proof of an improved Hardy inequality for mappings having support in a cone-like domain. We recall that for $\Sigma \subset \mathbb{S}^{N-1}$ we have set $\mathcal{C}^1_\Sigma = \{ r\sigma \mid r \in (0,1) \ , \ \sigma \in \Sigma \ \}$ and $\mu(\mathcal{C}^1_\Sigma) = (N-2)^2/4 + \lambda_1(\Sigma)$.

Proposition A.1 *Let Σ be a domain in $\mathbb{S}^{N-1}$. Then*

$$\text{(A.1)} \qquad \int_{\mathcal{C}^1_\Sigma} |\nabla u|^2 \ dx - \mu(\mathcal{C}_\Sigma) \int_{\mathcal{C}^1_\Sigma} |x|^{-2}|u|^2 \geq \frac{1}{4} \int_{\mathcal{C}^1_\Sigma} |x|^{-2} \left|\log |x|\right|^{-2} |u|^2 \ dx$$

for any $u \in C^\infty_c(\mathcal{C}^1_\Sigma)$.

Proof. We start by fixing an arbitrary function $v \in C^\infty_c(\mathbb{R}_+ \times \Sigma)$. We apply the Hardy inequality to the function $v(\cdot, \sigma) \in C^\infty_c(\mathbb{R}_+)$, for any fixed $\sigma \in \Sigma$, and then we integrate over Σ to get

$$\int_0^\infty \int_\Sigma |v_s|^2 \ dsd\sigma \geq \frac{1}{4} \int_0^\infty \int_\Sigma s^{-2}|v|^2 \ dsd\sigma \,.$$

On the other hand, notice that $v(s, \cdot) \in C^\infty_c(\Sigma)$ for any $s \in \mathbb{R}_+$. Thus, the Poincaré inequality for maps in Σ plainly implies

$$\int_0^\infty \int_\Sigma |\nabla_\sigma v|^2 \ dsd\sigma - \lambda_1(\Sigma) \int_0^\infty \int_\Sigma |v|^2 \ dsd\sigma \geq 0\,.$$

Adding these two inequalities we conclude that

$$\int_0^\infty \int_\Sigma \left[|v_s|^2 + |\nabla_\sigma v|^2 \right] \ dsd\sigma - \lambda_1(\Sigma) \int_0^\infty \int_\Sigma |v|^2 \ dsd\sigma \geq \frac{1}{4} \int_0^\infty \int_\Sigma s^{-2}|v|^2 \ dsd\sigma$$

for any $v \in C^\infty_c(\mathbb{R}_+ \times \Sigma)$. We use once more the Emden-Fowler transform T in (1.13) by letting $v := Tu \in C^\infty_c(\mathbb{R}_+ \times \Sigma)$ for $u \in C^\infty_c(\mathcal{C}^1_\Sigma)$. Since

$$\int_{B_1} \left[|\nabla u|^2 - \frac{(N-2)^2}{4} |x|^{-2}|u|^2 \right] \ dx = \int_0^\infty \int_{\mathbb{S}^{N-1}} \left[|v_s|^2 + |\nabla_\sigma v|^2 \right] \ dsd\sigma \,,$$

then (1.14) readily leads to the conclusion. $\square$

Remark A.2 *The arguments we have used to prove Proposition A.1 and the fact that the best constant in the Hardy inequality for maps in $C_c^\infty(\mathbb{R}_+)$ is not achieved show that the constants in inequality (A.1) are sharp, and not achieved.*

Remark A.3 *Notice that for $N \geq 1$, we have $\mathcal{C}_{\mathbb{S}^{N-1}} = \mathbb{R}^N \setminus \{0\}$ and $\mu(\mathcal{C}_{\mathbb{S}^{N-1}}) = (N-2)^2/4$. Thus A.1 gives (0.2) for $u \in C_c^\infty(B_1 \setminus \{0\})$.*

In the next proposition we extend the inequality (A.1) to cover the case $N = 1$.

Proposition A.4 *It holds that*

$$\int_0^1 |u'|^2 \, dt - \frac{1}{4} \int_0^1 t^{-2} |u|^2 \, dt \geq \frac{1}{4} \int_0^1 t^{-2} |\log t|^{-2} |u|^2 \, dt$$

for any $u \in C_c^\infty(0,1)$. The constants are sharp, and not achieved.

