Nonexistence of distributional supersolutions of a semilinear elliptic equation with Hardy potential

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Abstract. In this paper we study nonexistence of non-negative distributional supersolutions for a class of semilinear elliptic equations involving inverse-square potentials.

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Introduction

Let Ω define a domain of $\mathbb{R}^N$, $N \geq 3$. In this paper, we study nonnegative functions u satisfying

$$-\Delta u - b(x)\, u \geq u^p \quad \text{ in } \mathcal{D}'(\Omega), \tag{0.1}$$

with $p > 1$, $b \gneqq 0$ and $b \in L^1_{loc}(\Omega)$ is a singular potential of Hardy-type. More precisely, we are interested in distributional solutions to (0.1), that is, functions $u \in L^p_{\mathrm{loc}}(\Omega)$ such that $b(x)\, u \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega u(-\Delta\varphi - b(x)\varphi)\; dx \geq \int_\Omega u^p \varphi \; dx \qquad \forall \varphi \in C^\infty_c(\Omega),\, \varphi \geq 0.$$

The study of nonexistence results of (very) weak solution to problem (0.1) goes back to [4], where the authors were motivated by the failure of the Implicit Function

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Theorem. Further references in this direction are [5], [9], [12], [13]. We also quote [2], [3] [27], [29], [23], [21], [24], [20].

In this paper, we study nonexistence of solutions to (0.1) when $\partial\Omega$ possesses a conical singularity at 0 as well as when $\partial\Omega$ is of class C^2 at 0. Higher dimensional singularity will be also considered.

For any domain Σ in the unit sphere $\mathbb{S}^{N-1}$ we introduce the cone

$$\mathcal{C}_\Sigma := \left\{ r\sigma \in \mathbb{R}^N \mid r > 0 \ , \ \sigma \in \Sigma \ \right\} .$$

We recall that the best constant in the Hardy inequality for functions supported by $\mathcal{C}_\Sigma$ is given by

$$\mu(\mathcal{C}_\Sigma) := \inf_{u \in C^\infty_c(\mathcal{C}_\Sigma)} \frac{\displaystyle\int_{\mathcal{C}_\Sigma} |\nabla u|^2 \ dx}{\displaystyle\int_{\mathcal{C}_\Sigma} |x|^{-2} u^2 \ dx} = \frac{(N-2)^2}{4} + \lambda_1(\Sigma) \, ,$$

where $\lambda_1(\Sigma)$ is the first Dirichlet eigenvalue for the Laplace-Beltrami operator on Σ ([17], [28]). For a given radius $R > 0$ we introduce the cone-like domain

$$\mathcal{C}^R_\Sigma := \mathcal{C}_\Sigma \cap B_R = \{ r\sigma \mid r \in (0, R) \ , \ \sigma \in \Sigma \ \} \, ,$$

where B_R is the ball of radius R centered at 0. We study the inequality

$$-\Delta u - \frac{c}{|x|^2} \, u \ge u^p \quad \text{ in } \mathcal{D}'\left(\mathcal{C}^R_\Sigma\right) , \tag{0.2}$$

with

$$\lambda_1(\Sigma) < c \le \mu(\mathcal{C}_\Sigma) \, .$$

By homogeneity, an important role is played by

$$\alpha^-_\Sigma := \frac{N-2}{2} - \sqrt{\mu(\mathcal{C}_\Sigma) - c} \, ,$$

which is the smallest root of the equation

$$\alpha^2 - (N-2)\alpha + c - \lambda_1(\Sigma) = 0.$$

We notice that the restriction $c \le \mu(\mathcal{C}_\Sigma)$ is not restrictive (see Remark 1.5 below) and in addition $\alpha^-_\Sigma > 0$ when $c > \lambda_1(\Sigma)$. Finally we define

$$p_\Sigma = 1 + \frac{2}{\alpha^-_\Sigma} \, .$$

We observe that $p_\Sigma = \frac{N+2}{N-2}$ when $c = \mu(\mathcal{C}_\Sigma)$ while $p_\Sigma > \frac{N+2}{N-2}$ as soon as $c < \mu(\mathcal{C}_\Sigma)$. In [5], the authors have studied the case $\Omega = B_R \setminus \{0\} = \mathcal{C}^R_{\mathbb{S}^{N-1}}$. They proved that (0.2) has a non-trivial solution in $B_R \setminus \{0\}$ if and only if $p < p_{\mathbb{S}^{N-1}}$.
Our first result generalizes the nonexistence result in [5] to cone-like domains.

Theorem 0.1 *Let $\mathcal{C}^R_\Sigma$ be a cone-like domain of $\mathbb{R}^N$, $N \geq 3$. For $\lambda_1(\Sigma) < c \leq \mu(\mathcal{C}_\Sigma)$, let $u \in L^p_{loc}\left(\mathcal{C}^R_\Sigma\right)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'\left(\mathcal{C}^R_\Sigma\right).$$

If $p \geq p_\Sigma$ then $u \equiv 0$.

Theorem 0.1 improves a part of the nonexistence results obtained in [22], where more regular supersolutions were considered. We notice that the assumption $p \geq p_\Sigma$ is sharp (see the existence result in [22], Theorem 1.2).
We next consider the case where $0 \in \partial\Omega$ with $\partial\Omega$ is smooth at 0 and $b(x) = c|x|^{-2}$.

We define

$$\mu(\Omega) := \inf_{u \in C^\infty_c(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |x|^{-2} u^2 \, dx}.$$

Put $\Omega_r := \Omega \cap B_r(0)$. Recently it was proved in [15] that, there exits $r_0 = r_0(\Omega) > 0$ such that for all $r \in (0, r_0)$

$$\mu(\Omega_r) = \mu\left(\mathcal{C}_{\mathbb{S}^{N-1}_+}\right) = \frac{N^2}{4}, \tag{0.3}$$

with $\mathbb{S}^{N-1}_+$ is a hemisphere centered at 0 so that $\mathcal{C}_{\mathbb{S}^{N-1}_+}$ is a half-space. We have obtained:

Theorem 0.2 *Let Ω be a smooth domain of $\mathbb{R}^N$, $N \geq 3$, with $0 \in \partial\Omega$. Let $r > 0$ small so that* (0.3) *holds. For $N - 1 < c \leq \frac{N^2}{4}$, let $u \in L^p_{loc}(\Omega_r)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega_r).$$

If $p \geq p_{\mathbb{S}^{N-1}_+}$ then $u \equiv 0$.

Here also the nonexistence of nontrivial solution for $c \in (N-1, N^2/4]$ is sharp, see Proposition 3.2.

When we consider general domains, we face some obstacles in the restriction of the parameter c. This is due to the fact that $\mu(\Omega)$ is not in general smaller than $N-1$ for smooth domains Ω, with $0 \in \partial\Omega$, see [17]. A consequence of Theorem 0.2 is:

Corollary 0.3 *Let Ω be a smooth domain of $\mathbb{R}^N$, $N \geq 3$, with $0 \in \partial\Omega$. Assume that $N-1 < c \leq \mu(\Omega)$. Suppose that there exists $u \in L^p_{loc}(\Omega)$, $u \geq 0$ such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

If $p \geq p_{\mathbb{S}^{N-1}_+}$ then $u \equiv 0$.

In Corollary 0.3 above, we assume that the interval $[N-1, \mu(\Omega)]$ is not empty. This is not in general true (see Remark 0.6 below). However it holds for various domains or in higher dimensions. Indeed, we first observe that the inequality $\frac{(N-2)^2}{4} < \mu(\Omega) \leq \frac{N^2}{4}$ is valid for every smooth bounded domain Ω with $0 \in \partial\Omega$, see [17]. In particular $\mu(\Omega) > N-1$ whenever $N \geq 7$. Hence we get:

Corollary 0.4 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 7$, with $0 \in \partial\Omega$. Let $N-1 < c \leq \mu(\Omega)$ and $u \in L^p_{loc}(\Omega)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

If $p \geq p_{\mathbb{S}^{N-1}_+}$ then $u \equiv 0$.

When Ω is a smooth domain (not necessarily bounded), with $0 \in \partial\Omega$, is contained in the half-space $\mathcal{C}_{\mathbb{S}^{N-1}_+}$ then obviously $\mu(\Omega) = \frac{N^2}{4}$ by (0.3). In particular, thanks to Theorem 0.2, the restriction $N \geq 7$ in Corollary 0.4 and the boundedness of Ω can be removed. Indeed, we have:

Corollary 0.5 *Let Ω be a smooth domain of the half-space $\mathcal{C}_{\mathbb{S}^{N-1}_+}$, $N \geq 3$, with $0 \in \partial\Omega$. Let $N-1 < c \leq \frac{N^2}{4}$ and $u \in L^p_{loc}(\Omega)$ be non-negative such that*

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega).$$

If $p \geq p_{\mathbb{S}^{N-1}_+}$ then $u \equiv 0$.

Remark 0.6 *According to our argument, the assumption $N-1<\mu(\Omega)$ is crucial because it implies that $1<p_{\mathbb{S}^{N-1}_+}<\infty$ when $c>N-1$. However it is not valid for every smooth domain. In fact, one can construct a family of smooth bounded domains Ω^ε, for which $\mu(\Omega^\varepsilon)\le\frac{(N-2)^2}{4}+\varepsilon$, for $\varepsilon>0$ small, see [17], [16].*

Remark 0.7 *The conclusion in theorems 0.1, 0.2 still holds when u^p is replaced by $|x|^s u^q$ with $\lambda_1(\Sigma)<c\le\mu(\mathcal{C}_\Sigma)$. In this case one has to replace p_Σ with $q_\Sigma=1+\frac{2-s}{\alpha_\Sigma^-}$.*

We prove our nonexistence results via a linearization argument which were also used in [22]. However when working with weaker notion of solutions, further analysis are required. Our approach is to obtain a quite sharp lower estimate on u in such a way that u^{p-1} is somehow proportional to $b(x)$ and to look the problem as a linear problem: $-\Delta u-b(x)u-u^{p-1}\,u\ge 0$ in $\mathcal{D}'(\Omega)$. This leads to the inequality (see Lemma 1.4)

$$\int_\Omega|\nabla\varphi|^2-\int_\Omega b(x)\varphi^2\ge\int_\Omega u^{p-1}\varphi^2\quad\forall\varphi\in C_c^\infty(\Omega). \tag{0.4}$$

By using appropriate test functions in (0.4), we were able to contradict the existence of solutions. To lower estimate u, we construct sub-solutions for the operator $L:=-\Delta-b(x)$. On the other hand since we are working with "very weak" supersolutions in non-smooth domains, and the operator L does not in general satisfies the maximum principle, we have proved a comparison principle (see Lemma 1.3 in Section 1). We achieve this by requiring L to be coercive. Since in this paper the potential $b(x)$ is of Hardy-type, such coercivity is nothing but improvements of Hardy inequalities. The comparison principle allows us to put below u a more regular function v. Such function v turns out to be a supersolution for L and therefore can be lower estimated by the sub-solutions via standard arguments.

