On the Hardy-Poincaré inequality with boundary singularities

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Abstract. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 1$. In this paper we study the Hardy-Poincaré inequality with weight function singular at the boundary of Ω. In particular we provide sufficient and necessary conditions on the existence of minimizers.

Key Words: Hardy inequality, extremals, existence, non-existence.

1 Introduction

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 1$, with $0 \in \partial\Omega$. In the framework of Brezis and Marcus [1], we study the existence and non-existence of minima for the following quotient

$$(1.1) \qquad \mu_\lambda(\Omega) := \inf_{u \in H^1_0(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega |u|^2\, dx}{\displaystyle\int_\Omega |x|^{-2}|u|^2\, dx},$$

in terms of $\lambda \in \mathbb{R}$ and Ω. The existence and non-existence of extremals for (1.1) were studied in [3], [4], [6], [7], [8], [12], [13], [14] and the references there in. Especially in [7], the authors proved that for every smooth bounded domain Ω of $\mathbb{R}^N$, $N \geq 2$, with $0 \in \partial\Omega$

$$(1.2) \qquad \sup_{\lambda \in \mathbb{R}} \mu_\lambda(\Omega) = \frac{N^2}{4} = \mu_0\left(\mathbb{R}^N_+\right),$$

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where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x^1 > 0\}$, see also Lemma 3.4. In addition they showed that there exists $\lambda^* = \lambda^*(\Omega) \in [-\infty, +\infty)$ such that $\mu_\lambda(\Omega) < \frac{N^2}{4}$ and it is achieved for all $\lambda > \lambda^*$. If Ω is locally convex at 0, they proved that $\lambda^* \in \mathbb{R}$. Moreover if $\lambda^* \in \mathbb{R}$ and Ω is locally concave at 0 then there is no minimizer for $\mu_{\lambda^*}(\Omega) = \frac{N^2}{4}$.
The questions to know whether λ^* is finite for every smooth domain Ω and the non-existence of minimizers for $\mu_{\lambda^*}(\Omega)$ remained open.
We shall show that, indeed, the supremum in (1.2) is always attained by $\lambda^* \in \mathbb{R}$ and that there is no extremals for $\mu_{\lambda^*}(\Omega)$. Our main result is the following,

Theorem 1.1 *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, with $0 \in \partial\Omega$. Then there exists $\lambda^*(\Omega) \in \mathbb{R}$ such that $\mu_\lambda(\Omega)$ is attained if and only if $\lambda > \lambda^*(\Omega)$.*

We notice that if $N = 1$ then by [6] we have that $\lambda^*(\Omega) \geq 0$ and thus $\mu_{\lambda^*}(\Omega)$ is not achieved by [8]. We mention that, as observed in [7] and [6], there are various smooth bounded domains such that $\lambda^*(\Omega) < 0$.
The fact that $\lambda^*(\Omega) \in \mathbb{R}$ is a consequence of the following local Hardy inequality, for $r > 0$ small,

$$(1.3) \qquad \int_{\Omega \cap B_r(0)} |\nabla u|^2\, dx \geq \frac{N^2}{4} \int_{\Omega \cap B_r(0)} |x|^{-2}|u|^2\, dx \quad \forall\, u \in H^1_0(\Omega \cap B_r(0)).$$

On the other hand the above inequality implies that $\mu_0(\Omega \cap B_r(0)) = \frac{N^2}{4}$ by (1.2). In particular, even if a domain has negative principal curvatures at 0, its Hardy constant may be equal to $\frac{N^2}{4}$ the Hardy constant of the half-space $\mathbb{R}^N_+$. This is not the case for the Hardy-Sobolev constant, see Ghoussoub-Kang [10]. Hence the existence of extremals for μ_0 depends on all the geometry of the domain instead of the geometric quantities at the origin, see Proposition 4.2.
In Section 2, we introduce the system of *normal coordinates* and the modified ground states used in the hall paper. In section 3, we show that $\lambda^* \in \mathbb{R}$ and we provide an improvement of (1.3). In Section 4, we show that the problem

$$-\Delta u - \frac{N^2}{4}|x|^{-2}u = \lambda u, \qquad \text{in } \Omega$$

does not possess a non trivial and nonnegative supersolution in $H^1_0(\Omega) \cap C(\Omega)$. In Section 5 we prove Theorem 1.1. Finally in Section 6, we generalize Theorem 1.1 by studying variational problems of type (1.1) with some weights.

2 Preliminaries and Notations

For $N \geq 2$, we denote by $\{E_1, E_2, \dots, E_N\}$ the standard orthonormal basis of $\mathbb{R}^N$; $\mathbb{R}^N_+ = \{y \in \mathbb{R}^N \,:\, y^1 > 0\}$; $B_r(y_0) = \{y \in \mathbb{R}^N \,:\, |y - y_0| < r\}$; $B_r^+ = B_r(0) \cap \mathbb{R}^N_+$ and $S^{N-1}_+ = \partial B_1(0) \cap \mathbb{R}^N_+$.

