Calc. Var. (2009) 36:437–451
DOI 10.1007/s00526-009-0236-3

Calculus of Variations

Some local eigenvalue estimates involving curvatures

Mouhamed Moustapha Fall

Received: 21 July 2008 / Accepted: 30 March 2009 / Published online: 15 April 2009

Abstract We first establish a local Faber–Krahn isoperimetric comparison in terms of scalar curvature pinching. Secondly we derive estimates of Cheeger constants related to the Dirichlet and Neumann problems via the (relative) isoperimetric profiles which allow us to obtain, in particular, lower bounds for first non-zero eigenvalues of the problem of Dirichlet and Neumann. These estimates involve scalar curvature and mean curvature respectively.

Mathematics Subject Classification (2000) 15A42 · 35B05 · 35R45 · 52A40

1 Introduction

Let $(\mathcal{M}^{n+1}, g)$ be a Riemannian manifold. For $\Omega \subset \mathcal{M}$ a bounded smooth domain (open connected) we consider the Dirichlet and Neumann eigenvalue problems:

$$\Delta_g f + \lambda f = 0 \quad \Omega, \quad f = 0 \quad \partial\Omega;$$
$$\Delta_g f + \mu f = 0 \quad \Omega, \quad \langle \nabla f, \eta \rangle_g = 0 \quad \partial\Omega,$$

with $\Delta_g f = \mathrm{div}_g(\nabla f)$ and η is the outer unit normal to $\partial\Omega$.

The set of eigenvalues, counted with multiplicities, in the above eigenvalue problems consists of

$$0 < \lambda_1(\Omega, g) < \lambda_2(\Omega, g) \leq \cdots + \infty;$$
$$0 = \mu_1(\Omega, g) < \mu_2(\Omega, g) \leq \cdots + \infty.$$

By the Max–Min Theorem (see for example [3, p. 17]), one knows that the best possible constants in the Poincaré and Poincaré–Wirtinger inequalities are given by $\lambda_1(\Omega, g)$ and $\mu_2(\Omega, g)$. The purpose of the present paper is to obtain local lower bounds on $\lambda_1(\Omega, g)$ and $\mu_2(\Omega, g)$ via curvature pinching. Before stating our results, some preliminaries are required.

M. M. Fall (✉)
Sissa, Via Beirut 2-4, 34014 Trieste, Italy
e-mail: fall@sissa.it; mouhamed.m.fall@gmail.com

Letting $0 < v < |\mathcal{M}|_g$, define the *Faber–Krahn (FK) isoperimetric profile* of $\mathcal{M}$ as

$$FK_{\mathcal{M}}(v, g) := \inf_{\Omega \subset \mathcal{M},\, |\Omega|_g = v} \lambda_1(\Omega, g).$$

Here Ω ranges over the smooth domains of $\mathcal{M}$. For example by the variational characterization for the lowest Dirichlet eigenvalue and the properties of symmetric decreasing rearrangements of functions, the Faber–Krahn isoperimetric profile of $\mathbb{R}^{n+1}$ is

$$FK_{\mathbb{R}^{n+1}}(v) = \left(\frac{|B|}{v}\right)^{\frac{2}{n+1}} \lambda_1(B),$$

where B is the unit ball of $\mathbb{R}^{n+1}$.

Explicit lower bounds for the profile $FK_{\mathcal{M}}$ are very important in applications and are called *physical isoperimetric inequalities* for instance see [3].

Let

$$I_{\mathcal{M}}(v, g) := \inf_{\Omega \subset \mathcal{M},\, |\Omega|_g = v} |\partial\Omega|_g,$$

be the isoperimetric profile of $\mathcal{M}$. Lower bounds for the profile $I_{\mathcal{M}}$ are called *geometric isoperimetric inequalities*. It is well know that the two above profiles are intimately linked, one can see for example [3, Sect. 2] for more details.