The paper is organized as follows. In Section 1 we prove some preliminary results, which are mainly used in the paper. The proofs of Theorems 0.1, 0.2 will be carried out in Sections 2, 3 respectively. Finally in the last section, we study the problem

$$\begin{cases}-\Delta u-\frac{(N-k-2)^2}{4}\,\frac{1}{\mathrm{dist}(x,\Gamma)^2}\,q(x)\,u\ge u^p & \text{in } \mathcal{D}'(\Omega\setminus\Gamma),\\ u\in L^p_{loc}(\Omega\setminus\Gamma),\\ u\gneqq 0,\end{cases} \tag{0.5}$$

where Γ is a smooth closed submanifold of Ω and q is a nonnegative weight.

1 Preliminaries and comparison lemmata

Let Ω be a bounded open subset of $\mathbb{R}^N$. In this section we deal with comparison results involving a differential operator of the type

$$-\Delta - b(x)\,,$$

where $b \in L^1_{loc}(\Omega)$ is a given non-negative weight. We shall always assume that $-\Delta - b(x)$ is coercive, in the sense that there exists a constant $C(\Omega) > 0$ such that

$$\int_\Omega |\nabla u|^2\, dx - \int_\Omega b(x)u^2\, dx \geq C(\Omega)\int_\Omega u^2\, dx \quad \text{for any } u \in C_c^\infty(\Omega). \tag{1.1}$$

Following [10], we define the space $H(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the scalar product

$$(u,v) \mapsto \int_\Omega \nabla u \nabla v\, dx - \int_\Omega b(x)uv\, dx\,.$$

The scalar product in $H(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle_{H(\Omega)}$.
Clearly $H^1_0(\Omega) \hookrightarrow H(\Omega) \hookrightarrow L^2(\Omega)$ by (1.1), and hence $L^2(\Omega)$ embeds into the dual space $H(\Omega)'$. By the Lax Milligram theorem, for any $f \in L^2(\Omega)$ there exists a unique function $v \in H(\Omega)$ such that

$$-\Delta v - b(x)v = f \quad \text{in } H(\Omega),$$

that is,

$$\langle v,\varphi\rangle_{H(\Omega)} = \int_\Omega f\varphi\, dx \quad \text{for any } \varphi \in H(\Omega).$$

Remark 1.1 *Observe that if $b \in L^\infty(\Omega)$ then $H(\Omega) = H^1_0(\Omega)$ since*

$$C\int_\Omega |\nabla u|^2\, dx \leq \|u\|^2_{H(\Omega)} \leq \int_\Omega |\nabla u|^2\, dx\,,$$

where the constant $C > 0$ depends only on $C(\Omega)$, and on the L^∞ norm of b.

We start with the following technical result which will be useful in the sequel.

Lemma 1.2 *Let $u \in L^1_{loc}(\Omega)$ be non-negative and $g \in L^2(\Omega)$ such that*

$$-\Delta u \geq g \quad \text{in } \mathcal{D}'(\Omega).$$

Let $v \in H^1_0(\Omega)$ be the solution to

$$-\Delta v = g \quad \text{in } \Omega.$$

Then

$$v \leq u \quad \text{in } \Omega.$$

Proof. For $\varepsilon > 0$, define $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Let $\widetilde{\Omega}_\varepsilon$ be a smooth open set compactly contained in Ω and containing Ω_ε. Denote by ρ_n the standard mollifier and put $u_n = \rho_n * u$. Then for $\varepsilon > 0$ there exists N_ε such that u_n is smooth in $\widetilde{\Omega}_\varepsilon$ up to the boundary for all $n \geq N_\varepsilon$. Consider $v_{\varepsilon,n} \in H^1_0(\widetilde{\Omega}_\varepsilon)$ be the solution of $-\Delta v_{\varepsilon,n} = \rho_n * g = g_n$ in $\widetilde{\Omega}_\varepsilon$. Clearly $-\Delta(u_n - v_{\varepsilon,n}) \geq 0$ in $\widetilde{\Omega}_\varepsilon$ and $u_n - v_{\varepsilon,n} \geq 0$ on $\partial\widetilde{\Omega}_\varepsilon$, because u is non-negative. It turns out that $u_n - v_{\varepsilon,n} \geq 0$ in $\widetilde{\Omega}_\varepsilon$ by the maximum principle. Letting $v_\varepsilon \in H^1_0(\widetilde{\Omega}_\varepsilon)$ be the solution of $-\Delta v_\varepsilon = g$ in $\widetilde{\Omega}_\varepsilon$, by Hölder and Poincaré inequalities, we have that $\|v_{\varepsilon,n} - v_\varepsilon\|_{H^1_0(\widetilde{\Omega}_\varepsilon)} \leq C\|g_n - g\|_{L^2(\widetilde{\Omega}_\varepsilon)}$, with $C > 0$ is a constant independent on n. In particular $v_{\varepsilon,n}$ converges to v_ε in $\widetilde{\Omega}_\varepsilon$. Therefore $u \geq v_\varepsilon$ in $\widetilde{\Omega}_\varepsilon$. To conclude, it suffices to notice that $v_\varepsilon \to v$ weakly in $H^1_0(\Omega)$ and pointwise in Ω. □

We have the following comparison principle.

Lemma 1.3 *Let $u \in L^1_{loc}(\Omega)$ be non-negative with $b(x)u \in L^1_{loc}(\Omega)$ and let $f \in L^2(\Omega)$ with $f \geq 0$ such that*

$$-\Delta u - b(x)u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

Let $v \in H(\Omega)$ be the solution of

$$-\Delta v - b(x)v = f \quad \text{in } H(\Omega).$$

Then

$$v \leq u \quad \text{in } \Omega.$$

Proof. Step 1: *We first prove the result if* $b \in L^\infty(\Omega)$.
We let $v_0 \in H_0^1(\Omega)$ solving

$$-\Delta v_0 = f \quad \text{ in } \Omega.$$

Then $0 \le v_0 \le u$ in Ω by Lemma 1.2 and because $f \ge 0$. We define inductively the sequence $v_n \in H_0^1(\Omega)$ by

$$-\Delta v_1 = b(x)v_0 + f \quad \text{in } \Omega, \qquad -\Delta v_n = b(x)v_{n-1} + f \quad \text{in } \Omega.$$

Since $b \ge 0$, we have $-\Delta u \ge b(x)v_0 + f$ in $\mathcal{D}'(\Omega)$. Thus using once again Lemma 1.2, we obtain $v_0 \le v_1 \le u$ in Ω. By induction, we have

$$v_0 \le v_1 \le \cdots \le v_n \le u \quad \text{ in } \Omega \quad \forall n \in \mathbb{N}.$$

Since $v_{n-1} \le v_n$ in Ω, we have

$$\int_\Omega |\nabla v_n|^2 dx - \int_\Omega b(x)|v_n|^2 \le \int_\Omega f(x) v_n dx.$$

By Hölder inequality and (1.1) (see Remark 1.1) v_n is bounded in $H_0^1(\Omega)$. We conclude that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$ as $n \to \infty$ which is the unique solution to

$$-\Delta v = b(x)v + f \quad \text{in } \Omega.$$

Since $v_n \to v$ in $L^2(\Omega)$, we get $v \le u$ in Ω.

Step 2: *Conclusion of the proof.*
We put $b_k(x) = \min(b(x), k)$ for every $k \in \mathbb{N}$. We consider $v^k \in H_0^1(\Omega)$ be the unique solution to

$$\int_\Omega \nabla v^k \nabla \varphi - \int_\Omega \min\{b(x), k\}\, v^k \varphi = \int_\Omega f\varphi \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.2}$$

Thanks to **Step 1**, we have $v^k \le u$ in Ω.
Next, we check that such a sequence v^k, satisfying (1.2), converges to v in $L^2(\Omega)$ when $k \to \infty$. Indeed, we have

$$\begin{aligned} \|v^k\|^2_{H(\Omega)} &\le \|v^k\|^2_{H_0^1(\Omega)} - \int_\Omega \min\{b(x), k\}\, |v^k|^2 \, dx \\ &= \int_\Omega f v^k \, dx \le C \|v^k\|_{H(\Omega)} \end{aligned}$$

by Hölder inequality and by (1.1), where the constant C depends on f and Ω but not on k. Therefore the sequence v^k is bounded in $H(\Omega)$. We conclude that there exists $\widetilde{v} \in H(\Omega)$ such that, for a subsequence, $v^k \rightharpoonup \widetilde{v}$ in $H(\Omega)$. Now by (1.2), we have

$$\langle v^k, \varphi \rangle_{H(\Omega)} + \int_\Omega (b(x) - \min\{b(x), k\})\, v^k \varphi = \int_\Omega f\varphi.$$

Since for every $k \geq 1$ and any $\varphi \in C_c^\infty(\Omega)$

$$\left| (b(x) - \min\{b(x), k\})\, v^k \varphi \right| \leq (b(x) - \min\{b(x), k\})\, u|\varphi| \leq 2b(x)u|\varphi| \in L^1(\Omega),$$

the dominated convergence theorem implies that

$$\langle \widetilde{v}, \varphi \rangle_{H(\Omega)} = \int_\Omega f\varphi \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{1.3}$$

We therefore have that $\widetilde{v} = v$ by uniqueness. By (1.3), we have

$$\begin{aligned}
\|v - v^k\|^2_{H(\Omega)} &= \|v^k\|^2_{H(\Omega)} - \langle v, v^k \rangle_{H(\Omega)} + \langle v, v - v^k \rangle_{H(\Omega)} \\
&= \|v^k\|^2_{H(\Omega)} - \int_\Omega f v^k + \langle v, v - v^k \rangle_{H(\Omega)} \\
&\leq \|v^k\|^2_{H_0^1(\Omega)} - \int_\Omega \min\{b(x), k\}\, |v^k|^2 \, dx - \int_\Omega f v^k + \langle v, v - v^k \rangle_{H(\Omega)} \\
&= \langle v, v - v^k \rangle_{H(\Omega)}.
\end{aligned}$$

We thus obtain

$$C(\Omega) \int_\Omega |v - v^k|^2 \, dx \leq \langle v, v - v^k \rangle_{H(\Omega)} \to 0$$

by (1.1). Hence $v^k \to v$ pointwise and thus $v \leq u$ in Ω.

□

We conclude this section by pointing out the following Allegretto-Piepenbrink type result which is essentially contained in [18]. A version for distributional solutions is also contained in [[8], Theorem 2.12].