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^N$ with boundary $\mathcal{M} := \partial \mathcal{U}$ a smooth closed hypersurface of $\mathbb{R}^N$ and $0 \in \mathcal{M}$. We write $N_{\mathcal{M}}$ for the unit normal vector-field of $\mathcal{M}$ pointed into $\mathcal{U}$. Up to a rotation, we assume that $N_{\mathcal{M}}(0) = E_1$. For $x \in \mathbb{R}^N$, we let $d_{\mathcal{M}}(x) = \operatorname{dist}(\mathcal{M}, x)$ be the distance function of $\mathcal{M}$. Given $x \in \mathcal{U}$ and close to $\mathcal{M}$ then it can be written uniquely as $x = \sigma_x + d_{\mathcal{M}}(x)\, N_{\mathcal{M}}(\sigma_x)$, where σ_x is the projection of x on $\mathcal{M}$. We further use the Fermi coordinates $(y^2, \dots, y^N)$ on $\mathcal{M}$ so that for σ_x close to 0, we have

$$\sigma_x = \operatorname{Exp}_0 \left(\sum_{i=2}^{N} y^i E_i \right),$$

where $\operatorname{Exp}_0 : \mathbb{R}^N \to \mathcal{M}$ is the exponential mapping on $\mathcal{M}$ endowed with the metric induced by $\mathbb{R}^N$. In this way a neighborhood of 0 in $\mathcal{U}$ can be parameterized by the map

$$F_{\mathcal{M}}(y) = \operatorname{Exp}_0 \left(\sum_{i=2}^{N} y^i E_i \right) + y^1\, N_{\mathcal{M}} \left(\operatorname{Exp}_0 \left(\sum_{i=2}^{N} y^i E_i \right) \right), \quad y \in B_r^+,$$

for some $r > 0$. In this coordinates, the Laplacian Δ is given by

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{(\partial x^i)^2} = \frac{\partial^2}{(\partial y^1)^2} + h_{\mathcal{M}} \circ F_{\mathcal{M}} \frac{\partial}{\partial y^1} + \sum_{i,j=2}^{N} \frac{\partial}{\partial y^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial y^j} \right),$$

where $h_{\mathcal{M}}(x) = \Delta\, d_{\mathcal{M}}(x)$; for $i, j = 2 \dots, N$, $g_{ij} = \langle \frac{\partial F_{\mathcal{M}}}{\partial y^i}, \frac{\partial F_{\mathcal{M}}}{\partial y^j} \rangle$; the quantity $|g|$ is the determinant of g and g^{ij} is the component of the inverse of the matrix $(g_{ij})_{2 \leq i,j \leq N}$.

Since $g_{ij} = \delta_{ij} + O(y^1) + O(|y|^2)$, we have the following Taylor expansion

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{(\partial y^i)^2} + h_{\mathcal{M}} \circ F_{\mathcal{M}} \frac{\partial}{\partial y^1} + \sum_{i=2}^{N} O_i(|y|) \frac{\partial}{\partial y^i} + \sum_{i,j=2}^{N} O_{ij}(|y|) \frac{\partial^2}{\partial y^i \partial y^j}. \tag{2.1}$$

For $a \in \mathbb{R}$, we put $X_a(t) := |\log t|^a$, $t \in (0, 1)$. Let

$$\overline{\omega}_a(y) := y^1 |y|^{-\frac{N}{2}} X_a(|y|) \quad \forall y \in \mathbb{R}^N_+$$

and put

$$L_y := -\sum_{i=1}^{N}\frac{\partial^2}{(\partial y^i)^2} - \frac{N^2}{4}|y|^{-2} + a(a-1)|y|^{-2}X_{-2}(|y|).$$

Then one easily verifies that

$$\begin{cases} L_y\,\overline{\omega}_a = 0 \text{ in } \mathbb{R}^N_+,\\ \overline{\omega}_a = 0 \text{ on } \partial\mathbb{R}^N_+\setminus\{0\},\\ \overline{\omega}_a \in H^1(B^+_R)\ \forall R>0,\ a<-\dfrac{1}{2}.\end{cases}$$

For $K\in\mathbb{R}$, we define

$$\omega_{a,K}(y) = e^{Ky^1}\,\overline{\omega}_a(y).$$

This function satisfies similar boundary and integrability conditions as $\overline{\omega}_a$. In addition it holds that

$$L_y\,\omega_{a,K} = -\frac{2K}{y^1}\omega_{a,K} + 2K\left(\frac{N}{2} + aX_{-1}(|y|)\right)\frac{y^1}{|y|^2}\omega_{a,K} - K^2\omega_{a,K}. \tag{2.2}$$

Furthermore for all $a\in\mathbb{R}$

$$\begin{aligned}\sum_{i=2}^{N} O_i(|y|)\frac{\partial\omega_{a,K}}{\partial y^i} + \sum_{i,j=2}^{N} O_{ij}(|y|)\frac{\partial^2\omega_{a,K}}{\partial y^i\partial y^j} &= y^1\,e^{Ky^1}O\left(|y|^{-\frac{N}{2}-1}X_a(|y|)\right)\\ &= \mathcal{O}_{a,K}(|y|^{-1})\,\omega_{a,K}(y).\end{aligned}$$