Proof. Write $u(t) = t^{1/2}\psi(|\log t|) = t^{1/2}\psi(s)$ for a function $\psi \in C_c^\infty(\mathbb{R}_+)$ and then apply the Hardy inequality to ψ. □

Next, let $\theta \in \mathbb{R}$ be a given parameter and let Σ be a Lipschitz domain in $\mathbb{S}^{N-1}$, with $N \geq 2$. For an arbitrary $u \in C_c^\infty(\mathcal{C}_\Sigma^1)$ we put $v = |x|^{-\theta/2} u$. Then the Hardy-Leray inequality (A.1) and integration by parts plainly imply that

$$\int_{\mathcal{C}_\Sigma^1} |x|^\theta |\nabla v|^2 \, dx - \mu(\mathcal{C}_\Sigma; \theta) \int_{\mathcal{C}_\Sigma^1} |x|^{\theta-2} |v|^2 \geq \frac{1}{4} \int_{\mathcal{C}_\Sigma^1} |x|^{\theta-2} \left|\log |x|\right|^{-2} |v|^2 \, dx$$

for any $v \in C_c^\infty(\mathcal{C}_\Sigma^1)$, where

$$\mu(\mathcal{C}_\Sigma; \theta) := \frac{(N-2+\theta)^2}{4} + \lambda_1(\Sigma) \,. \tag{A.2}$$

It is well known that

$$\frac{(N-2+\theta)^2}{4} = \inf_{\substack{u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \\ u \neq 0}} \frac{\displaystyle\int_{B_1} |x|^\theta |\nabla u|^2 \, dx}{\displaystyle\int_{B_1} |x|^{\theta-2} |u|^2 \, dx}$$

is the Hardy constant relative to the operator $L_\theta v = -\mathrm{div}(|x|^\theta \nabla v)$. For the case $N = 1$ one can obtain in a similar way the inequality

$$\int_0^1 t^\theta |v'|^2 \, dt - \frac{(\theta-1)^2}{4} \int_0^1 t^{\theta-2} |v|^2 \, dt \geq \frac{1}{4} \int_0^1 t^{\theta-2} |\log t|^{-2} |v|^2 \, dt$$

which holds for any $\theta \in \mathbb{R}$ and for any $v \in C_c^\infty(0,1)$.

B A general necessary condition

In this appendix we show in particular that a necessary condition for the existence of non-trivial and non-negative solutions to (0.1) and (2.2) is that $\alpha \leq 1/4$. We need the following general lemma, which naturally fits into the classical Allegretto-Piepenbrink theory (see for instance [3] and [16]).

Lemma B.1 *Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$. Let $a \in L^\infty_{loc}(\Omega)$ and $a > 0$ in Ω. Assume that $u \in L^1_{loc}(\Omega)$ is a non-negative, non-trivial solution to*

$$-\Delta u \geq a(x)u \quad \mathcal{D}'(\Omega).$$

Then

$$\int_\Omega |\nabla\phi|^2 \, dx \geq \int_\Omega a(x)\,|\phi|^2 \, dx, \quad \text{for any } \phi \in C^\infty_c(\Omega).$$

Proof. Let $A \subset \Omega$ be a measurable set such that $|A| > 0$ and $u > 0$ in A. Fix any function $\phi \in C^\infty_c(\Omega)$ and choose a domain $\widetilde{\Omega} \subset\subset \Omega$ such that $|\widetilde{\Omega} \cap A| > 0$ and $\phi \in C^\infty_c(\widetilde{\Omega})$. For any integer k large enough put $f_k = \min\{a(x)u, k\} \in L^\infty(\widetilde{\Omega})$. Let $v_k \in H^1_0(\widetilde{\Omega})$ be the unique solution to

$$\text{(B.1)} \qquad \begin{cases} -\Delta v_k = f_k & \text{in } \widetilde{\Omega}, \\ v_k = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