Recently Druet [6], has established the following (local) geometric isoperimetric inequality : let $x \in \mathcal{M}$ and scalar curvature satisfies $\mathbf{S}(x) < (n+1)n\,k$ for some $k \in \mathbb{R}$. There exists $r_x > 0$ such that for any $v \in (0\,,\, |B_g(x, r_x)|_g)$ and any ball E of volume v in $(\mathbb{M}^{n+1}, g_k)$, the space of constant sectional curvature k, there holds

$$I_{B_g(x,r_x)}(v, g) > |\partial E|_{g_k},$$

where $B_g(x, r)$ is the geodesic sphere centered at x with radius r. We prove here the counterpart for the Faber–Krahn profile.

Theorem 1.1 *Let $(\mathcal{M}^{n+1}, g)$ be a complete Riemannian manifold, $n \geq 2$ and let $x \in \mathcal{M}$. Assume that the scalar curvature at x satisfies $\mathbf{S}(x) < (n+1)n\,k$ for some $k \in \mathbb{R}$. Then there exists $r_x > 0$ such that for any $v \in (0\,,\, |B_g(x, r_x)|_g)$ and any ball E of volume v in $(\mathbb{M}^{n+1}, g_k)$, there holds*

$$FK_{B_g(x,r_x)}(v, g) > \lambda_1(E, g_k),$$

where $B_g(x, r)$ is the geodesic sphere centered at x with radius r in $\mathcal{M}$ and $(\mathbb{M}^{n+1}, g_k)$ is the space of constant sectional curvature k and dimension $n+1$.

Cheeger [4], provided lower bound of λ_1 and μ_2 by isoperimetric constants: the *Cheeger constants*. Therefore to obtain lower bound for these eigenvalues, one may rely on estimating the Cheeger constants. As we will see in Sect. 4, it turns out that the latter can be estimated via the (relative) isoperimetric profiles. And, using such argument, we have lower bounds for λ_1 in terms of volume.

Theorem 1.2 *Let $(\mathcal{M}^{n+1}, g)$ be a compact Riemannian manifold, $n \geq 2$. For any $\varepsilon > 0$, there exists $v_\varepsilon > 0$ such that for any $v \in (0, v_\varepsilon)$, there holds*

$$FK_{\mathcal{M}}(v, g) \geq \left(1 - (\beta_n\,\mathbf{S}_m + \varepsilon)\left(\frac{v}{|B|}\right)^{\frac{2}{n+1}}\right)^2 \frac{(n+1)^2}{4} \left(\frac{|B|}{v}\right)^{\frac{2}{n+1}},$$

where $\mathbf{S}_m$ is the maximum of the scalar curvature of $\mathcal{M}$ and $\beta_n = \frac{1}{(n+1)(n+3)}$.

Now concerning the Neumann problem, we let $\mathcal{M}_1$ be a smooth bounded domain of a Riemannian manifold $(\mathcal{M}^{n+1}, g)$, $n \geq 2$. We ask for a lower bound for the quantity

$$\inf_{\Omega \subset \mathcal{M}_1,\, |\Omega|_g = v} \mu_2(\Omega, g).$$

Taking into account the role of the mean curvature of $\partial\mathcal{M}_1$ in the expansion of the relative isoperimetric profile which is:

$$I^N_{\mathcal{M}_1}(v, g) := \inf_{\Omega \subset \mathcal{M}_1,\, |\Omega|_g = v} |\partial\Omega \cap \mathcal{M}_1|_g,$$

we can prove the following result.

Theorem 1.3 *Let $\mathcal{M}_1$ be a smooth bounded domain of a Riemannian manifold $(\mathcal{M}^{n+1}, g)$, $n \geq 2$. For any $\varepsilon > 0$, there exists $v_\varepsilon > 0$ such that for any $v \in (0, v_\varepsilon)$, there holds*

$$\inf_{\Omega \subset \mathcal{M}_1,\, |\Omega|_g = v} \mu_2(\Omega, g) \geq \left(1 - (\alpha_n \boldsymbol{H}_m + \varepsilon)\left(\frac{v}{|B_+|}\right)^{\frac{1}{n+1}}\right)^2 \frac{(n+1)^2}{4}\left(\frac{|B_+|}{v}\right)^{\frac{2}{n+1}},$$

where $\boldsymbol{H}_m$ is the maximum of the mean curvature of $\partial\mathcal{M}_1$, $\alpha_n = \frac{n}{(n+1)(n+2)}\frac{|B^n|}{|B_+|}$, B_+ is a half-ball of $\mathbb{R}^{n+1}$ and B^n is a unit ball of $\mathbb{R}^n$.