Here the error term $\mathcal{O}_{a,K}$ has the property that for any $A>0$, there exist positive constants $c=c(\Omega,A,K)$ and $s_0=s_0(\Omega,A,K)$ such that

$$|\mathcal{O}_{a,K}(s)| \le c\,s \quad \forall s\in(0,s_0),\ \forall a\in[-A,A]. \tag{2.3}$$

Let

$$W_{a,K}(x) := \omega_{a,K}(F^{-1}_{\mathcal{M}}(x)), \qquad \forall\, x\in\mathcal{B}^+_r := F_{\mathcal{M}}(B^+_r).$$

Then using (2.1), (2.2) and the fact that $|x| = |y| + O(|y|^2)$ we obtain the following expansion

$$L\,W_{a,K} = -\left(\frac{2K + h_{\mathcal{M}}(x)}{d_{\mathcal{M}}(x)}\right)W_{a,K} + \mathcal{O}_{a,K}(|x|^{-1})\,W_{a,K} \quad \text{in } \mathcal{B}^+_r, \tag{2.4}$$

with $L := -\Delta - \frac{N^2}{4}|x|^{-2} + a(a-1)|x|^{-2}X_{-2}(|x|)$. Moreover it is easy to see that

$$
\begin{cases} W_{a,K} > 0 & \text{in } \mathcal{B}_r^+, \\ W_{a,K} = 0 & \text{on } \mathcal{M} \cap \partial\mathcal{B}_r^+ \setminus \{0\}, \\ W_{a,K} \in H^1(\mathcal{B}_r^+), \ \forall a < -\frac{1}{2}. \end{cases} \tag{2.5}
$$

3 $\lambda^*(\Omega)$ is finite

We start with the following local improved Hardy inequality.

Lemma 3.1 *Let $\mathcal{U} = \mathbb{R}^N \setminus \overline{B_1(-E_1)}$. Then there exist constants $c = c(N) > 0$ and $r_0 = r_0(N) > 0$ such that for all $r \in (0, r_0)$ the inequality*

$$
\int_{\mathcal{B}_r^+} |\nabla u|^2\, dx - \frac{N^2}{4}\int_{\mathcal{B}_r^+} \frac{|u|^2}{|x|^2}\, dx \geq c\int_{\mathcal{B}_r^+} \frac{|u|^2}{|x|^2|\log|x||^2}\, dx + (N-1)\int_{\mathcal{B}_r^+} \frac{|u|^2}{d_{\mathcal{M}}(x)}\, dx
$$

holds for all $u \in H_0^1(\mathcal{B}_r^+)$.

Proof. It is easy to see that $h_{\mathcal{M}}(x) = \frac{N-1}{1+d_{\mathcal{M}}(x)}$ and thus

$$
-\frac{2(1-N) + h_{\mathcal{M}}(x)}{d_{\mathcal{M}}(x)} \geq \frac{N-1}{d_{\mathcal{M}}(x)} \quad \forall x \in \mathcal{U}. \tag{3.1}
$$

For $r > 0$ small, we set

$$
\tilde{w}(x) = \omega_{\frac{1}{2}, 1-N}(F_{\mathcal{M}}^{-1}(x)), \quad \forall x \in \mathcal{B}_r^+.
$$

By (2.4) and (3.1), we have

$$
-\frac{\Delta\tilde{w}}{\tilde{w}} \geq \frac{N^2}{4}|x|^{-2} + \frac{1}{4}|x|^{-2}X_{-2}(|x|) + \frac{N-1}{d_{\mathcal{M}}(x)} + O(|x|^{-1}) \ \text{ in } \mathcal{B}_r^+.
$$

Hence there exists $r_0 = r_0(N) > 0$ such that for all $r \in (0, r_0)$

$$
-\frac{\Delta\tilde{w}}{\tilde{w}} \geq \frac{N^2}{4}|x|^{-2} + c|x|^{-2}X_{-2}(|x|) + \frac{N-1}{d_{\mathcal{M}}(x)} \quad \text{in } \mathcal{B}_r^+, \tag{3.2}
$$

for some positive constant c depending only on N. Fix $r \in (0, r_0)$ and let $u \in C_c^\infty(\mathcal{B}_r^+)$. We put $\psi = \frac{u}{\tilde{w}}$. Then one has $|\nabla u|^2 = |\tilde{w}\nabla\psi|^2 + |\psi\nabla\tilde{w}|^2 + \nabla(\psi^2) \cdot \tilde{w}\nabla\tilde{w}$. Therefore $|\nabla u|^2 = |\tilde{w}\nabla\psi|^2 + \nabla\tilde{w} \cdot \nabla(\tilde{w}\psi^2)$. Integrating by parts, we get

$$
\int_{\mathcal{B}_r^+} |\nabla u|^2\, dx = \int_{\mathcal{B}_r^+} |\tilde{w}\nabla\psi|^2\, dx + \int_{\mathcal{B}_r^+} \left(-\frac{\Delta\tilde{w}}{\tilde{w}}\right) u^2\, dx.
$$

The proof is then complete by (3.2) and a desnsity argument. □

As a consequence, we have

Corollary 3.2 *Let Ω be Lipschitz domain and of class C^2 at $0 \in \partial\Omega$. Then there exist constants $c = c(\Omega) > 0$ and $r_0 = r_0(\Omega) > 0$ such that for all $r \in (0, r_0)$, the inequality*

$$\int_{\Omega \cap B_r(0)} |\nabla u|^2 \, dx - \frac{N^2}{4} \int_{\Omega \cap B_r(0)} \frac{|u|^2}{|x|^2} \, dx \geq c \int_{\Omega \cap B_r(0)} \frac{|u|^2}{|x|^2 |\log |x||^2} \, dx$$

holds for all $u \in H_0^1(\Omega \cap B_r(0))$.