Notice that $v \in C^{1,\beta}(\widetilde{\Omega})$ for any $\beta \in (0,1)$. Since for k large enough the function f_k is non-negative and non-trivial then $v \geq 0$. Actually it turns out that $v^{-1} \in L^\infty_{loc}(\widetilde{\Omega})$ by the Harnack inequality. Finally, a convolution argument and the maximum principle plainly give

$$\text{(B.2)} \qquad u \geq v_k > 0 \quad \text{almost everywhere in } \widetilde{\Omega}.$$

Since $v_k^{-1}\phi \in L^\infty(\widetilde{\Omega})$ then we can use $v_k^{-1}\phi^2$ as test function for (B.1) to get

$$\int_\Omega \nabla v_k \cdot \nabla \left(v_k^{-1}\phi^2\right) \, dx = \int_\Omega f_k v_k^{-1}\phi^2 \, dx \geq \int_\Omega f_k u^{-1}\phi^2 \, dx$$

by (B.2). Since $\nabla v_k \cdot \nabla \left(v_k^{-1}\phi^2\right) = |\nabla\phi|^2 - \left|v_k\nabla(v_k^{-1}\phi)\right|^2 \leq |\nabla\phi|^2$, we readily infer

$$\int_\Omega |\nabla\phi|^2 \, dx \geq \int_\Omega f_k u^{-1}\phi^2 \, dx$$

and Fatou's lemma implies that

$$\int_\Omega |\nabla \phi|^2 \ dx \geq \int_\Omega a(x)\, \phi^2 \ dx\,.$$

The conclusion readily follows. □

The sharpness of the constants in (0.2) (compare with Remark A.2) and Lemma B.1 plainly imply the following result.

Theorem B.2 *Let $N \geq 1$, $R \in (0,1]$ and $c, \alpha \geq 0$. Let $u \in L^1_{loc}(B_R \setminus \{0\})$ be a non-negative distributional solution to*

$$-\Delta u - c\,|x|^{-2}\, u \geq \alpha |x|^{-2}\, |\log|x||^{-2}\ u \quad \text{in } \mathcal{D}'(B_R \setminus \{0\}).$$

i) If $c > \frac{(N-2)^2}{4}$ then $u \equiv 0$.
ii) If $c = \frac{(N-2)^2}{4}$ and $\alpha > \frac{1}{4}$ then $u \equiv 0$.

We notice that proposition *i)* in Theorem B.2 was already proved in [4] (see also [10]).

Finally, from Remark A.2 and Lemma B.1, we obtain the next nonexistence result.

Theorem B.3 *Let Σ be a domain properly contained in $\mathbb{S}^{N-1}$, $R \in (0,1]$ and $c, \alpha \geq 0$. Let $u \in L^1_{loc}(\mathcal{C}^R_\Sigma)$ be a non-negative distributional solution to*

$$-\Delta u - c\,|x|^{-2}\, u \geq \alpha |x|^{-2}\, |\log|x||^{-2}\ u \quad \text{in } \mathcal{D}'(\mathcal{C}^R_\Sigma).$$

i) If $c > \mu(\mathcal{C}_\Sigma)$ then $u \equiv 0$.
ii) If $c = \mu(\mathcal{C}_\Sigma)$ and $\alpha > \frac{1}{4}$ then $u \equiv 0$.

C Extensions

In this appendix we state some nonexistence theorems that can proved by using a suitable functional change $u \mapsto \psi$ and Theorem 1.3. We shall also point out some corollaries of our main results.

C.1 The k-improved weights

We define a sequence of radii $R_k \to 0$ by setting $R_1 = 1$, $R_k = e^{-\frac{1}{R_{k-1}}}$. Then we use induction again to define two sequences of radially symmetric weights $X_k(x) \equiv X_k(|x|)$ and z_k in B_{R_k} by setting $X_1(|x|) = |\log|x||^{-1}$ for $|x| < 1 = R_1$ and

$$X_{k+1}(|x|) = X_k\left(|\log|x||^{-1}\right) \ , \quad z_k(x) = |x|^{-1}\prod_{i=1}^{k} X_i(|x|)$$

for all $x \in B_{R_k} \setminus \{0\}$. It can be proved by induction that z_k is well defined on B_{R_k} and $z_k \in L^2_{\rm loc}(B_{R_k})$. We are interested in distributional solutions to

$$-\Delta u - \frac{(N-2)^2}{4}|x|^{-2}u \geq \alpha z_k^2\, u \quad \mathcal{D}'(B_R \setminus \{0\}) \tag{C.1}$$

for $R \in (0, R_k]$. The next result includes Theorem 0.1 by taking $k = 1$.