The proof of the first theorem relies on PDE techniques and fine asymptotic analysis of solutions of elliptic equations this is carried out in Sect. 3. The second and third theorem are consequences of estimates of Cheeger isoperimetric profiles. To achieve this, we use the fine expansions of the isoperimetric profile near zero by Druet [6] and Narduli [12] in compact Riemannian manifolds and the relative isoperimetric profile near zero by the author in [7], see Sect. 4. The relevance of the scalar curvature when studying the sharpness of the FK isoperimetric inequality was first noticed by [13] where the authors prove the existence of critical domains closed to geodesic balls centered near non-degenerate critical points of the scalar curvature.

Let us mention that when this manuscript was already typed, we have learned that Druet [5], by using the isoperimetric inequality in [6] and symmetrization arguments, gave the asymptotic expansions of the FK isoperimetric prole for compact Riemannian as v tends to zero.

2 Preliminaries and notations

We start by recalling the well known *FK isoperimetric inequality* which is an immediate consequence of the spherical symmetric rearrangement argument, see for instance Pölya-Szegö [15].

Theorem 2.1 *for any $f \in H^{1,2}(\mathbb{R}^{n+1})$ with $|\{f > 0\}| < \infty$, there holds*

$$|\{f > 0\}|^{\frac{2}{n+1}} \int_{\mathbb{R}^{n+1}} |\nabla f|^2\, dx \geq |B|^{\frac{2}{n+1}} \lambda_1(B) \int_{\mathbb{R}^{n+1}} f^2\, dx. \tag{1}$$

In particular for any open subset Ω of $\mathbb{R}^{n+1}$ with finite measure,

$$\lambda_1(\Omega) \geq FK_{\mathbb{R}^{n+1}}(|\Omega|), \tag{2}$$

where

$$FK_{\mathbb{R}^{n+1}}(v) = \left(\frac{|B|}{v}\right)^{\frac{2}{n+1}} \lambda_1(B).$$

Equality holds in (2)*, for some* Ω*, if and only if* Ω *is a ball.*

We will denote by $\phi_1(|x|) > 0$ the first L^2-normalized radial positive eigenfunction corresponding to $\lambda_1(B)$ with $\phi_1(1) = 0$ where B is the unit ball of $\mathbb{R}^{n+1}$ centered at the origin.

Let $y_0 \in \mathcal{M}$ and $E_1, \ldots, E_{n+1}$ an orthonormal basis of $T_{y_0}\mathcal{M}$. We consider the geodesic coordinate system centered at y_0 defined by

$$f^{y_0}(y) := \mathrm{Exp}_{y_0}\left(y^i E_i\right) \quad \text{for } y = \left(y^1, \ldots, y^{n+1}\right) \in \mathbb{R}^{n+1}.$$

We define geodesic balls of $\mathcal{M}$ centered at y_0 with radius $r > 0$ as $B_g(y_0, r) = f^{y_0}(rB)$. If $q = f^{y_0}(y) \in \mathcal{M}$ near the point $y_0 = f(0) \in \mathcal{M}$, one can expand the metric $g_{ij}(q) = \langle Y_i, Y_j \rangle$ in y.

The proof of the following results can be found in [14].