Proof. Since Ω is of class C^2 at $0 \in \partial\Omega$, there exits a ball with $0 \in \partial B$ and $\Omega \subset \mathcal{U} = \mathbb{R}^N \setminus \overline{B}$. Therefore by Lemma 3.1, we get the result. $\square$

Remark 3.3 *We should notice that Lemma 3.1 implies that "Ω is locally concave at $0 \in \partial\Omega$" does not necessarly implies that $\mu(\Omega) < \frac{N^2}{4}$ as it happens in the Hardy-Sobolev case, see [10], [11], [5].*

For sake of completeness, we include the proof of (1.2) in the following lemma.

Lemma 3.4 *Let Ω be a Lipschitz domain and of class C^2 at $0 \in \partial\Omega$. Then there exists $\lambda^*(\Omega) \in \mathbb{R}$ such that*

$$\mu_\lambda(\Omega) = \frac{N^2}{4}, \qquad \forall \lambda \leq \lambda^*(\Omega),$$
$$\mu_\lambda(\Omega) < \frac{N^2}{4}, \qquad \forall \lambda > \lambda^*(\Omega).$$

Proof. Claim: $\sup_{\lambda \in \mathbb{R}} \mu_\lambda \leq \frac{N^2}{4}$.
It is well known that $\mu_0(\mathbb{R}^N_+) = \frac{N^2}{4}$, see for instance [9] or [14]. So for any $\delta > 0$, we let $u_\delta \in C_c^\infty(\mathbb{R}^N_+)$ such that

$$\int_{\mathbb{R}^N_+} |\nabla u_\delta|^2 \, dy \leq \left(\frac{N^2}{4} + \delta \right) \int_{\mathbb{R}^N_+} |y|^{-2} u_\delta^2 \, dy.$$

We let B a ball contained in Ω and such that $0 \in \partial B$. If $\varepsilon > 0$, put

$$v(x) = \varepsilon^{\frac{2-N}{2}} u_\delta \left(\varepsilon^{-1} F_{\partial B}^{-1}(x) \right).$$

Clearly, provided ε is small enough, we have that $v \in C_c^\infty(\Omega)$ thus by the change of variable formula

$$\mu_\lambda(\Omega) \le \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx + \lambda \int_\Omega v^2\,dx}{\displaystyle\int_\Omega |x|^{-2} v^2\,dx} \le (1+c\varepsilon)\,\frac{\displaystyle\int_{\mathbb{R}^N_+} |\nabla u_\delta|^2\,dy}{\displaystyle\int_{\mathbb{R}^N_+} |y|^{-2} u_\delta^2\,dy} + c\varepsilon^2|\lambda|,$$

where we have used the fact that $F_{\partial B}^{-1}(x) = x + O(|x|^2)$ and c is a constant depending only on Ω. We conclude that

$$\mu_\lambda(\Omega) \le (1+c\varepsilon)\left(\frac{N^2}{4} + \delta\right) + c\varepsilon^2|\lambda|.$$

Taking the limit in ε and then in δ, the claim follows.

Claim : There exists $\tilde\lambda \in \mathbb{R}$ such that $\mu_{\tilde\lambda} = \frac{N^2}{4}$

For $\delta > 0$ small, we let $\psi \in C^\infty(B_\delta(0))$ be a cut-off function, satisfying

$$0 \le \psi \le 1\,, \quad \psi \equiv 0 \text{ in } \mathbb{R}^N \setminus B_{\frac{\delta}{2}}(0)\,, \quad \psi \equiv 1 \text{ in } B_{\frac{\delta}{4}}(0)\,.$$

We write any $u \in H_0^1(\Omega)$ as $u = \psi u + (1-\psi)u$, to get

$$\int_\Omega |x|^{-2}|u|^2\,dx \le \int_\Omega |x|^{-2}|\psi u|^2\,dx + c\int_\Omega |u|^2\,dx\,, \tag{3.3}$$

where the constant c depends only on δ. Since $\psi u \in H_0^1(\Omega \cap B_\delta(0))$, if δ is sufficiently small, Corollary 3.2 implies that

$$\frac{N^2}{4}\int_\Omega |x|^{-2}|\psi u|^2\,dx \le \int_\Omega |\nabla(\psi u)|^2\,dx. \tag{3.4}$$

In addition, we have

$$\int_\Omega |\nabla(\psi u)|^2\,dx \le \int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega \nabla(\psi^2)\cdot\nabla(u^2)\,dx + c\int_\Omega |u|^2\,dx\,.$$