Theorem C.1 *Let $k \geq 1$, $R \in (0, R_k]$ and let $u \geq 0$ be a distributional solution to (C.1). Assume that there exists $\gamma \leq 1$ such that*

$$u \in L^2_{\rm loc}(B_R; z_k^2 X_k^{2(\gamma-1)}\ dx)\ , \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2\,.$$

Then $u = 0$ almost everywhere in B_R.

Proof. We start by introducing the k^{th} Emden-Fowler transform $u \mapsto T_k u$,

$$u(x) = z_k(|x|)^{-\frac{1}{2}}|x|^{\frac{1-N}{2}}X_k(|x|)^{\frac{1}{2}}(T_k u)\left(X_k(|x|)^{-1}, \frac{x}{|x|}\right)\,.$$

Notice that for any $R < R_k$ it results

$$\int_{B_R} z_k^2\, X_k^{2(\gamma-1)}|u|^2\ dx = \int_{X_k(R)^{-1}}^{\infty} s^{-2\gamma}\int_{\mathbb{S}^{N-1}} |T_k u|^2\ ds d\sigma\,, \tag{C.2}$$

so that $T_k u \in L^2(I_a \times \mathbb{S}^{N-1}; s^{-2\gamma}\ dsd\sigma)$ for any $a > X_k(R)^{-1}$. This can be easily checked by noticing that $X_k' = z_k X_k$. Next we set

$$\psi_u(s) := \int_{\mathbb{S}^{N-1}} (T_k u)(s,\sigma)d\sigma\,.$$

By (C.2) we have that $\psi \in L^2(I_a; s^{-2\gamma}\ ds)$ for any $a > X_k(R)^{-1}$. Thanks to Theorem 1.3, to conclude the proof it suffices to show that ψ is a distributional solution to

$-\psi'' \geq \alpha s^{-2}\psi$ in the interval $I_{\tilde{a}}$, where $\tilde{a} = X_k(R)^{-1}$. To this end, fix any test function $\varphi \in C^\infty(I_{\tilde{a}})$, and define the radially symmetric mapping $\tilde{\varphi} \in C_c^\infty(B_R \setminus \{0\})$ such that $T_k\tilde{\varphi} = \varphi$. By direct computation one can prove that

$$\Delta\tilde{\varphi} + \frac{(N-2)^2}{4}|x|^{-2}\tilde{\varphi} = \omega\tilde{\varphi} + |x|^{\frac{1-N}{2}} z_k^{\frac{3}{2}} X_k^{-\frac{3}{2}} \varphi'' \left(X_k(|x|)^{-1}\right)$$

where $\omega \equiv 0$ if $k = 1$, and

$$\omega = \frac{1}{2}\left[\left(\sum_{i=1}^{k-1} z_i\right)^2 - \frac{1}{2}\sum_{i=1}^{k-1} z_i^2\right]$$

if $k \geq 2$. Since $\omega \geq 0$ then

$$\int_{B_R} u\left(\Delta\tilde{\varphi} + \frac{(N-2)^2}{4}|x|^{-2}\tilde{\varphi}\right)\, dx \geq \int_{\tilde{a}}^{\infty} \psi\varphi''\, ds$$

provided that φ is non-negative. In addition it results

$$\int_{B_R} z_k^2\, u\tilde{\varphi}\ dx = \int_{\tilde{a}}^{\infty} s^{-2}\psi\varphi\ ds\,.$$

Since φ was arbitrarily chosen, the conclusion readily follows. □

By similar arguments as above and in Section 2, we can prove a nonexistence result of positive solutions to the problem

$$-\Delta u - \mu(\mathcal{C}_\Sigma)|x|^{-2}u \geq \alpha z_k^2\, u \quad \mathcal{D}'(\mathcal{C}_\Sigma^R)\,, \tag{C.3}$$

where $\mathcal{C}_\Sigma$ is a Lipschitz proper cone in $\mathbb{R}^N$, $N \geq 1$, and $C_\Sigma^R = \mathcal{C}_\Sigma \cap B_R$. We shall skip the proof the following result.