Lemma 2.2 *In the above notations, for any* $i, j = 1, \ldots, n+1$*, we have*

$$g_{ij}(q) = \delta_{ij} + \frac{1}{3}\langle R_{y_0}(Y, E_i)Y, E_j\rangle_g + O_{y_0}(|y|^3);$$
$$dv_g = \left(1 - \frac{1}{6} Ric_{y_0}(Y, Y) + O_{y_0}(|y|^3)\right) dy,$$

where $Y := y^i E_i$ *and* dv_g *(resp.* dy*) is the volume element of* $\mathcal{M}$ *(resp.* $\mathbb{R}^{n+1}$*).*

There holds

$$\left|B_g(y_0, r)\right|_g = r^{n+1}|B|\left(1 - \frac{1}{6(n+3)} r^2 \boldsymbol{S}(y_0) + O_{y_0}(r^3)\right);$$

where $\boldsymbol{S}$ *is the scalar curvature of* $\mathcal{M}$.

If $B_g(y_0, r)$ is a geodesic ball centered at some point $y_0 \in \mathcal{M}$ with radius r, it is shown in Chavel [3, Sect. 8] that the first Dirichlet eigenvalue for geodesic ball has the following expansion in terms of its radius:

$$\lambda_1(B_g(y_0, r), g) = \frac{\lambda_1(B)}{r^2} - \frac{\mathbf{S}(y_0)}{6} + O_{y_0}(r).$$

Using the expansion of geodesic balls (see the above lemma), one has the expansion in terms of volume

$$\lambda_1(\Omega_0, g) = \left(1 - \frac{2\lambda_1(B) + (n+3)(n+1)}{6(n+3)(n+1)\lambda_1(B)}\left(\frac{|\Omega_0|_g}{|B|}\right)^{\frac{2}{n+1}} \mathbf{S}(y_0) + O\left(|\Omega_0|_g^{\frac{3}{n+1}}\right)\right) \times FK_{\mathbb{R}^{n+1}}(|\Omega_0|_g),$$

for any small geodesic ball Ω_0 of $\mathcal{M}$ centered at y_0.

If then E is a ball in $(\mathbb{M}^{n+1}, g_k)$, one has that

$$\lambda_1(E, g_k) = \left(1 - \gamma_n\left(\frac{|E|_{g_k}}{|B|}\right)^{\frac{2}{n+1}} n(n+1)k + O\left(|E|_{g_k}^{\frac{3}{n+1}}\right)\right) FK_{\mathbb{R}^{n+1}}(|E|_{g_k}),$$

where $\gamma_n = \frac{2\lambda_1(B)+(n+3)(n+1)}{6(n+3)(n+1)\lambda_1(B)}$. Therefore to prove Theorem 1.1 it is enough to show that for every $\varepsilon > 0$ there exits $r_\varepsilon > 0$ such that

$$FK_{B_g(x_0,r_\varepsilon)}(v,g) \geq \left(1 - (\gamma_n\, \mathbf{S}(x_0) + \varepsilon) \left(\frac{v}{|B|}\right)^{\frac{2}{n+1}}\right) FK_{\mathbb{R}^{n+1}}(v),$$

for every $0 < v < \left|B_g(x_0, r_\varepsilon)\right|_g$.

3 Proof of Theorem 1.1

Proposition 3.1 *Let $x_0 \in \mathcal{M}$. For any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that*

$$FK_{B_g(x_0,r_\varepsilon)}(v,g) \geq \left(1 - (\gamma_n\, \mathbf{S}(x_0) + \varepsilon) \left(\frac{v}{|B|}\right)^{\frac{2}{n+1}}\right) FK_{\mathbb{R}^{n+1}}(v),$$

for all $0 < v < \left|B_g(x_0, r)\right|_g$ with $\gamma_n = \frac{2\lambda_1(B)+(n+3)(n+1)}{6(n+3)(n+1)\lambda_1(B)}$ and $\mathbf{S}$ is the scalar curvature of $\mathcal{M}$.