Using integration by parts we get

$$\int_\Omega |\nabla(\psi u)|^2\,dx \le \int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega \Delta(\psi^2)|u|^2\,dx + c\int_\Omega |u|^2\,dx.$$

Combining this with (3.3) and (3.4) we infer that there exits a positive constant c depending only on δ and Ω such that

$$\frac{N^2}{4}\int_\Omega |x|^{-2}|u|^2\,dx \le \int_\Omega |\nabla u|^2\,dx + c\int_\Omega |u|^2\,dx \quad \forall u \in H_0^1(\Omega).$$

This together with the first calim implies that $\mu_{-c}(\Omega) = \frac{N^2}{4}$.
Finally, noticing that $\mu_\lambda(\Omega)$ is decreasing in λ, we can set

$$\lambda^*(\Omega) := \sup\left\{\lambda \in \mathbb{R} \,:\, \mu_\lambda(\Omega) = \frac{N^2}{4}\right\} \tag{3.5}$$

so that $\mu_\lambda(\Omega) < \frac{N^2}{4}$ for all $\lambda > \lambda^*(\Omega)$. □

4 Non-existence result

In this section we prove the following non-existence result.

Theorem 4.1 *Let Ω be a bounded Lipschitz domain of class C^2 at $0 \in \partial\Omega$ and let $\lambda \geq 0$. Suppose that $u \in H^1_0(\Omega) \cap C(\Omega)$ is a non-negative function satisfying*

$$-\Delta u - \frac{N^2}{4}|x|^{-2}u \geq -\lambda u \quad \text{in } \Omega. \tag{4.1}$$

Then $u \equiv 0$.

Proof. Up to scaling and rotation, we may assume that Ω contains the ball $B = B_1(E_1)$ such that $\overline{B} \cap \overline{\Omega} = \{0\}$. We will use the coordinates in Section 2 with $\mathcal{U} = B$ and $\mathcal{M} = \partial B$. For $r > 0$ small we define $G_r^+ := F_{\partial B}(B_r^+)$.
We suppose that u does not identically vanish near 0 and satisfies (4.1) so that $u > 0$ in $\Omega \cap B_{r_0}(0)$ by the maximum principle, for some $r_0 > 0$.
We define

$$w_a(x) := \omega_{a,N-1}(F_{\partial B}^{-1}(x)), \quad \forall x \in G_r^+.$$

Letting $L := -\Delta - \frac{N^2}{4}|x|^{-2} + \lambda$ then by (2.4)

$$L\, w_a \leq -\frac{2(N-1) + h_{\partial B}}{d_{\partial B}}\, w_a + \left(\lambda - \frac{3}{4}\,|x|^{-2}\, X_{-2}(|x|)\right) w_a + \mathcal{O}_a(|x|^{-1})\, w_a,$$

for every $a < -\frac{1}{2}$. Since $-h_{\partial B}(x) = (N-1)\,(1 + O(|x|))$ in G_r^+, by (2.3) we can choose $r > 0$ small, independent on $a \in (-1, -\frac{1}{2})$, so that

$$L\, w_a \leq 0 \quad \text{in } G_r^+, \quad \forall a \in (-1, -\frac{1}{2}). \tag{4.2}$$

Let $R > 0$ so that

$$R\, w_a \le u \quad \text{on } \overline{F_{\partial B}\left(r S_+^{N-1}\right)} \quad \forall a < -\frac{1}{2}.$$

By (2.5), setting $v_a = R\, w_a - u$, it turns out that $v_a^+ = \max(v_a, 0) \in H_0^1(G_r^+)$ because $w_a = 0$ on $\partial B \cap \partial G_r^+$. Moreover by (4.1) and (4.2),

$$L\, v_a \le 0 \quad \text{in } G_r^+, \quad \forall a \in (-1, -\frac{1}{2}).$$

Multiplying the above inequality by v_a^+ and integrating by parts yields

$$\int_{G_r^+} |\nabla v_a^+|^2\, dx - \frac{N^2}{4} \int_{G_r^+} |x|^{-2} |v_a^+|^2\, dx + \lambda \int_{G_r^+} |v_a^+|^2\, dx \le 0.$$

But then Corollary 3.2 implies that $v_a^+ = 0$ in G_r^+. Therefore $u \ge R\, w_a$ for all $a \in (-1, -\frac{1}{2})$ and this contradicts the fact that $\frac{u}{|x|} \in L^2(\Omega)$ because $\int_{G_r^+} \frac{{w_a}^2}{|x|^2} \ge c \int_{B_r^+} \frac{\omega_{a,N-1}^2}{|y|^2} \ge \frac{c}{2a+1} |\log r|^{2a+1}$, for some positive constant c depending only on B. Consequently u vanish identically in G_r^+ and thus by the maximum principle $u \equiv 0$ in Ω. □