Lemma 1.4 *Let Ω be a domain (possibly unbounded) in $\mathbb{R}^N$, $N \geq 1$. Let $V \in L^1_{loc}(\Omega)$ and $V > 0$ in Ω. Assume that $u \in L^1_{loc}(\Omega)$, $V(x)u \in L^1_{loc}(\Omega)$ and that u is a non-negative, non-trivial solution to*

$$-\Delta u \geq V(x)u \quad \mathcal{D}'(\Omega).$$

Then

$$\int_\Omega |\nabla \phi|^2 \, dx \geq \int_\Omega V(x)\, \phi^2 \, dx \quad \textit{for any } \phi \in C_c^\infty(\Omega).$$

Proof. Put $V_k(x) = \min\{V(x), k\}$ then Lemma B.1 in [18] yields

$$\int_\Omega |\nabla \phi|^2 \, dx \geq \int_\Omega V_k(x)\, \phi^2 \, dx \quad \text{for any } \phi \in C_c^\infty(\Omega).$$

To conclude, it suffices to use Fatou's lemma. □

Remark 1.5 *Given Ω any domain in $\mathbb{R}^N$, $N \geq 1$. Define*

$$\mu(\Omega) := \inf_{u \in C_c^\infty(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega |x|^{-2} u^2 \, dx}.$$

Then Lemma 1.4 clearly implies that if $c > \mu(\Omega)$ there is no non-negative and non-trivial $u \in L^1_{loc}(\Omega)$ that satisfies $-\Delta u - \frac{c}{|x|^2} u \geq 0$ in $\mathcal{D}'(\Omega)$.

Suppose that Ω is a smooth bounded domain and that the potential $b(x)$ satisfies

$$\int_\Omega |\nabla \varphi|^2 \, dx - \int_\Omega b(x) \varphi^2 \, dx \geq C(b) \left(\int_\Omega |\varphi|^r \, dx \right)^{\frac{2}{r}} \quad \text{for any } \varphi \in C_c^\infty(\Omega)$$

for some $C(b) > 0$ and $2 < r$. By [[10] Lemma 7.2], we can let $G \in L^1(\Omega \times \Omega)$ be the Green function associated to $-\Delta - b(x)$:

$$\begin{cases} -\Delta G(\cdot, y) - b(x) G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where δ_y denotes the Dirac measure at some $y \in \Omega$. Define

$$\zeta_0(x) := \int_\Omega G(x, y) \, dy$$

which is the $H(\Omega)$-solution to $-\Delta \zeta_0 - b(x)\, \zeta_0 = 1$. By using Lemma 1.3 and Lemma 1.4, we can prove the following

Proposition 1.6 *Suppose that $\int_\Omega \zeta_0^{p+1} \, dx = \infty$ for some $p > r$ then there is no nonnegative and nontrivial u satisfying $-\Delta u - b(x)\, u \geq u^p$ in $\mathcal{D}'(\Omega)$.*

Proof. If such u exists, it is positive by the maximum principle therefore, we can define $v \in H(\Omega)$ be the solution of $-\Delta v - b(x)\, v = \min(u^p, 1)$ so that by Lemma 1.3 we have $u \geq v$ in Ω. Thanks to [[10] Corollary 2.4], we have $u \geq v \geq C\zeta_0$. By applying Lemma 1.4 with $V(x) = b(x) + (C\zeta_0)^{p-1}$ we conclude that

$$\infty > \|\zeta_0\|_{H(\Omega)} \geq C^{p+1} \int_\Omega \zeta_0^{p+1}\, dx.$$

$\square$

2 Proof of Theorem 0.1

We state the following lemma which is a consequence of Lemma 1.3 and [[22], Theorem 4.2].

Lemma 2.1 *Let $u \in L^1_{loc}(\Omega)$ be positive and let $f \in L^1_{loc}(\mathcal{C}^r_\Sigma)$ with $f \gneqq 0$ such that*

$$-\Delta u - V\left(\frac{x}{|x|}\right) |x|^{-2}\, u \geq f \quad \text{in } \mathcal{D}'(\mathcal{C}^r_\Sigma),$$

where $\|V\|_{L^\infty(\Sigma)} \leq \mu(\mathcal{C}_\Sigma)$ and $V \geq 0$. Then for every $\widetilde{\Sigma} \subset\subset \Sigma$ there exists a constant $C > 0$ such that

$$u(x) \geq C|x|^{\frac{2-N}{2}+\sqrt{\frac{(2-N)^2}{4}+\lambda_{1,V}}} \quad \text{in } \mathcal{C}^{r/2}_{\widetilde{\Sigma}}, \tag{2.1}$$

where $\lambda_{1,V}$ is the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}\Phi - V\Phi = \lambda_{1,V}\Phi$ on Σ .

Proof. Up to a scaling, we can assume that $r = 1$. We recall the following improved Hardy inequality

$$\int_{\mathcal{C}^1_\Sigma} |\nabla\varphi|^2\, dx - \mu(\mathcal{C}_\Sigma) \int_{\mathcal{C}^1_\Sigma} |x|^{-2}|\varphi|^2 \geq C_0 \int_{\mathcal{C}^1_\Sigma} |\varphi|^2\, dx \quad \forall\varphi \in C^\infty_c(\mathcal{C}^1_\Sigma), \tag{2.2}$$

for some $C_0 > 0$ (see for instance [17]). We can therefore pick $v \in H\left(\mathcal{C}^1_\Sigma\right)$ solves

$$-\Delta v - V\left(\frac{x}{|x|}\right) |x|^{-2} v = \min(f, 1) \quad \text{in } H\left(\mathcal{C}^1_\Sigma\right). \tag{2.3}$$

Then by the maximum principle and Lemma 1.3, we have $0 < v \le u$ in $\mathcal{C}^1_\Sigma$. Approximating v by smooth functions compactly supported in $\mathcal{C}^1_\Sigma$ with respect to the $H\left(\mathcal{C}^1_\Sigma\right)$-norm, we infer that

$$-\Delta v - V\left(\frac{x}{|x|}\right)|x|^{-2}v = \ \min(f,1) \quad \text{in } \mathcal{D}'\left(\mathcal{C}^1_\Sigma\right).$$

Elliptic regularity theory then implies that $v \in C^{1,\gamma}_{loc}(\mathcal{C}^1_\Sigma) \subset H^1_{loc}(\mathcal{C}^1_\Sigma)$. By applying [[22], Theorem 4.2] (up to Kelvin transform), we get the lower estimate (2.1) for v and hence for u.

□

Proof of Theorem 0.1

Up to a scaling, we can assume that $R = 1$. We argue by contradiction. If $u \neq 0$ then by the maximum principle $u > 0$ in $\mathcal{C}^1_\Sigma$. We will show that appropriate lower bound of u and an application of Lemma 1.4 will lead to a contradiction.

Case 1: $c < \mu(\mathcal{C}_\Sigma)$**.**

By Lemma 2.1

$$u(x) \ge C_0|x|^{\frac{2-N}{2}+\sqrt{\mu(\mathcal{C}_\Sigma)-c}} \quad \forall x \in \mathcal{C}^{1/2}_{\widetilde{\Sigma}},$$

where C_0 is a positive constant and $\widetilde{\Sigma} \subset\subset \Sigma$. By assumption $u^{p-1}(x)|x|^2 \ge C_0^{p-1}$. In particular for every $\varepsilon \in (0,1)$, we have

$$-\Delta u - (c+\varepsilon V)|x|^{-2}u \ge \frac{1}{2}u^p \quad \text{ in } \mathcal{D}'(\mathcal{C}^{1/2}_{\widetilde{\Sigma}}),$$

where $V = \frac{C_0^{p-1}}{2}\chi_{\widetilde{\Sigma}}$. We notice that for ε small, $c+\varepsilon V < \mu(\mathcal{C}_\Sigma)$. We apply once more Lemma 2.1 to get

$$u(x) \ge C_1|x|^{\frac{2-N}{2}+\sqrt{(N-2)^2/4+\lambda_{1,\varepsilon}}} \quad \forall x \in \mathcal{C}^{1/4}_{\widetilde{\Sigma}}, \tag{2.4}$$

where $\lambda_{1,\varepsilon}$ is the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}\Phi - (c+\varepsilon V)\Phi = \lambda_{1,\varepsilon}\Phi$ on Σ. We observe that, for ε small, $\lambda_{1,\varepsilon} < \lambda_1(\Sigma) - c < 0$ and thus

$$p-1 \ge \frac{2}{\alpha^-_\Sigma} > \frac{2}{\frac{N-2}{2} - \sqrt{(N-2)^2/4+\lambda_{1,\varepsilon}}} > 0.$$

Recalling that $-\Delta u \geq u^{p-1}u$, we deduce from (2.4) that

$$-\Delta u - \rho(x)|x|^{-2}u \geq 0 \quad \text{ in } \mathcal{D}'(\mathcal{C}^{1/4}_{\tilde{\Sigma}}),$$

where $\rho(x) = \frac{C_1^{p-1}}{2}|x|^{(\frac{2-N}{2}+\sqrt{(N-2)^2/4+\lambda_{1,\varepsilon}})(p-1)+2}$. Since $\rho(x) \to +\infty$ as $|x| \to 0$, applying Lemma 1.4, we contradict the sharpness of the Hardy constant $\mu(\mathcal{C}_{\tilde{\Sigma}})$.

Case 2: $c = \mu(\mathcal{C}_\Sigma)$**.**
We consider the function $v \in H(\mathcal{C}^1_\Sigma)$ solving

$$-\Delta v - \mu(\mathcal{C}_\Sigma)|x|^{-2}v = \min(u^p, 1).$$

Then by Lemma 1.3 and the maximum principle $0 < v \leq u$ in $\mathcal{C}^1_\Sigma$. By Lemma 2.1,

$$v(x) \geq C|x|^{\frac{2-N}{2}} \qquad \text{for } x \in \mathcal{C}^{1/2}_{\tilde{\Sigma}}.$$

Since $-\Delta u - \mu(\mathcal{C}_\Sigma)|x|^{-2}u = u^{p-1}u$ in $\mathcal{D}'(\mathcal{C}^1_\Sigma)$, by Lemma 1.4 and the above estimate, we have

$$\begin{aligned}
\infty > \|v\|^2_{H(\mathcal{C}^1_\Sigma)} &\geq \int_{\mathcal{C}^1_\Sigma} u^{p-1}v^2\,dx \geq \int_{\mathcal{C}^1_\Sigma} v^{p+1}\,dx \\
&\geq C\int_{\mathcal{C}^{1/2}_{\tilde{\Sigma}}} v^{p+1}\,dx \geq C\int_{\mathcal{C}^{1/2}_{\tilde{\Sigma}}} |x|^{-N}\,dx = \infty.
\end{aligned}$$

This readily leads to a contradiction. Theorem 0.1 is completely proved. □

3 Smooth domains

In this section, we introduce a system of coordinates near $0 \in \partial\Omega$ that flattens $\partial\Omega$, see [19]. This will allows us to construct a (super-) sub-solution via the function $y^1|y|^{-\frac{N}{2}+\sqrt{\frac{N^2}{4}-c}}$ which is the (virtual) ground state for the operator $\Delta + c|y|^{-2}$ in the half-space $\mathbb{R}^N_+$.