Proof Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for every $r > 0$ there exits $v_r \in (0\,,\ |B_g(x_0, r)|_g)$ such that

$$FK_{B_g(x_0,r)}(v_r, g) < \left(1 - (\gamma_n\, \mathbf{S}(x_0) + \varepsilon_0) \left(\frac{v_r}{|B|}\right)^{\frac{2}{n+1}}\right) FK_{\mathbb{R}^{n+1}}(v_r).$$

We define $\rho_r \to 0$ as $r \to 0$ by $v_r = |\rho_r B|$. (To simplify notations, we will set $r = r_j$, where r_j is a sequence decreasing to 0 when $j \to +\infty$.) Also there exist $\Omega_r \subset B_g(x_0, r)$ smooth domains with $|\Omega_r|_g = |\rho_r B|$ such that

$$\lambda_1(\Omega_r, g) < \left(1 - (\gamma_n\, \mathbf{S}(x_0) + \varepsilon_0)\ \rho_r^2\right)\ FK_{\mathbb{R}^{n+1}}(|\rho_r B|).$$

We let $u_r \in \mathcal{C}^{1,\alpha}(\overline{\Omega_r})$ be the positive eigenfunction corresponding to $\lambda_1(\Omega_r, g)$, which is normalized by

$$\int_{\Omega_r} u_r^2\, dv_g = 1, \quad \int_{\Omega_r} |\nabla u_r|_g^2\, dv_g = \lambda_1(\Omega_r, g).$$

Letting $\Omega_r' = \mathrm{Exp}_{x_0}^{-1}(\Omega_r) \subset \mathbb{R}^{n+1}$ and $\bar{g}(y) = \mathrm{Exp}_{x_0}^{*} g(y)$, one has that

$$|\Omega_r|_g = |\Omega_r'|_{\bar{g}}, \quad \lambda_1(\Omega_r, g) = \lambda_1(\Omega_r', \bar{g}),$$

so that using Lemma 2.2,

$$\begin{aligned}
\lambda_1(\Omega_r') &\leq \frac{\int_{\Omega_r'} |\nabla u_r \circ \mathrm{Exp}_{x_0}|^2\, dy}{\int_{\Omega_r'} |u_r \circ \mathrm{Exp}_{x_0}|^2\, dy} \\
&\leq \left(1 + c\, r^2\right) \frac{\int_{\Omega_r'} |\nabla u_r \circ \mathrm{Exp}_{x_0}|_{\bar{g}}^2\, dv_{\bar{g}}}{\int_{\Omega_r'} |u_r \circ \mathrm{Exp}_{x_0}|^2\, dv_{\bar{g}}} \\
&\leq \left(1 + c\, r^2\right) \frac{\int_{\Omega_r} |\nabla u_r|_g^2\, dv_g}{\int_{\Omega_r} |u_r|^2\, dv_g} \\
&\leq \left(1 + c\, r^2\right)\ \lambda_1(\Omega_r, g).
\end{aligned}$$

Therefore by the assumption together with Lemma 2.2, we get

$$\lambda_1(\Omega_r') \le \left(1 + c\, r^2\right)\; FK_{\mathbb{R}^{n+1}}(|\rho_r B|), \quad |\Omega_r'| = \left(1 + O(r^2)\right) |\Omega_r|_g. \tag{3}$$

We define $U_r := \frac{1}{\rho_r}\, \Omega_r' \subset \frac{r}{\rho_r}\, B$. For any $x \in \frac{1}{\rho_r}\, B$, we set

$$g_r(x) = \bar{g}(\rho_r x).$$

Using Lemma 2.2, we have

$$\begin{aligned} (g_r)_{ij}(x) &= \delta_{ij} - \frac{\rho_r^2}{3} \langle R_{x_0}(X, E_i)X, E_j\rangle + O_{x_0}(\rho_r^3);\\ dv_{g_r} &= \left(1 + \frac{\rho_r^2}{6}\, Ric_{x_0}(X, X) + O_{x_0}(\rho_r^3)\right) dx, \end{aligned} \tag{4}$$

Clearly

$$g_r \to euc \quad \text{on } K \text{ compact subset of } \mathbb{R}^{n+1}, \tag{5}$$

where euc denote the Euclidean metric on $\mathbb{R}^{n+1}$.