As in [6], starting from exterior domains, we can see that, in general, existence of extremals for μ_0 depends on all the geometry of the domain rather than the geometric constants at the origin. Indeed, let G be a smooth bounded domain of $\mathbb{R}^N$, $N \ge 2$ with $0 \in \partial G$. For $r > 0$, set $\Omega_r = B_r(0) \cap (\mathbb{R}^N \setminus \overline{G})$. It was shown in [6] that there exits $r_1 > 0$ such that $\mu_0(\Omega_r) < \frac{N^2}{4}$ for all $r \in (r_1, \infty)$ and $\mu_0(\Omega_r)$ is achieved. But Corollary 3.2 and (1.2) yields $\mu_0(\Omega_r) = \frac{N^2}{4}$ for $r \in (0, r_0)$. In particular by Theorem 4.1, we get,

Proposition 4.2 *There exit r_0, $r_1 > 0$ such that the problem*

$$\begin{cases} \Delta u + \mu_0(\Omega_r)\, |x|^{-2} u = 0, & \text{in } \Omega_r, \\ u \in H_0^1(\Omega_r), & \\ u \gneqq 0 & \text{in } \Omega_r \end{cases}$$

has a solution for all $r \in (r_1, \infty)$ and does not have a solution for every $r \in (0, r_0)$.

Remark 4.3 *Let Ω be as in Theorem 4.1. Then by similar argument, one can show that there is no positive function $u \in H_0^1(\Omega) \cap C(\Omega)$ that satisfies*

$$-\Delta u - \frac{N^2}{4} \frac{u}{|x|^2} \geq -\frac{\eta(x)}{|x|^2} u \quad \text{in } \Omega,$$

with η is continuous, non-negative and $|\log|x||^2 \eta(x) \to 0$ as $|x| \to 0$.

Remark 4.4 *We should mention that some sharp non-existence results of distributional solution was obtained in [8]. Indeed assume that Ω contains a half-ball centered at $0 \in \partial\Omega$ and that $u \in L^2(\Omega; |x|^{-2}\,dx)$ satisfies*

$$-\int_\Omega u \left(\Delta\varphi + \frac{N^2}{4} \frac{\varphi}{|x|^2} \right) dx \geq -\frac{3}{4} \int_\Omega u \frac{\varphi}{|x|^2 |\log|x||^2}\, dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

then u vanish in a neighborhood of 0.

5 Proof of Theorem 1.1

The proof of the "if" part is similar to the one given in [1], see also [7]. Secondly, since the mapping $\lambda \mapsto \mu_\lambda(\Omega)$ is constant on $(0, \lambda^*(\Omega)]$, it is not difficult to see that $\mu_\lambda(\Omega)$ is not achieved for all $\lambda < \lambda^*(\Omega)$. Now we assume that $\mu_{\lambda^*}(\Omega)$ is attained by a mapping $u \in H_0^1(\Omega)$. Then it is also achieved by $|u|$ so we can assume that $u \gneqq 0$. Furthermore since u solves

$$-\Delta u - \frac{N^2}{4} |x|^{-2} u = \lambda^* u \quad \text{in } \Omega,$$

by standard elliptic regularity theory, u is smooth in Ω. Therefore, Theorem 4.1 implies that $u = 0$ in Ω which is not possible. $\square$

6 Hardy inequality with weight

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$ with $0 \in \partial\Omega$. Following [1] and [2], we study the existence of extremals of the following quotient:

$$J_\lambda := \inf_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 p\, dx - \lambda \int_\Omega |x|^{-2} |u|^2 \eta\, dx}{\displaystyle\int_\Omega |x|^{-2} |u|^2 q\, dx}, \tag{6.1}$$

where the weights p, q and η are nonnegative, nontrivial and satisfy

$$p \in C^1(\overline{\Omega}), \quad q, \eta \in C(\overline{\Omega}), \quad p, \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta(0) = 0. \tag{6.2}$$

We have the following generalization of Theorem 1.1:

Theorem 6.1 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$ with $0 \in \partial\Omega$. Assume that the weight functions in* (6.1) *satisfy* (6.2) *and that*

$$p(0) = q(0) > 0. \tag{6.3}$$

Then, there exists $\lambda^* = \lambda^*(p, q, \eta, \Omega)$ such that

$$J_\lambda = \frac{N^2}{4}, \quad \forall \lambda \leq \lambda^*,$$
$$J_\lambda < \frac{N^2}{4}, \quad \forall \lambda > \lambda^*.$$

Furthermore J_λ is achieved if and only if $\lambda > \lambda^*$.

Proof. Step I: We first show that

$$\sup_{\lambda \in \mathbb{R}} J_\lambda \leq \frac{N^2}{4}. \tag{6.4}$$

Recall the notation in Section 2. For $\rho > 0$ small, we will put $\mathcal{B}_\rho^+ = F_{\partial\Omega}(B_\rho^+)$. By (6.3), for any $\varepsilon > 0$ we can let $r_\varepsilon > 0$ such that

$$p \leq (1+\varepsilon)p(0), \quad q \geq (1-\varepsilon)p(0), \quad \eta \leq \varepsilon \quad \text{in } \overline{\mathcal{B}_{r_\varepsilon}^+}.$$