3.1 Fermi coordinates

We denote by $\{E_1, E_2, \dots, E_N\}$ the standard orthonormal basis of $\mathbb{R}^N$ and we put

$$\mathbb{R}^N_+ = \{y \in \mathbb{R}^N \,:\, y^1 > 0\}, \quad \mathbb{S}^{N-1}_+ = \mathbb{S}^{N-1} \cap \mathbb{R}^N_+,$$
$$B_r(y_0) = \{y \in \mathbb{R}^N \,:\, |y - y_0| < r\}, \quad B_r^+ = B_r(0) \cap \mathbb{R}^N_+.$$

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^N$ with boundary $\mathcal{M} := \partial\mathcal{U}$ a smooth closed hypersurface of $\mathbb{R}^N$ and $0 \in \mathcal{M}$. We write $N_{\mathcal{M}}$ for the unit normal vector-field of $\mathcal{M}$ pointed into $\mathcal{U}$. Up to a rotation, we assume that $N_{\mathcal{M}}(0) = E_1$. For $x \in \mathbb{R}^N$, we let $d_{\mathcal{M}}(x) = \operatorname{dist}(\mathcal{M}, x)$ be the distance function of $\mathcal{M}$. Given $x \in \mathcal{U}$ and close to $\mathcal{M}$ then it can be written uniquely as $x = \sigma_x + d_{\mathcal{M}}(x)\, N_{\mathcal{M}}(\sigma_x)$, where σ_x is the projection of x on $\mathcal{M}$. We further use the Fermi coordinates $(y^2, \dots, y^N)$ on $\mathcal{M}$ so that for σ_x close to 0, we have

$$\sigma_x = \operatorname{Exp}_0\left(\sum_{i=2}^{N} y^i E_i\right),$$

where $\operatorname{Exp}_0 : \mathbb{R}^{N-1} \to \mathcal{M}$ is the exponential mapping on $\mathcal{M}$ endowed with the metric induced by $\mathbb{R}^N$, see [11]. In this way a neighborhood of 0 in $\mathcal{U}$ can be parameterize by the map

$$F_{\mathcal{M}}(y) = \operatorname{Exp}_0\left(\sum_{i=2}^{N} y^i E_i\right) + y^1\, N_{\mathcal{M}}\left(\operatorname{Exp}_0\left(\sum_{i=2}^{N} y^i E_i\right)\right), \quad y \in B_r^+,$$

for some $r > 0$. In this coordinates, the Laplacian Δ is given by

$$\sum_{i=1}^{N} \frac{\partial^2}{(\partial x^i)^2} = \frac{\partial^2}{(\partial y^1)^2} + h_{\mathcal{M}} \circ F_{\mathcal{M}}\, \frac{\partial}{\partial y^1} + \sum_{i,j=2}^{N} \frac{\partial}{\partial y^i}\left(\sqrt{|g|}\, g^{ij}\, \frac{\partial}{\partial y^j}\right),$$

where $h_{\mathcal{M}}(x) = \Delta\, d_{\mathcal{M}}(x)$; for $i, j = 2\dots, N$, $g_{ij} = \langle \frac{\partial F_{\mathcal{M}}}{\partial y^i}, \frac{\partial F_{\mathcal{M}}}{\partial y^j} \rangle$; the quantity $|g|$ is the determinant of g and g^{ij} is the component of the inverse of the matrix $(g_{ij})_{2 \le i,j \le N}$.

Since $g_{ij} = \delta_{ij} + O(y^1) + O(|y|^2)$ (see [19]), we have the following Taylor expansion

$$\sum_{i=1}^{N} \frac{\partial^2}{(\partial x^i)^2} = \sum_{i=1}^{N} \frac{\partial^2}{(\partial y^i)^2} + h_{\mathcal{M}} \circ F_{\mathcal{M}}\, \frac{\partial}{\partial y^1} + \sum_{i=2}^{N} O_i(|y|) \frac{\partial}{\partial y^i} + \sum_{i,j=2}^{N} O_{ij}(|y|) \frac{\partial^2}{\partial y^i \partial y^j}. \tag{3.1}$$

For $a \in \mathbb{R}$, we put $X_a(t) := |\log t|^a$, $t \in (0, 1)$ and for $c \le \frac{N^2}{4}$, set

$$\overline{\omega}_a(y) := y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} X_a(|y|) \quad \forall y \in \mathbb{R}^N_+$$

and put

$$L_y := -\sum_{i=1}^{N} \frac{\partial^2}{(\partial y^i)^2} - c\,|y|^{-2} + a(a-1)|y|^{-2} X_{-2}(|y|).$$

Then one easily verifies that

$$\begin{cases} L_y\,\overline{\omega}_a = 2a\sqrt{\dfrac{N^2}{4}-c}\,|y|^{-2}X_{-1}(|y|)\,\overline{\omega}_a & \text{in } \mathbb{R}^N_+, \\ \overline{\omega}_a = 0 & \text{on } \partial\mathbb{R}^N_+\setminus\{0\}, \\ \overline{\omega}_a \in H^1(B_R^+)\ \forall R>0,\ \forall c<\dfrac{N^2}{4},\ \forall a\le 0. & \end{cases}$$

For $K\in\mathbb{R}$, we define

$$\omega_{a,K}(y) = e^{Ky^1}\,\overline{\omega}_a(y).$$

This function satisfies similar boundary and integrability conditions as $\overline{\omega}_a$. In addition it holds that

(3.2)

$$\begin{aligned} L_y\,\omega_{a,K} = &-\frac{2K}{y^1}\omega_{a,K} + 2a\sqrt{\frac{N^2}{4}-c}\,|y|^{-2}X_{-1}(|y|)\,\overline{\omega}_a \\ &+2K\left(\tfrac{N}{2}-\sqrt{\tfrac{N^2}{4}-c}+aX_{-1}(|y|)\right)\tfrac{y^1}{|y|^2}\omega_{a,K} - K^2\omega_{a,K}. \end{aligned}$$

Furthermore for all $a\in\mathbb{R}$

$$\begin{aligned} \sum_{i=2}^N O_i(|y|)\frac{\partial\omega_{a,K}}{\partial y^i} + \sum_{i,j=2}^N O_{ij}(|y|)\frac{\partial^2\omega_{a,K}}{\partial y^i\partial y^j} &= y^1\,e^{Ky^1}O\left(|y|^{-\frac{N}{2}-1+\sqrt{\frac{N^2}{4}-c}}X_a(|y|)\right) \\ &= \mathcal{O}_{a,K}(|y|^{-1})\,\omega_{a,K}(y). \end{aligned}$$

Here the error term $\mathcal{O}_{a,K}$ has the property that for any $A>0$ and $c_0<\frac{N^2}{4}$, there exit some constants $C>0$ and $s_0>0$ such that

(3.3) $$|\mathcal{O}_{a,K}(s)| \le C\,s \quad \forall s\in(0,s_0),\ \forall a\in[-A,A]\ \forall c\in\left[c_0,\frac{N^2}{4}\right].$$

Let

(3.4) $$W_{a,K}(x) := \omega_{a,K}(F_{\mathcal{M}}^{-1}(x)), \qquad \forall\,x\in\mathcal{B}_r^+ := F_{\mathcal{M}}(B_r^+).$$

Then using (3.1), (3.2) and the fact that $|x| = |y| + O(|y|^2)$ we obtain the following expansions

(3.5)

$$L_x\,W_{a,K} = -\left(\frac{2K+h_{\mathcal{M}}(x)}{d_{\mathcal{M}}(x)}\right)W_{a,K} + 2a\sqrt{\frac{N^2}{4}-c}\,|x|^{-2}X_{-1}(|x|)\,W_{a,K} + \mathcal{O}_{a,K}(|x|^{-1})\,W_{a,K},$$

with $L_x := -\Delta - c\,|x|^{-2} + a(a-1)|x|^{-2}X_{-2}(|x|)$. Moreover it is easy to see that

$$
\begin{cases}
W_{a,K} > 0 & \text{in } \mathcal{B}_r^+, \\
W_{a,K} = 0 & \text{on } \mathcal{M} \cap \partial\mathcal{B}_r^+ \setminus \{0\}, \\
W_{a,K} \in H^1(\mathcal{B}_r^+) \quad \forall c < \frac{N^2}{4}, \text{ and } \forall a \leq 0.
\end{cases} \tag{3.6}
$$

3.2 Non-existence

We start by recalling the following local improved Hardy inequality. Given a domain $\Omega \subset \mathbb{R}^N$, of class C^2 at $0 \in \partial\Omega$, there exist two constants $C(\Omega) > 0$ and $r_0 = r_0(\Omega) > 0$ such that

$$
\int_{\Omega_{r_0}} |\nabla u|^2\,dx - c\int_{\Omega_{r_0}} |x|^{-2}u^2\,dx \geq C(\Omega)\int_{\Omega_{r_0}} u^2\,dx \quad \forall u \in C_c^\infty(\Omega_{r_0}), \tag{3.7}
$$

for every $c \in \left(-\infty, \frac{N^2}{4}\right]$, with $\Omega_{r_0} := \Omega \cap B_{r_0}(0)$, see [15]. From this we can define the space $H(\Omega_{r_0})$ to be the completion of $C_c^\infty(\Omega_{r_0})$ with respect to the scalar product

$$
\int_{\Omega_{r_0}} \nabla u \nabla v - c\int_{\Omega_{r_0}} |x|^{-2}uv \quad \forall u, v \in C_c^\infty(\Omega_{r_0}).
$$

In the sequel we will assume that Ω contains the ball $B = B_1(E_1)$ such that $\partial B \cap \partial\Omega = \{0\}$. Recalling the notations in Section 3.1, we state the following result

Lemma 3.1 *Let $c_0 \in \left(-\infty, \frac{N^2}{4}\right]$ and $f \in L^\infty(\Omega_{r_0})$ be a non-negative and non-trivial function. For $c \in \left[c_0, \frac{N^2}{4}\right]$, let $v \in H(\Omega_{r_0})$ be the unique solution of the problem*

$$
\int_{\Omega_{r_0}} \nabla v \nabla \phi\,dx - c\int_{\Omega_{r_0}} |x|^{-2}v\phi\,dx = \int_{\Omega_{r_0}} f\phi\,dx \quad \forall \phi \in C_c^\infty(\Omega_{r_0}).
$$

Then there exist $R > 0$ and $r > 0$ such that

$$
v(F_{\partial B}(y)) \geq R\,y^1|y|^{-\frac{N}{2}+\sqrt{\frac{N^2}{4}-c}} \quad \forall y \in B_r^+ \quad \forall c \in \left[c_0, \frac{N^2}{4}\right].
$$

Proof. For $a \leq 0$ and $r > 0$ small we define $G_r^+ := F_{\partial B}(B_r^+)$ and

$$
w_a(x) := \omega_{a,N-1}(F_{\partial B}^{-1}(x)), \quad \forall x \in G_r^+.
$$

Letting $L := -\Delta - c\,|x|^{-2}$, by (3.5),

$$\begin{aligned} L\,(w_0 + w_{-1}) \;&\le\; -\frac{2(N-1) + h_{\partial B}}{d_{\partial B}}\,(w_0 + w_{-1}) - 2\,|x|^{-2}\,X_{-2}(|x|)\,w_{-1} \\ &- \;2\sqrt{\frac{N^2}{4} - c}|x|^{-2}X_{-1}(|x|)\,w_{-1} + O(|x|^{-1})\,(w_0 + w_{-1}). \end{aligned}$$

We observe that

$$w_{-1}(x) = w_0(x)|\log|F^{-1}_{\partial B}(x)||^{-1} = w_0(x)\left(X_{-1}(|x|) + O(|x|)\right). \tag{3.8}$$

Since $-h_{\partial B}(x) = (N-1)\,(1 + O(|x|))$ in G_r^+, we have using (3.8),

$$L\,(w_0 + w_{-1}) \le 0 \quad \text{ in } G_r^+, \tag{3.9}$$

for r positive small.