We also let

$$w_r(x) = \rho_r^{\frac{n+1}{2}}\, u_r \circ \mathrm{Exp}_{x_0}(\rho_r x) \quad \text{for } x \in U_r, \quad w_r(x) = 0 \quad \text{for } x \in \mathbb{R}^{n+1} \setminus U_r.$$

One easily sees that w_r satisfies

$$\Delta_{g_r} w_r + \lambda_1(U_r, g_r)\, w_r = 0 \quad \mathbb{R}^{n+1}; \quad \{w_r > 0\} = U_r \tag{6}$$

and

$$\int_{\mathbb{R}^{n+1}} |w_r|^2\, dv_{g_r} = 1. \tag{7}$$

By (22), and recalling that $FK_{\mathbb{R}^{n+1}}(|B|) = \lambda_1(B)$,

$$\lambda_1(U_r, g_r) = \int_{\mathbb{R}^{n+1}} |\nabla w_r|_{g_r}^2\, dv_{g_r} < \left(1 - (\gamma_n\, \mathbf{S}(x_0) + \varepsilon_0)\; \rho_r^2\right) \lambda_1(B). \tag{8}$$

Clearly since the functional $E \mapsto |E|_g^{\frac{2}{n+1}} \lambda_1(E, g)$ is scale invariant, we have

$$|\Omega_r|_g^{\frac{2}{n+1}} \lambda_1(\Omega_r, g) = |U_r|_{g_r}^{\frac{2}{n+1}} \lambda_1(U_r, g_r).$$

Dividing the second equation in (3) by ρ_r^{n+1} yields

$$|U_r| \to |B| \quad \text{when } r \to 0. \tag{9}$$

Multiplying the first inequality (3) by ρ_r^2, gives

$$\lambda_1(U_r) \le \left(1 + c\, r^2\right)\, \lambda_1(B).$$

We have thus by stability result in [8] that, if r is small, there exists $x_r \in \mathbb{R}^{n+1}$ such that

$$|(U_r + x_r) \,\triangle\, B| \to 0. \tag{10}$$

Up to changing w_r by $w_r(\cdot - x_r)$ we will assume that $x_r = 0$.

Now by [11, Lemma 4.1], (5), (8) and (10), we may assume that w_r is uniformly bounded on compact sets of $\mathbb{R}^{n+1}$. Therefore for any compact sets $K \in \mathbb{R}^{n+1}$, $\|\Delta_{g_r} w_r\|_{L^\infty(K)} \le C_K$

so by [10], $\|\nabla w_r\|_{\mathcal{C}^{1,\alpha}(K)}$ is uniformly bounded. By Ascoli's theorem, this leads to the existence of some $\psi \in \mathcal{C}^0(\mathbb{R}^{n+1})$ such that after passing to a sub-sequence, and for any compact subset K of $\mathbb{R}^{n+1}$,

$$\lim_{r\to 0} w_r = \psi. \tag{11}$$

We first make the following observation that $\psi \equiv 0$ on $\mathbb{R}^{n+1}\backslash\overline{B}$. In fact if there is $y_0 \in \mathbb{R}^{n+1}\backslash\overline{B}$ such that $\psi(y_0) > 0$ then by continuity, we have $\psi > 0$ on $B(y_0,\eta) \subset \mathbb{R}^{n+1}\backslash\overline{B}$ for some $\eta > 0$. But then for every r less than some $r(y_0)$, one has $w_r > 0$ on $B(y_0,\eta)$ because of the uniform convergence on compact sets. Namely $B(y_0,\eta) \subset U_r\backslash B$ because $U_r = \{w_r > 0\}$ and then

$$0 < C \le |B(y_0,\eta)| \le |U_r\backslash B| \le |U_r \triangle B| \to 0$$

which is not possible.

We conclude that by continuity $\{\psi > 0\}$ is an open subset of B.

Claim: $\psi = \phi_1$.