By Corollary 3.2 and Lemma 3.4, $\mu_0(\mathcal{B}_{r_\varepsilon}^+) = \frac{N^2}{4}$, so for any $\delta > 0$ we can let $u \in C_c^\infty(\mathcal{B}_{r_\varepsilon}^+)$ such that

$$\int_{\mathcal{B}_{r_\varepsilon}^+} |\nabla u|^2 \leq \left(\frac{N^2}{4} + \delta\right) \int_{\mathcal{B}_{r_\varepsilon}^+} |x|^{-2} u^2.$$

It turns out that

$$J_\lambda \leq \frac{\displaystyle\int_\Omega |\nabla u|^2 p - \lambda \int_\Omega |x|^{-2} u^2 \eta}{\displaystyle\int_\Omega |x|^{-2} u^2 q} \leq \frac{1+\varepsilon}{1-\varepsilon}\left(\frac{N^2}{4} + \delta\right) + \frac{\varepsilon|\lambda|}{(1-\varepsilon)q(0)}.$$

Sending δ and ε to zero, (6.4) follows immediately.
Step II: There exists $\tilde{\lambda} \in \mathbb{R}$ such that $J_{\tilde{\lambda}} = \frac{N^2}{4}$.
We fix $r_0 > 0$ positive small and put

$$K_0 = \frac{1}{2} \min_{\mathcal{B}^+_{r_0}} \left(-\nabla p \cdot \nabla d_{\partial\Omega} - h_{\partial\Omega} \right). \tag{6.5}$$

For every $r \in (0, r_0)$, we set

$$\tilde{w}(x) = \omega_{\frac{1}{2}, K_0}(F^{-1}_{\partial\Omega}(x)), \quad \forall x \in \mathcal{B}^+_r.$$

Notice that $\mathrm{div}(p\nabla \tilde{w}) = p\Delta\tilde{w} + \nabla p \cdot \nabla \tilde{w}$. For $r > 0$ small, using (2.4) we get, in $\mathcal{B}^+_r$,
(6.6)

$$-\mathrm{div}(p\nabla\tilde{w}) = p\frac{N^2}{4}|x|^{-2}\tilde{w} + \frac{p}{4}|x|^{-2}X_{-2}(|x|)\tilde{w} + \frac{-\nabla p\cdot\nabla d_{\partial\Omega} - h_{\partial\Omega} - 2K_0}{d_{\partial\Omega}}\tilde{w} + O(|x|^{-1})\tilde{w}.$$

Hence by (6.3) and (6.5) there exist constants $c > 0$ and $r_1 > 0$ (depending on p, q, η and Ω) such that for all $r \in (0, r_1)$

$$-\mathrm{div}(p\nabla\tilde{w}) \geq q\frac{N^2}{4}|x|^{-2}\tilde{w} + c|x|^{-2}X_{-2}(|x|)\tilde{w} \quad \mathcal{B}^+_r. \tag{6.7}$$

Fix $r \in (0, r_1)$ and let $u \in C^\infty_c(\mathcal{B}^+_r)$. We put $\psi = \frac{u}{\tilde{w}}$. Then one has $|\nabla u|^2 = |\tilde{w}\nabla\psi|^2 + |\psi\nabla\tilde{w}|^2 + \nabla(\psi^2)\cdot\tilde{w}\nabla\tilde{w}$. Therefore $|\nabla u|^2 p = |\tilde{w}\nabla\psi|^2 p + p\nabla\tilde{w}\cdot\nabla(\tilde{w}\psi^2)$. Integrating by parts, we get

$$\int_{\mathcal{B}^+_r} |\nabla u|^2 p\, dx = \int_{\mathcal{B}^+_r} |\tilde{w}\nabla\psi|^2 p\, dx + \int_{\mathcal{B}^+_r} \left(-\frac{\mathrm{div}(p\nabla\tilde{w})}{\tilde{w}} \right) u^2\, dx.$$

This together with (6.7) yields

$$\int_{\mathcal{B}^+_r} |\nabla u|^2 p\, dx \geq \frac{N^2}{4}\int_{\mathcal{B}^+_r} |x|^{-2}u^2 q\, dx + c\int_{\mathcal{B}^+_r} |x|^{-2}X_{-2}(|x|)u^2. \tag{6.8}$$

We can now proceed as in the proof of Lemma 3.4 (since $\eta > 0$ in Ω) to conclude that there exists a constant $C = C(p, q, \eta, \Omega) > 0$ such that

$$\frac{N^2}{4}\int_\Omega |x|^{-2}u^2 q\ dx \leq \int_\Omega |\nabla u|^2 p\ dx + C\int_\Omega |x|^{-2}u^2\eta\ dx \quad \forall u \in H^1_0(\Omega).$$

Therefore we can define λ^* as in (3.5) to end the proof of this step.
Step III: Let $u \in H^1_0(\Omega) \cap C(\Omega)$ is a non-negative function satisfying

$$-\mathrm{div}(p\nabla u) - \frac{N^2}{4}q|x|^{-2}u \geq -\lambda|x|^{-2}\eta u \quad \text{in } \Omega. \tag{6.9}$$

Then $u \equiv 0$.
Here, we assume that Ω contains the ball $B = B_1(E_1)$ such that $\overline{B} \cap \overline{\Omega} = \{0\}$ and set $G_r^+ = F_{\partial B}(B_r^+)$. As in the previous step, we put

$$K_1 = \frac{1}{2} \max_{\overline{G^+_{r_0}}} \left(-\nabla p \cdot \nabla d_{\partial\Omega} - h_{\partial\Omega}\right). \tag{6.10}$$