Case $c \in \left[c_0, \frac{N^2}{4}\right)$.

We put $U = w_0 + w_{-1}$. Then $U \in H^1(G_r^+) \cap C(\overline{G_r^+})$ by (3.6) and $v \in H^1_0(\Omega_{r_0}) \cap C(\Omega_{r_0})$ by elliptic regularity theory and Remark 1.1. Moreover $v > 0$ in Ω_{r_0} by the maximum principle. Therefore since $F_{\partial B}\left(\overline{r\mathbb{S}^{N-1}_+}\right) \subset \Omega_{r_0}$, we can let

$$R = r^{\frac{N-2}{2}}\,e^{-(N-1)r} \inf_{y \in \overline{r\mathbb{S}^{N-1}_+}} v > 0 \tag{3.10}$$

so that

$$R\,U \le v \quad \text{ on } F_{\partial B}\left(\overline{r\mathbb{S}^{N-1}_+}\right).$$

By (3.6) and setting $\varphi = R\,U - v$, we get $\varphi^+ := \max(\varphi, 0) \in H^1_0(G_r^+)$ because $U = 0$ on $\partial B \cap \partial G_r^+$. Since $Lv \ge 0$, we have

$$L\,\varphi \le 0 \quad \text{ in } G_r^+,$$

by (3.9). Multiplying the above inequality by φ^+ and integrating by parts yields

$$\int_{G_r^+} |\nabla\varphi^+|^2\,dx - c\int_{G_r^+} |x|^{-2}|\varphi^+|^2\,dx \le 0.$$

This implies that $\varphi^+ \equiv 0$ in G_r^+ for all r positive small. We conclude that $v \ge R(w_0 + w_{-1})$ in G_r^+ and thus

$$v(F_{\partial B}(y)) \ge R\,w_0(F_{\partial B}(y)) \ge R\,y^1|y|^{-\frac{N}{2}+\sqrt{\frac{N^2}{4}-c}}, \quad \forall y \in G_r^+, \quad \forall c \in \left[c_0, \frac{N^2}{4}\right).$$

Case $c = \frac{N^2}{4}$.
In this case, we notice that the solutions v_k to the problem

$$\int_{\Omega_{r_0}} \nabla v_k \nabla \phi \, dx - \left(\frac{N^2}{4} - \frac{1}{k} \right) \int_{\Omega_{r_0}} |x|^{-2} v_k \phi \, dx = \int_{\Omega_{r_0}} f \phi \, dx \quad \forall \phi \in H(\Omega_{r_0})$$

are H^1_0-solutions if r_0 is small enough (independent on k) and they are monotone increasing to v as $k \to \infty$. Hence by (3.10) and from the above argument we deduce that there exist an integer $k_0 \geq 1$ and a constant R (possibly depending on k_0) such that

$$v(F_{\partial B}(y)) \geq v_k(F_{\partial B}(y)) \geq R \, y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{1}{k}}}, \quad \forall y \in G_r^+, \quad \forall k \geq k_0.$$

Passing to the limit as $k \to \infty$, we get the result. □

3.2.1 Proof of Theorem 0.2

Recall that

$$\alpha^-_{\mathbb{S}^{N-1}_+} = \frac{N-2}{2} - \sqrt{\frac{N^2}{4} - c}, \qquad p \geq p_{\mathbb{S}^{N-1}_+} := 1 + \frac{2}{\alpha^-_{\mathbb{S}^{N-1}_+}}, \tag{3.11}$$

Suppose that $u \neq 0$ near 0 thus we can find a bounded function f with $f \neq 0$ and $0 \leq f \leq u^p$. By Lemma 1.3 and the maximum principle, there exits $v \in H(\Omega_{r_0})$ such that $u \geq v > 0$ and

$$\int_{\Omega_{r_0}} \nabla v \nabla \phi \, dx - c \int_{\Omega_{r_0}} |x|^{-2} v \phi \, dx = \int_{\Omega_{r_0}} f \phi \, dx \quad \forall \phi \in C_c^\infty(\Omega_{r_0}),$$

for some $r_0 > 0$ small. In addition Lemma 3.1 yields

$$v(F_{\partial B}(y)) \geq R \, y^1 |y|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} \quad \forall y \in B_r^+. \tag{3.12}$$

Case 1: $c \in \left(N - 1, \frac{N^2}{4} \right)$.
Since $-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c} < 0$, (3.12) implies that

$$u(x) \geq v(x) \geq C \, d_{\partial B}(x) \, |x|^{-\frac{N}{2} + \sqrt{\frac{N^2}{4} - c}} \quad \forall x \in G_r^+, \tag{3.13}$$

where we recall that $d_{\partial B}(F_{\partial B}(y)) = y^1$ and $|F^{-1}_{\partial B}(x)| \leq C|x|$.
Let $\gamma \in (0,1)$ then for every $x \in B_\gamma(\gamma E_1) \subset B_1(E_1) = B$, we have

$$d_{\partial B}(x) = 1 - |x - E_1| > (1-\gamma)\, x^1.$$

Using this together with (3.11) and (3.13), we obtain

$$u^{p-1}(x) \geq C \left(\frac{x^1}{|x|}\right)^{p-1} |x|^{-2} \quad \forall x \in G_r^+. \tag{3.14}$$

Since for $\gamma > 0$ small u satisfies

$$-\Delta u - c|x|^{-2}u \geq \frac{1}{2}u^{p-1}u + \frac{1}{2}u^p \quad \text{ in } \mathcal{D}'(B_\gamma(\gamma E_1)),$$

we thus have from (3.14)

$$-\Delta u - V_\varepsilon\left(\frac{x}{|x|}\right)|x|^{-2}u \geq \frac{1}{2}u^p \quad \text{ in } \mathcal{D}'(B_\gamma(\gamma E_1)),$$

where $V_\varepsilon\left(\frac{x}{|x|}\right) = c + \frac{\varepsilon}{2}C\left(\frac{x}{|x|}\cdot E_1\right)^{p-1}$ for every $\varepsilon \in (0,1)$. From now on, we will fix ε so small that $V_\varepsilon < \frac{N^2}{4}$.
Given $\delta \in (0,1)$, consider the cone $\mathcal{C}_\delta := \{x \in \mathbb{R}^N \,:\, x^1 > \delta|x|\}$ and define $\Sigma_\delta = \mathcal{C}_\delta \cap \mathbb{S}^{N-1}$. We observe that for every $\delta \in (0,1)$, there exists $r_\delta > 0$ such that the cone-like domain

$$\mathcal{C}^{r_\delta}_{\Sigma_\delta} \subset B_\gamma(\gamma E_1).$$

It follows that

$$-\Delta u - V_\varepsilon\left(\frac{x}{|x|}\right)|x|^{-2}u \geq \frac{u^p}{2} \quad \text{ in } \mathcal{D}'(\mathcal{C}^{r_\delta}_{\Sigma_\delta}).$$

Let $\lambda_{1,\delta,\varepsilon}$ be the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}\Phi - V_\varepsilon\Phi = \lambda_{1,\delta,\varepsilon}\Phi$ on Σ_δ. Since $\lambda_1(\Sigma_\delta) \searrow N-1 = \lambda_1(\mathbb{S}^{N-1}_+)$ as $\delta \to 0$, we can choose a $\delta_\varepsilon \in (0,1)$ such that

$$\lambda_{1,\delta,\varepsilon} < N - 1 - c < 0 \quad \forall \delta \in (0,\delta_\varepsilon). \tag{3.15}$$

Since $V_\varepsilon \leq \frac{N^2}{4} < \frac{(N-2)^2}{4} + \lambda_1(\Sigma_\delta) = \mu(\mathcal{C}_{\Sigma_\delta})$ for every $\delta \in (0,\delta_\varepsilon)$, we can apply Lemma 2.1 to have $\forall \delta \in (0,\delta_\varepsilon)$

$$u(x) \geq C|x|^{\frac{2-N}{2}+\sqrt{\frac{(2-N)^2}{4}+\lambda_{1,\delta,\varepsilon}}}, \quad \text{ in } \mathcal{C}^{r_\delta/2}_{\widetilde{\Sigma}_\delta}$$

where $\widetilde{\Sigma}_\delta \subset\subset \Sigma_\delta$. We get from (3.15)

$$p-1 \geq \frac{2}{\alpha^-_{\mathbb{S}^{N-1}_+}} > \frac{2}{\frac{N-2}{2} - \sqrt{(N-2)^2/4 + \lambda_{1,\delta,\varepsilon}}} > 0.$$

Since $-\Delta u \geq u^{p-1}u$, we deduce that $\forall \delta \in (0, \delta_\varepsilon)$

$$-\Delta u - \rho(x)|x|^{-2}u \geq 0 \quad \text{ in } \mathcal{D}'(\mathcal{C}^{r_\delta/2}_{\widetilde{\Sigma}_\delta}),$$

where $\rho(x) \geq C|x|^{(\frac{2-N}{2}+\sqrt{(N-2)^2/4+\lambda_{1,\delta,\varepsilon}})(p-1)+2}$. Since $\rho(x) \to +\infty$ as $|x| \to 0$, applying Lemma 1.4, we contradict the sharpness of the Hardy constant $\mu(\mathcal{C}_{\widetilde{\Sigma}_\delta})$.