Observe that by (3),

$$\int_{U_r} |\nabla w_r|^2\,dx = \rho_r^2 \int_{\Omega'} |\nabla u_r \circ \mathrm{Exp}_{x_0}|^2\,dy$$
$$\le \rho_r^2\left(1+c\,r^2\right)\int_{\Omega_r} |\nabla u_r|_{\bar g}^2\,dv_{\bar g} \le \left(1+c\,r^2\right)\lambda_1(B), \tag{12}$$

similarly one can verify easily, by Lemma 2.2, that

$$\int_{U_r} |w_r|^2\,dx = \int_{\Omega'} |u_r \circ \mathrm{Exp}_{x_0}|^2\,dy = \left(1+c\,r^2\right)\int_{\Omega_r'} |u_r|^2\,dv_{\bar g} = \left(1+c\,r^2\right). \tag{13}$$

By (12), (w_r) is bounded in $H^{1,2}(\mathbb{R}^{n+1})$ thus we get the existence of some $w \in H^{1,2}(\mathbb{R}^{n+1})$ such that

$$w_r \rightharpoonup w \quad \text{in} \quad H^{1,2}(\mathbb{R}^{n+1}). \tag{14}$$

In particular using (12), we conclude that

$$\int_{\mathbb{R}^{n+1}} |\nabla w|^2\,dx \le \limsup_{r\to 0} \int_{\mathbb{R}^{n+1}} |\nabla w_r|^2\,dx \le \lambda_1(B). \tag{15}$$

We show that w_r converges strongly in $L^2(\mathbb{R}^{n+1})$. To this end, thanks to the uniform bound on the L^2 norm of the Dirichlet energy in (12), it will suffice to show that no L^2-mass is concentrated by w_r at infinity, i.e. that for every $\delta > 0$ there exists $R > 0$ such that

$$\sup_r \|w_r\|_{L^2(\mathbb{R}^{n+1}\backslash RB)} < \delta.$$

By Hölder inequality (recall that $\{w_r > 0\} = U_r$),

$$\int_{\mathbb{R}^{n+1}\backslash RB} |w_r|^2\,dx \le \left(\int_{\mathbb{R}^{n+1}\backslash RB} |w_r|^{2^*}\,dx\right)^{\frac{2}{2^*}} |U_r\backslash RB|^{\frac{2^*-2}{2^*}},$$

where $2^* = \frac{2(n+1)}{n-1}$. Since by Sobolev embeddings

$$\int_{\mathbb{R}^{n+1}} |w_r|^{2^*}\, dx \leq c(n) \int_{\mathbb{R}^{n+1}} |\nabla w_r|^2\, dx,$$

we get using (9), (10) and (12)

$$\limsup_{r\to 0} \int_{\mathbb{R}^{n+1}\setminus RB} |w_r|^2\, dx \leq\ c(n)\ |B\setminus RB|^{\frac{2^*-2}{2^*}}$$

as desired.

Now up to a sub-sequence w_r converges to w pointwise therefore it must be $w = \psi$ by (11).

Clearly using (10), (13) and the strong convergence,

$$\int_{\mathbb{R}^{n+1}} |w_r|^2\, 1_{U_r}\, dx \to \int_B |w|^2\, dx = 1. \tag{16}$$

Recall that $\{w > 0\}$ is an open subset of B thus $|\{w > 0\}| \leq |B|$. If we now use (1) and (15), we conclude that $|\{w > 0\}| \geq |B|$. Therefore

$$|\{w > 0\}| = |B|.$$

The minimizing property of first eigenvalue imply, using (15) and (16), that

$$\lambda_1(\{w > 0\}) \leq \frac{\int_{\mathbb{R}^{n+1}} |\nabla w|^2\, dx}{\int_{\mathbb{R}^{n+1}} w^2\, dx} \leq \lambda_1(B).$$

Therefore equality holds in the FK isoperimetric inequality (2) thus $\{w > 0\} = B + z_0$ for some $z_0 \in \mathbb{R}^{n+1}$ and with the normalization (16), $w = \phi_1(|\cdot - z_0|)$. Up to translation, we will also assume that $z_0 = 0$.

Pointwise estimates of w_r on $U_r\setminus B$.