For $r \in (0, r_0)$ and $a < -\frac{1}{2}$, we set

$$w_a(x) = \omega_{a,K_1}(F_{\partial B}^{-1}(x)), \quad \forall x \in G_r^+ = F_{\partial B}(B_r^+).$$

Letting $L = -\mathrm{div}(p\nabla\cdot) - \frac{N^2}{4} q|x|^{-2} + |\lambda||x|^{-2}\eta$ then by (6.10) and (6.3), we get

$$L w_a \leq \left(|\lambda||x|^{-2}\eta - \frac{3}{4} p|x|^{-2} X_{-2}(|x|)\right) w_a + \mathcal{O}_a(|x|^{-1}) w_a \quad \text{ in } G_r^+.$$

Therefore by (2.3) we can choose $r > 0$ small, independent on $a \in (-1, -\frac{1}{2})$, so that

$$L\, w_a \leq 0 \quad \text{in } G_r^+, \quad \forall a \in (-1, -\frac{1}{2}). \tag{6.11}$$

If $u \gneqq 0$ near the origin then by the maximum principle, we can assume that $u > 0$ in G_{2r}^+. Hence we can let $R > 0$ so that

$$R\, w_a \leq u \quad \text{ on } \overline{F_{\partial B}\left(r S_+^{N-1}\right)} \quad \forall a < -\frac{1}{2}.$$

By (2.5), setting $v_a = R\, w_a - u$, it turns out that $v_a^+ = \max(v_a, 0) \in H_0^1(G_r^+)$. Moreover by (6.9) and (6.11),

$$L\, v_a \leq 0 \quad \text{in } G_r^+, \quad \forall a \in (-1, -\frac{1}{2}).$$

Multiplying the above inequality by v_a^+ and integrating by parts yields

$$\int_{G_r^+} |\nabla v_a^+|^2 p \, dx - \frac{N^2}{4} \int_{G_r^+} |x|^{-2} |v_a^+|^2 q \, dx + |\lambda| \int_{G_r^+} |x|^{-2} |v_a^+|^2 \eta \, dx \leq 0.$$

But then (6.8) implies that $v_a^+ = 0$ in G_r^+. Therefore $u \geq R\, w_a$ for all $a \in (-1, -\frac{1}{2})$ and this contradicts the fact that $\frac{u}{|x|} \in L^2(\Omega)$. Consequently u vanish identically in G_r^+ and thus by the maximum principle $u \equiv 0$ in Ω.
Step IV: If $J_\lambda < \frac{N^2}{4}$ then it is achieved.
The proof of the existence part, since $\eta(0) = 0$, is similar to the one given in [1] so we skip it. □

Remark 6.2 *Let Ω be a smooth smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$. Let Σ_k be a smooth compact sub-manifold of $\partial\Omega$ with dimension $0 \leq k \leq N-1$. Here Σ_0 is a single point. Consider the problem (P_k^λ) of finding minimizers for the quotient:*

$$J_\lambda^k := \inf_{u \in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2 p\, dx - \lambda \int_\Omega dist(x, \Sigma_k)^{-2} |u|^2 \eta\, dx}{\displaystyle\int_\Omega dist(x, \Sigma_k)^{-2} |u|^2 q\, dx}, \tag{6.12}$$

where the weights p, q and η are smooth positive in $\overline{\Omega}$ with $\eta = 0$ on Σ_k and the following normalization

$$\min_{\Sigma_k} \frac{p}{q} = 1 \tag{6.13}$$

holds. We put

$$I_k = \int_{\Sigma_k} \frac{d\sigma}{\sqrt{1 - (q(\sigma)/p(\sigma))}}, \quad 1 \leq k \leq N-1 \quad \text{and} \quad I_0 = \infty. \tag{6.14}$$

It was shown in [1] that there exists λ^* such that if $\lambda > \lambda^*$ then $J_\lambda^{N-1} < \frac{1}{4}$ and (P_{N-1}^λ) has a solution while for $\lambda \leq \lambda^*$, $J_\lambda^{N-1} = \frac{1}{4}$ and (P_{N-1}^λ) does not have a solution whenever $\lambda < \lambda^*$. The critical case $(P_{N-1}^{\lambda^*})$ was treated in [2], where the authors proved that $(P_{N-1}^{\lambda^*})$ admits a solution if and only if $I_{N-1} < \infty$. This clearly holds here for $(P_0^{\lambda^*})$ by Theorem 6.1. We believe that such type of results remain true for all k by taking in to account that in the flat case,

$$\inf_{u \in H_0^1(\mathbb{R}_+^N)} \frac{\displaystyle\int_{\mathbb{R}_+^N} |\nabla u|^2\, dx}{\displaystyle\int_{\mathbb{R}_+^N} \frac{u^2}{x_1^2 + \cdots + x_{N-k}^2}\, dx} = \frac{(N-k)^2}{4},$$

see [9], with $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x^1 > 0\}$.

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