Case 2: $c = \frac{N^2}{4}$.
Here, we recall that $p \geq \frac{N+2}{N-2}$. By (3.7) we can let $\zeta \in H(G_r^+)$ be the unique solution to the problem

$$\int_{G_r^+} \nabla\zeta\nabla\phi\, dx - \frac{N^2}{4}\int_{G_r^+} |x|^{-2}\zeta\phi\, dx = \int_{G_r^+} 1\,\phi\, dx \quad \forall \phi \in C_c^\infty(G_r^+).$$

We put $\Phi(y) = \frac{y^1}{|y|}$. Then by Lemma 1.4, Lemma 3.1 and (3.12), we have

$$\begin{aligned}
\|\zeta\|^2_{H(G_r^+)} &\geq \int_{G_r^+} v^{p-1}|\zeta|^2\, dx \\
&\geq C\int_{B_r^+} |y|^{\frac{2-N}{2}(p+1)}\Phi^{p+1}\sqrt{|\hat{g}|}(y)dy \\
&\geq C\int_{B_r^+} |y|^{\frac{2-N}{2}(p+1)}\Phi^{p+1}\, dy \\
&= C\int_{\mathbb{S}^{N-1}_+} \Phi^{p+1}\, d\sigma \int_0^r t^{\frac{2-N}{2}(p+1)}t^{N-1}\, dt \\
&\geq C\int_{\mathbb{S}^{N-1}_+} \Phi^{p+1}\, d\sigma \int_0^r t^{-1}\, dt = \infty.
\end{aligned}$$

This clearly contradicts the fact that $\zeta \in H(G_r^+)$. ☐

3.3 Existence

Let Ω be a domain of $\mathbb{R}^N$, $N \geq 3$ which is of class C^2 at $0 \in \Omega$, we shall show that for some $r > 0$ small, there exists a positive function $u \in L^p(\Omega \cap B_r(0))$,

$1 < p < p_{\mathbb{S}^{N-1}_+} = 1 + \frac{2}{\alpha^-_{\mathbb{S}^{N-1}_+}}$ and

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \quad \text{in } \mathcal{D}'(\Omega \cap B_r(0)).$$

Letting B be a unit ball with $0 \in \partial B$, call $\mathcal{U} = \mathbb{R}^N \setminus \overline{B}$ and $\mathcal{M} = \partial \mathcal{U}$. Under the notations in Section 3.1, the above existence result is a consequence of the following

Proposition 3.2 *Let $1 < p < p_{\mathbb{S}^{N-1}_+}$ and $N-1 < c \leq \frac{N^2}{4}$. Then there exists $r > 0$ small such that the problem*

$$\begin{cases} -\Delta w - \frac{c}{|x|^2} w = w^p & \textit{in } \mathcal{D}'(\mathcal{B}^+_r), \\ w \in L^p(\mathcal{B}^+_r), \\ w > 0 \quad \mathcal{B}^+_r \end{cases} \tag{3.16}$$

has a supersolution, with $\mathcal{B}^+_r = F_{\mathcal{M}}(B^+_r)$.

Proof. Notice that $h_{\mathcal{M}}(x) = \frac{N-1}{1+d_{\mathcal{M}}(x)}$ and thus

$$-\frac{2(1-N) + h_{\mathcal{M}}(x)}{d_{\mathcal{M}}(x)} \geq \frac{N-1}{d_{\mathcal{M}}(x)} \quad \forall x \in \mathcal{U}. \tag{3.17}$$

Define (see (3.4))

$$w(x) = \omega_{\frac{1}{2p}, 1-N}(F^{-1}_{\mathcal{M}}(x)) \quad \forall x \in \mathcal{B}^+_r.$$

By (3.5), (3.17) and using the fact that $|x| = |y| + O(|y|^2)$, we have

$$\begin{aligned} -\Delta w(F_{\mathcal{M}}(y)) - c|F_{\mathcal{M}}(y)|^{-2} w(F_{\mathcal{M}}(y)) \quad \geq \quad & \frac{2p-1}{4p^2} |y|^{-2} X_{-2}(|y|)\, w(F_{\mathcal{M}}(y)) \\ & + O(|y|^{-1})\, w(F_{\mathcal{M}}(y)). \end{aligned}$$

In particular if $r > 0$ small

$$-\Delta w(F_{\mathcal{M}}(y)) - c|F_{\mathcal{M}}(y)|^{-2} w(F_{\mathcal{M}}(y)) \quad \geq \quad C|y|^{-2} X_{-2}(|y|)\, w(F_{\mathcal{M}}(y)) \quad \forall y \in B^+_r,$$

with $C > 0$ a constant depending only on p and N. Therefore w is a supersolution provided

$$C|y|^{-2} X_{-2}(|y|)\, w(F_{\mathcal{M}}(y)) \geq w(F_{\mathcal{M}}(y))^p \quad \forall y \in B^+_r$$

or equivalently

$$C\frac{y^1}{|y|}e^{(1-N)y^1}|y|^{-\alpha^-_{\mathbb{S}^{N-1}_+}-2}|\log|y||^{-2+\frac{1}{2p}} \geq \left(\frac{y^1}{|y|}e^{(1-N)y^1}|y|^{-\alpha^-_{\mathbb{S}^{N-1}_+}}|\log|y||^{\frac{1}{2p}}\right)^p \quad \forall y \in B_r^+.$$

Since $0 < \frac{y^1}{|y|}e^{(1-N)y^1} \leq 1$ and $p \geq 1$, the above holds if

$$C|y|^{-\alpha^-_{\mathbb{S}^{N-1}_+}-2}|\log|y||^{-2+\frac{1}{2p}} \geq \left(|y|^{-\alpha^-_{\mathbb{S}^{N-1}_+}}|\log|y||^{\frac{1}{2p}}\right)^p \quad \forall y \in B_r^+.$$

The previous inequality is true provided

$$C|y|^{-\alpha^-_{\mathbb{S}^{N-1}_+}-2+p\alpha^-_{\mathbb{S}^{N-1}_+}}|\log|y||^{-2+\frac{1}{2p}-\frac{1}{2}} \geq 1 \quad \forall y \in B_r^+.$$

This is clearly possible whenever $p < p_{\mathbb{S}^{N-1}_+} = 1 + \frac{2}{\alpha^-_{\mathbb{S}^{N-1}_+}}$ and $r > 0$ is small enough.

Finally, we notice that $\int_{\mathcal{B}_r^+} w^p\,dx \leq C\int_0^r t^{\frac{N-4}{2}-\sqrt{\frac{N^2}{4}-c}}|\log t|^{\frac{1}{2}}\,dt < \infty$, when $N-1 < c \leq \frac{N^2}{4}$. This concludes the proof. $\square$

4 Problem with perturbation

We let $\Gamma \subset \mathbb{R}^N$ be a smooth closed submanifold of dimension k with $1 \leq k < N-2$. Let Ω be a smooth domain in $\mathbb{R}^N$ containing Γ. We study the problem

$$\begin{cases} -\Delta u - \frac{(N-k-2)^2}{4}\frac{1}{\operatorname{dist}(x,\Gamma)^2}\,q(x)\,u \geq u^p & \text{in } \mathcal{D}'(\Omega\setminus\Gamma), \\ u \in L^p_{loc}(\Omega\setminus\Gamma), \\ u \gneqq 0 & \text{in } \Omega\setminus\Gamma, \end{cases} \tag{4.1}$$

where $q \in C^2(\Omega)$, $q \geq 0$ in Ω and normalized as

$$\max_{\sigma\in\Gamma} q(\sigma) = 1. \tag{4.2}$$

We obtain the following result:

Theorem 4.1 *Suppose that $p \geq \frac{N-k+2}{N-k-2}$ and that (4.2) holds. Then problem (4.1) does not have a solution.*

The above supercriticality assumption on p is sharp as we will see in Section 4.2 below.

Remark 4.2

- *It was observed in [[5], Remark 3] that if* $0 < \max_\Gamma q < 1$ *or* $q \equiv 1$ *then* (4.1) *does not have a solution when*

$$p \geq p^+ := 1 + \frac{2}{\frac{N-k-2}{2} - \sqrt{\frac{(N-k-2)^2}{4} - c}} \geq \frac{N-k+2}{N-k-2},$$

with $c = \frac{(N-k-2)^2}{4} \max_\Gamma q$.

- *We should mention that extremals for weighted Hardy inequality was studied in [6], [7] and [14] when* Γ *is a submanifold of* $\partial\Omega$ *and* $k = 1, \dots, N-1$. *In these papers, the finiteness of the integral* $\displaystyle\int_\Gamma \frac{1}{\sqrt{1-q(\sigma)}}\, d\sigma$ *was necessary and sufficient to obtain the existence of an eigenfunction in some function space corresponding to some "critical" eigenvalue.*
  We belive that the argument in this paper and the results in [14] might be used to study problem (4.1) *but with* $\Gamma \subset \partial\Omega$.

In the sequel, we denote by $\delta(x) := \mathrm{dist}(x\,, \Gamma)$. For $\beta > 0$, we consider the interior of the tube around Γ of radius β defined as $\Gamma_\beta := \{x \in \Omega \,:\, \delta(x) < \beta\}$. It is well known that if β is positive small, the function δ is smooth in $\Gamma_\beta \setminus \Gamma$. If β is small then for all $x \in \Gamma_\beta$, there exists a unique projection $\sigma(x) \in \Gamma$ given by

$$\sigma(x) = x - \frac{1}{2}\nabla(\delta^2)(x) = x - \delta(x)\nabla\delta(x). \tag{4.3}$$

In addition the function σ is also smooth in Γ_β, see for instance [1].

From now on, we will consider β's for which the projection function σ is smooth. Set

$$\omega_0(x) = \delta^{-\alpha(x)}, \tag{4.4}$$

with

$$\alpha(x) = \alpha_q(x) = \frac{N-k-2}{2} - \sqrt{\tilde{\alpha}(x)} \tag{4.5}$$

and where

$$\tilde{\alpha}(x) = \left(\frac{N-k-2}{2}\right)^2 (1 - q(\sigma(x)) + \delta(x)).$$

Clearly α is well defined as soon as $q \leq 1$ on Γ. Recall that $X_a(t) = |\log t|^a$, $t \in (0,1)$ and $a \in \mathbb{R}$. We define

$$\omega_a(x) := \omega_0(x)\, X_a(\delta(x)).$$

We will need the following result which will be useful in the proof of Theorem 4.1.