Theorem C.2 *Let $k \geq 1$, $R \in (0, R_k]$ and let $u \geq 0$ be a distributional solution to* (C.3)*. Assume that there exists $\gamma \leq 1$ such that*

$$u \in L^2(\mathcal{C}_\Sigma^R; z_k^2 X_k^{2(\gamma-1)}\ dx)\ , \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2\,.$$

Then $u = 0$ almost everywhere in $\mathcal{C}_\Sigma^R$.

Some related improved Hardy inequalities involving the weight z_k and which motivate the interest of problems (C.1) and (C.3) can be found in [2], [8], [12] and also [5].

C.2 Exterior cone-like domains

The Kelvin transform

$$u(x) \mapsto |x|^{2-N} u\left(\frac{x}{|x|^2}\right)$$

can be used to get nonexistence results for exterior domains in $\mathbb{R}^N$.

Let Σ be a domain in $\mathbb{S}^{N-1}$, $N \geq 2$, and let $\mathcal{C}_\Sigma$ be the cone defined in Section 2. We recall that $\mu(\mathcal{C}_\Sigma) = (N-2)^2/4 + \lambda_1(\Sigma)$. Since the inequality in (0.1) is invariant with respect to the Kelvin transform, then Theorems 0.1 and 2.1 readily lead to the following nonexistence result.

Theorem C.3 *Let Σ be a Lipschitz domain in $\mathbb{S}^{N-1}$, with $N \geq 2$. Let $R > 1$, $\alpha \in \mathbb{R}$ and let $u \geq 0$ be a distributional solution to*

$$-\Delta u - \mu(\mathcal{C}_\Sigma)|x|^{-2}\, u \geq \alpha |x|^{-2}\, |\log |x||^{-2}\, u \quad \text{in } \mathcal{D}'(\mathcal{C}_\Sigma \setminus \overline{B}_R).$$

Assume that there exists $\gamma \leq 1$ such that

$$u \in L^2(\mathcal{C}_\Sigma \setminus \overline{B}_R; |x|^{-2}|\log |x||^{-2\gamma}\, dx)\ , \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2\,.$$

Then $u = 0$ almost everywhere in $\mathcal{C}_\Sigma \setminus \overline{B}_R$.

A similar statement holds in case $N = 1$ for ordinary differential inequalities in unbounded intervals $(R, 0)$ with $R > 0$, and for problems involving the weight z_k^2.

C.3 Degenerate elliptic operators

Let $\theta \in \mathbb{R}$ be a given real parameter. We notice that u is a distributional solution to (2.2) if and only if $v = |x|^{-\theta/2} u$ is a distributional solution to

$$\text{(C.4)} \qquad -\text{div}(|x|^\theta \nabla v) - \mu(\mathcal{C}_\Sigma; \theta)|x|^{\theta-2}\, v \geq \frac{1}{4}\, |x|^{\theta-2}|\log |x||^{-2}\, v \quad \text{in } \mathcal{D}'(\mathcal{C}_\Sigma^R)\ ,$$

where $\mu(\mathcal{C}_\Sigma; \theta)$ is defined in Remark A.2. Therefore Theorem 0.1 and Theorem 2.1 imply the following nonexistence result for linear inequalities involving the weighted Laplace operator $L_\theta v = -\text{div}(|x|^\theta \nabla v)$.

Theorem C.4 *Let Σ be a Lipschitz domain in $\mathbb{S}^{N-1}$. Let $\theta \in \mathbb{R}$, $R \in (0,1]$, $\alpha \in \mathbb{R}$ and let $v \geq 0$ be a distributional solution to (C.4). Assume that there exists $\gamma \leq 1$ such that*

$$v \in L^2(\mathcal{C}_\Sigma^R; |x|^{\theta-2} |\log|x||^{-2\gamma}\ dx)\ , \quad \alpha \geq \frac{1}{4} - (1-\gamma)^2\,.$$

Then $v = 0$ almost everywhere in $\mathcal{C}_\Sigma^R$.

A nonexistence result for the operator $-\mathrm{div}(|x|^\theta \nabla v)$ similar to Theorem C.3 or to Theorem C.1 can be obtained from Theorem C.4, via suitable functional changes.

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