Lemma 4.3 *Put* $L_q := -\Delta - \left(\frac{N-k-2}{2}\right)^2 \delta^{-2}\, q$. *Then there exit* $C, \beta_0 > 0$ *depending only on* Γ, a *and* $\|q\|_{C^2(\overline{\Omega})}$ *such that*

(4.6)

$$\left| L_q\, \omega_a - 2\, a \sqrt{\tilde{\alpha}} \delta^{-2}\, X_{-1}\, \omega_a + a(a-1)\delta^{-2}\, X_{-2}\, \omega_a \right| \leq C |\log(\delta)|\, \delta^{-\frac{3}{2}}\, \omega_a, \qquad \text{in } \Gamma_{\beta_0}.$$

Proof. We start by noticing that

$$\Delta\omega_0 = \omega_0 \Bigg(\Delta \log(\omega_0) + |\nabla \log(\omega_0)|^2 \Bigg) \tag{4.7}$$

and that

$$-\Delta \log(\omega_0) = \Delta\alpha \log(\delta) + 2\nabla\alpha \cdot \nabla(\log(\delta)) + \alpha\Delta \log(\delta). \tag{4.8}$$

We have

$$-\Delta\alpha = \Delta\sqrt{\tilde{\alpha}} = \sqrt{\tilde{\alpha}} \left(\frac{1}{2}\Delta \log(\tilde{\alpha}) + \frac{1}{4}|\nabla \log(\tilde{\alpha})|^2 \right). \tag{4.9}$$

By simple computations we get

$$\sqrt{\tilde{\alpha}} \nabla \log(\tilde{\alpha}) = \frac{\nabla\tilde{\alpha}}{\sqrt{\tilde{\alpha}}} = \left(\frac{N-k-2}{2}\right)^2 \frac{-\nabla(q \circ \sigma) + \nabla\delta}{\sqrt{\tilde{\alpha}}}$$

and

$$\sqrt{\tilde{\alpha}}\, |\Delta \log(\tilde{\alpha})| \leq \frac{|\Delta\tilde{\alpha}|}{\sqrt{\tilde{\alpha}}} + \frac{|\nabla\tilde{\alpha}|^2}{\sqrt{\tilde{\alpha}}}.$$

We deduce that there exits a constant $\beta_0 > 0$ depending only on Γ and $\|q\|_{C^2(\overline{\Omega})}$ such that

$$|\Delta\alpha| \leq \frac{C}{\delta^{\frac{3}{2}}} \quad \text{in } \Gamma_{\beta_0}. \tag{4.10}$$

Similar we have

$$|\nabla\alpha\cdot\nabla\log\delta|\le\frac{C}{\delta^{\frac{3}{2}}}\quad\text{in }\Gamma_{\beta_0}. \tag{4.11}$$

Recall that (see for instance [9])

$$\alpha\Delta\log(\delta)=\alpha\,\frac{N-k-2}{\delta^2}\,(1+O(\delta)). \tag{4.12}$$

Using (4.9), (4.10), (4.11) and (4.12) in the formula (4.8), we obtain the following estimate:

$$\left|\Delta\log(\omega_0)+\alpha\,\frac{N-k-2}{\delta^2}\right|\le C\,\frac{|\log\delta|}{\delta^{\frac{3}{2}}}\quad\text{in }\Gamma_{\beta_0}. \tag{4.13}$$

We also have

$$-\nabla(\log(\omega_0))=\nabla(\alpha\log(\delta))=\alpha\frac{\nabla\delta}{\delta}+\log(\delta)\nabla\alpha$$

and thus

$$\left||\nabla(\log(\omega_0))|^2-\frac{\alpha^2}{\delta^2}\right|\le C\,\frac{|\log\delta|}{\delta^{\frac{3}{2}}}\quad\text{in }\Gamma_{\beta_0}. \tag{4.14}$$

By using (4.13), (4.14) in the identity (4.7), we conclude that

$$\left|\frac{\Delta\omega_0}{\omega_0}+\alpha\,\frac{N-k-2}{\delta^2}-\frac{\alpha^2}{\delta^2}\right|\le C\,\frac{|\log\delta|}{\delta^{\frac{3}{2}}}\quad\text{in }\Gamma_{\beta_0}.$$

We use the fact that $|q(x)-q(\sigma(x))|\le C\delta(x)$ to deduce that

$$\Delta\omega_0\;=\;\frac{(N-k-2)^2}{4}\delta^{-2}\,q(x)\,\omega_0+O(|\log(\delta)|\,\delta^{-\frac{3}{2}})\,\omega_0\quad\text{in }\Gamma_{\beta_0}.$$

To conclude, we write

$$\omega_a(x):=\omega_0(x)\,(-\log(\delta(x)))^a$$

and the proof of (4.6) follows with some little computations. We skip the details. □

4.1 Proof of Theorem 4.1

Step I: The following inequality holds:

$$\int_{\Gamma_{\beta_0}}|\nabla\varphi|^2dx-\frac{(N-k-2)^2}{4}\int_{\Gamma_{\beta_0}}\delta^{-2}q\varphi^2dx\ge C\int_{\Gamma_{\beta_0}}\delta^{-2}X_{-2}\varphi^2dx \tag{4.15}$$

for any $\varphi \in C_c^\infty(\Gamma_{\beta_0})$, with $\beta_0 > 0$ small depending only on K and $\|q\|_{C^2(\overline{\Omega})}$ and $C > 0$ is a constant.
Indeed, observe that by (4.6),

$$-\frac{\Delta\omega_{\frac{1}{2}}}{\omega_{\frac{1}{2}}} - \frac{(N-k-2)^2}{4}\delta^{-2}\, q \geq \frac{1}{4}\delta^{-2}\, X_{-2} - C|\log(\delta)|\,\delta^{-\frac{3}{2}} \qquad \text{in } \Gamma_\beta \setminus \Gamma.$$

Hence, there exist $\beta_0 > 0$ small and a constant $C > 0$ such that

$$-\Delta\omega_{\frac{1}{2}} - \frac{(N-k-2)^2}{4}\delta^{-2}\, q\,\omega_{\frac{1}{2}} - C\delta^{-2}\, X_{-2}\,\omega_{\frac{1}{2}} \geq 0 \qquad \text{in } \Gamma_{\beta_0} \setminus \Gamma. \tag{4.16}$$

Since $\omega_{\frac{1}{2}} \in L^1(\Gamma_{\beta_0})$, the inequality (4.16) holds in $\mathcal{D}'(\Gamma_{\beta_0})$ thus by Lemma 1.4, (4.15) follows.
Step II: Set $\theta_a := \omega_0 + \omega_a$, with $a < -1/2$. There exist positive constants C and β_0 depending only on a, Γ and $\|q\|_{C^2(\overline{\Omega})}$ such that

$$\|\theta_a\|^2_{H^1(\Gamma_{\beta_0})} \leq C\int_\Gamma \frac{1}{\sqrt{1-q(\sigma)}}\, d\sigma., \tag{4.17}$$

First of all it is easy to see that, since $X_a \leq 1$ for a negative, we can estimate

$$|\nabla\theta_a|^2 \leq C\delta^{-2\alpha-2} \qquad \text{in } \Gamma_{\beta_0}. \tag{4.18}$$

Following [10], there exits a family of disjoint open sets W_i, $i = 1, \dots, m_0$ of Γ such that

$$\Gamma = \bigcup_{i=1}^{m_0} \overline{W_i}, \qquad |\overline{W_i} \cap \overline{W_j}| = 0, \quad i \neq j.$$

Moreover by (4.18),

$$\|\theta_a\|^2_{H^1(\Gamma_{\beta_0})} \leq C\int_{\Gamma_{\beta_0}} \delta^{-2\alpha-2} = C\sum_{i=1}^{m_0}\int_{W_i \times B^{N-k}_{\beta_0}} \delta^{-2\alpha-2}\,(1 + O_i(\delta))\, d\delta\, d\sigma, \tag{4.19}$$

where B^{N-k}_β is the ball of $\mathbb{R}^{N-k}$ with radius β. Therefore, we have

$$\begin{aligned}
\|\theta_a\|^2_{H^1(\Gamma_{\beta_0})} &\leq C\sum_{i=1}^{m_0}\int_{W_i}\int_{\mathbb{S}^{N-k-1}}\int_0^{\beta_0} \delta^{-1}\delta^{(N-k-2)\sqrt{1-q(\sigma)+\delta}}\, d\delta d\sigma \\
&\leq C\sum_{i=1}^{m_0}\int_{W_i}\int_{\mathbb{S}^{N-k-1}}\int_0^{\beta_0} \delta^{-1}\delta^{(N-k-2)\sqrt{1-q(\sigma)}}\, d\delta d\sigma \\
&\leq C\sum_{i=1}^{m_0}\int_{W_i} \frac{1}{\sqrt{1-q(\sigma)}}\, d\sigma = C\int_\Gamma \frac{1}{\sqrt{1-q(\sigma)}}\, d\sigma.
\end{aligned}$$

This ends the proof of this step.

Step III: Let u satisfies (4.1) and $\theta_a = \omega_0 + \omega_a$, for $a < -1/2$. For any $\beta > 0$ small, there exists a constant $C > 0$ such that

$$u \geq C\theta_a \quad \text{in } \Gamma_\beta. \tag{4.20}$$

Indeed, define $q_n(x) := q(x) - \frac{1}{n}$ with $n \in \mathbb{N}^*$ and we put $\theta_{a,n} = \delta^{-\alpha_{q_n}} + \delta^{-\alpha_{q_n}} X_a(\delta)$. Recalling (4.5), by (4.6) there exit constants $\beta_0, C > 0$ (independent on n) such that

$$L_{q_n}\theta_{a,n} \leq -\frac{3}{4}\delta^{-2}\,|\log\delta|^{-2+a}\,\delta^{-\alpha_{q_n}} + C|\log(\delta)|\,\delta^{-\frac{3}{2}}\,\delta^{-\alpha_{q_n}} \quad \text{in } \Gamma_\beta,$$

for any $\beta \in (0, \beta_0)$. Therefore for all $\beta > 0$ small we obtain

$$-\Delta\theta_{a,n} - \frac{(N-k-2)^2}{4}\delta^{-2}\,q_n(x)\,\theta_{a,n} \leq 0 \quad \text{in } \Gamma_\beta \quad \forall n \geq 1. \tag{4.21}$$

By [[5], Lemma 1], $u \in L^p_{loc}(\Omega)$. In addition, it is nonnegative and non-trivial in Ω and satisfies

$$-\Delta u - \frac{(N-k-2)^2}{4}\delta^{-2}q(x)\,u \geq u^p \quad \text{in } \mathcal{D}'(\Omega). \tag{4.22}$$

Hence by the maximum principle, $u > 0$ in Ω. For $\beta > 0$ small (independent on n), by (4.15) we can pick $v_n \in H^1_0(\Gamma_\beta)$ solution to

$$-\Delta v_n - \frac{(N-k-2)^2}{4}\delta^{-2}q_n(x)\,v_n = \min(u^p, 1) \quad \text{in } \Gamma_\beta. \tag{4.23}$$

By Lemma 1.3 the sequence $(v_n)_n$ is monotone increasing and converging pointwise to $v \in H(\Gamma_\beta)$ solution to $-\Delta v - \frac{(N-k-2)^2}{4}\delta^{-2}q(x)\,v = \min(u^p, 1)$. By Lemma 1.3 we have that $u \geq v \geq v_n > 0$ in Γ_β for any $n \geq 1$. By elliptic regularity theory v_n is continuous in $\Gamma_\beta \setminus \Gamma$. We choose $M_n > 0$ such that

$$M_n \sup_{\partial\Gamma_{\frac{\beta}{2}}} \theta_a = \inf_{\partial\Gamma_{\frac{\beta}{2}}} v_n. \tag{4.24}$$

Clearly, we have $M_n\,\theta_{a,n} \leq v_n$ on $\partial\Gamma_{\frac{\beta}{2}}$. It follows form (4.17) that $(M_n\,\theta_{a,n} - v_n)^+ \in H^1_0\left(\Gamma_{\frac{\beta}{2}}\right)$. On the other hand by (4.21) and (4.23),

$$-\Delta\left(M_n\,\theta_{a,n} - v_n\right) - \frac{(N-k-2)^2}{4}\delta^{-2}q(x)\left(M_n\,\theta_{a,n} - v_n\right) \leq 0 \quad \text{in } \Gamma_{\frac{\beta}{2}}.$$

Multiplying this inequality by $(M_n\,\theta_{a,n} - v_n)^+$ and integrating by parts yields $M_n\,\theta_{a,n} \le v_n$ on $\Gamma_{\frac{\beta}{2}}$ by (4.15). Since v_n is monotone increasing to v, by the choice of M_n in (4.24), there exists an integer $n_0 \ge 1$ such that $M_{n_0}\,\theta_{a,n} \le v_n$ for all $n \ge n_0$. Passing to the limit, we get (4.20).
Step IV: There is no u satisfying (4.1) with $p \ge \frac{N-k+2}{N-k-2}$.

By using (4.20) we have that

$$u^{p-1} \ge C\theta_a^{p-1} \ge C\omega_0^{p-1} \ge C\delta^{-2+2\sqrt{1-q\circ\sigma}} \quad \text{ in } \Gamma_\beta,$$

for some $C > 0$ and provided β is small. This together with (4.22) give

$$-\Delta u - \left(q + C_0\delta^{2\sqrt{1-q\circ\sigma}}\right)\frac{(N-k-2)^2}{4}\delta^{-2}\,u \ge 0 \qquad \text{in } \mathcal{D}'(\Gamma_\beta), \tag{4.25}$$

for some $C_0 > 0$. By Lemma 1.4 we have, $\forall \varphi \in C_c^\infty(\Gamma_\beta)$

$$\frac{(N-k-2)^2}{4} \le \frac{\displaystyle\int_{\Gamma_\beta} |\nabla\varphi|^2\,dx}{\displaystyle\int_{\Gamma_\beta}\left(q + C_0\delta^{2\sqrt{1-q(\sigma)}}\right)\delta^{-2}\,\varphi^2\,dx}. \tag{4.26}$$

Our aim is to construct appropriate test functions in (4.26) supported in a neighborhood of the maximum point of q on Γ in order to get a contradiction.

By (4.2), we can let $\sigma_0 \in \Gamma$ be such that

$$q(\sigma_0) = \max_{\sigma\in\Gamma} q(\sigma) = 1. \tag{4.27}$$

For $y \in \mathbb{R}^N$, we write $y = (\tilde{y}, \bar{y}) \in \mathbb{R}^{N-k} \times \mathbb{R}^k$ with $\tilde{y} = (y^1, \dots, y^{N-k})$ and $\bar{y} = (y^{N-k+1}, \dots, y^N)$. Consider $f : \mathbb{R}^k \to \Gamma$ a normal parameterization of a neighborhood of σ_0 with $f(0) = \sigma_0$. In a neighborhood of σ_0, we consider $\mathcal{N}_i$, $i = 1, \dots, N-k$ an orthonormal frame filed on the normal bundle of Γ. We can therefore define a parameterization of a neighborhood, in $\mathbb{R}^N$, of σ_0 by the mapping $Y : B_r(0) \to \Gamma_\beta$ as

$$y \mapsto Y(y) = f(\bar{y}) + \sum_{i=1}^{N-k} y^i \mathcal{N}_i(f(\bar{y})) \in \Gamma_\beta,$$

for some $r > 0$ small. By identification using (4.3), we get for some $r > 0$ small

$$\delta(Y(y)) = |\tilde{y}|, \qquad \sigma(Y(y)) = f(\bar{y}) \qquad \forall y \in B_r(0). \tag{4.28}$$

Denoting by g the metric induced by Y with component $g_{ij}(y) = \langle \partial_i Y(y), \partial_j Y(y) \rangle$, it is not difficult to verify that for all $y \in B_r(0)$

$$g_{ij}(y) = \delta_{ij} + O(|y|) \qquad \text{for } i, j = 1, \ldots, N. \tag{4.29}$$

Next we let $w \in C_c^\infty(\mathbb{R}^{N-k} \setminus \{0\} \times \mathbb{R}^k)$. We choose $\varepsilon_0 > 0$ small such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\varepsilon \, \mathrm{Supp} w \subset B_r(0).$$

We define the following test function

$$\varphi_\varepsilon(x) = \varepsilon^{\frac{2-N}{2}} w\left(\varepsilon^{-1} Y^{-1}(x)\right), \quad x \in Y(\varepsilon \, \mathrm{Supp} w).$$

Clearly, for every $\varepsilon \in (0, \varepsilon_0)$, we have that $\varphi_\varepsilon \in C_c^\infty(\Gamma_\beta)$ and thus by (4.26), we have (summations over repeated indices is understood)

$$\begin{aligned}
\frac{(N-k-2)^2}{4} &\leq \frac{\displaystyle\int_{\Gamma_\beta} |\nabla \varphi_\varepsilon|^2 \, dx}{\displaystyle\int_{\Gamma_\beta} \left(q + C_0 \delta^{2\sqrt{1-q(\sigma)}}\right) \delta^{-2} \, \varphi_\varepsilon^2 \, dx} \\
&= \frac{\displaystyle\varepsilon^{2-N} \int_{\mathbb{R}^N} \varepsilon^{-2} (g^\varepsilon)^{ij} \partial_i w \partial_j w \sqrt{|g^\varepsilon|} \, dy}{\displaystyle\varepsilon^{2-N} \int_{\mathbb{R}^N} \left(q(Y(\varepsilon y)) + C_0 |\varepsilon \tilde{y}|^{2\sqrt{1-q(f(\varepsilon \bar{y}))}}\right) |\varepsilon \tilde{y}|^{-2} \, w^2 \sqrt{|g^\varepsilon|} \, dy}, \\
&= \frac{\displaystyle\int_{\mathbb{R}^N} (g^\varepsilon)^{ij} \partial_i w \partial_j w \sqrt{|g^\varepsilon|} \, dy}{\displaystyle\int_{\mathbb{R}^N} \left(q(Y(\varepsilon y)) + C_0 |\varepsilon \tilde{y}|^{2\sqrt{1-q(f(\varepsilon \bar{y}))}}\right) |\tilde{y}|^{-2} \, w^2 \sqrt{|g^\varepsilon|} \, dy},
\end{aligned}$$

where g^ε is the metric with component $g^\varepsilon_{ij}(y) = g_{ij}(\varepsilon y)$ with $(g^\varepsilon)^{ij}(y)$ denotes the component of the inverse matrix of g^ε and $|g^\varepsilon|$ stands for the determinant of g_ε.

Observe that the scaled metric g^ε expands a $g^\varepsilon = Id + O(\varepsilon)$ on the support of w by (4.29). In addition since q is of class C^1, decreasing ε_0 if necessary, there exits $c > 0$ such that

$$1 - q(f(\varepsilon \bar{y})) \leq c\varepsilon \qquad \forall \bar{y} \in \mathrm{Supp} w \cap \mathbb{R}^k, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

by (4.28). From this we deduce that

$$|\varepsilon\tilde{y}|^{2\sqrt{1-q(f(\varepsilon\bar{y}))}} \to 1 \qquad \text{as } \varepsilon \to 0,$$

uniformly in $y \in \mathrm{Supp} w$. We then have from the dominated convergence theorem and using (4.27) together with (4.28)

$$\frac{(N-k-2)^2}{4} \quad \le \quad \frac{1}{1+C_0} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} |\tilde{y}|^{-2}\, w^2\, dy} \qquad \forall w \in C_c^\infty(\mathbb{R}^{N-k} \setminus \{0\} \times \mathbb{R}^k).$$

This is in contradiction with the well know fact that

$$\inf_{w\in C_c^\infty(\mathbb{R}^{N-k}\setminus\{0\}\times\mathbb{R}^k)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} |\tilde{y}|^{-2}\, w^2\, dy} = \inf_{w\in C_c^\infty(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w|^2\, dy}{\displaystyle\int_{\mathbb{R}^N} |\tilde{y}|^{-2}\, w^2\, dy} = \frac{(N-k-2)^2}{4}$$

because $N-k>2$, see for instance [[25], Section 2.1.6] and [[26], Lemma 1.1]. □

4.2 Existence

Proposition 4.4 *Let $1 \le p < \frac{N-k+2}{N-k-2}$. Then if β is small, there exists $u \in L^p(\Gamma_\beta)$ satisfying*

$$\begin{cases} -\Delta u - \frac{(N-k-2)^2}{4}\, \delta^{-2}\, q\, u \ge u^p & \text{in } \Gamma_\beta \setminus \Gamma, \\ u > 0 & \text{in } \Gamma_\beta. \end{cases} \tag{4.30}$$

Proof. Set

$$u = \omega_0 - \omega_{-1} = \omega_0(1 - X_{-1}(\delta)).$$

Then by Lemma 4.3 there exits $C>0$ such that

$$L_q u \ge 2\delta^{-2}\, X_{-3}(\delta)\, \delta^{-\alpha} - C X_1(\delta)\, \delta^{-\frac{3}{2}}\, \delta^{-\alpha} \qquad \text{in } \Gamma_\beta \setminus \Gamma.$$

Hence, provided β is small, we have $u>0$ and

$$-\Delta u - \frac{(N-k-2)^2}{4}\, \delta^{-2}\, q\, u \ge \delta^{-2} X_{-5}(\delta) u \quad \text{in } \Gamma_\beta \setminus \Gamma.$$

We thus want

$$\delta^{-2}X_{-5}(\delta)u \geq u^p \quad \text{ in } \Gamma_\beta \setminus \Gamma.$$

Or equivalently

$$\delta^{-2}X_{-5}(\delta)\delta^{-\alpha}(1 - X_{-1}(\delta)) \geq \delta^{-p\alpha}(1 - X_{-1}(\delta))^p \quad \text{ in } \Gamma_\beta \setminus \Gamma.$$

That is

$$\delta^{-2+(p-1)\alpha}X_{-5}(\delta)(1 - X_{-1}(\delta))^{1-p} \geq 1 \quad \text{ in } \Gamma_\beta \setminus \Gamma. \tag{4.31}$$

We observe that for $1 \leq p < \frac{N-k+2}{N-k-2}$ we have for every $x \in \Gamma_\beta \setminus \Gamma$

$$-2 + (p-1)\alpha(x) \leq -2 - (p-1)\frac{N-k-2}{2} < 0.$$

This implies that if β is small enough, (4.31) holds so that u satisfies (4.30). The fact that $u \in L^p(\Gamma_\beta)$ is easy to check, we skip the details. $\square$

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