Letting q_r a sequence with $q_r \nearrow 2$, consider the function

$$f_r(x) := |x|^{-\frac{1}{2-q_r}}.$$

A calculation yields, for small r,

$$|x|^{\frac{1}{2-q_r}}\left(-\Delta_{g_r} f_r - \lambda_1(U_r, g_r)\, f_r\right) \geq |x|^{-2}\left[\left(\frac{1}{2-q_r}\right)^2 - cr^2\right] - \lambda_1(U_r, g_r),$$

in $U_r\setminus B$, for some constant c independent of r. Thanks to (7) and the fact that $q_r \nearrow 2$ as $r \to 0$, one gets that

$$-\Delta_{g_r} f_r - \lambda_1(U_r, g_r)\, f_r \geq 0 \quad \text{in } U_r\setminus B$$

while $f_r = 1$ on ∂B. Now since w_r converges uniformly to zero on ∂B and satisfies (6), by the maximum principle (see for instance [1, Lemma 3.4]), we get that if r is small enough,

$$w_r(x) \leq |x|^{-\frac{1}{2-q_r}} \quad \text{in } U_r\setminus B. \tag{17}$$

We shall finish the proof of the proposition. From (4) one has

$$|\nabla w_r|^2_{g_r} = |\nabla w_r|^2\left\{1 - \frac{2\rho_r^2}{6}\, |\nabla w_r|^{-2}\, \langle R_{x_0}(X, \nabla w_r)X, \nabla w_r\rangle_g + \rho_r^3\, O(|x|^3)\right\}$$

again by (4) we get

$$\lambda_1(U_r, g_r) = \int\limits_{U_r} |\nabla w_r|^2_{g_r}\, dv_{g_r}$$

$$= \int\limits_{U_r} |\nabla w_r|^2\, dx - \frac{\rho_r^2}{6} \int\limits_{U_r} Ric_{x_0}(X, X)|\nabla w_r|^2\, dx$$

$$- \frac{2\rho_r^2}{6} \int\limits_{U_r} \langle R_{x_0}(X, \nabla w_r)X, \nabla w_r\rangle_g\, dx + O\left(\rho_r^3 \int\limits_{U_r} |x|^3 |\nabla w_r|^2\, dx\right).$$

Since by Ascoli's theorem, ∇w_r converges to $\nabla\phi_1$ pointwise, by (10),

$$\int\limits_{U_r\cap B} Ric_{x_0}(X, X)\, |\nabla w_r|^2\, dx \longrightarrow \int\limits_{B} Ric_{x_0}(X, X)\, |\nabla \phi_1|^2\, dx.$$

We estimate $\int_{U_r\setminus B} Ric_{x_0}(X, X)\, |\nabla w_r|^2\, dx$ (this argument will be used very often in the sequel). First note that by Lemma 2.2 as in (12), we have the estimate

$$\int\limits_{U_r\setminus B} |\nabla w_r|^2 \left|Ric_{x_0}(X, X)\right|\, dx \le C \int\limits_{U_r\setminus B} |\nabla w_r|^2_{g_r} |x|^2\, dv_{g_r}.$$

Call $F(x) = |x|^2$ and multiply the equality (6) by $F w_r$ and integrate by parts to get

$$\int\limits_{U_r\setminus B} |\nabla w_r|^2_{g_r} F\, dv_{g_r} = - \int\limits_{U_r\setminus B} w_r \langle \nabla w_r, \nabla F\rangle_{g_r}\, dv_{g_r} + \lambda_1(U_r, g_r) \int\limits_{U_r\setminus B} |w_r|^2 F\, dv_{g_r}$$

$$+ \oint\limits_{\partial B} w_r F \langle \nabla w_r, X\rangle_{g_r}\, d\sigma.$$

Here $d\sigma$ is the surface measure on ∂B and X, the position vector, is the outer unit normal to ∂B. By Hölder inequality,

$$\left|\int\limits_{U_r\setminus B} |\nabla w_r|^2_{g_r} F\, dv_{g_r}\right| \le \left(\int\limits_{U_r\setminus B} |\nabla w_r|^2_{g_r}\, dv_{g_r}\right)^{\frac{1}{2}} \left(\int\limits_{U_r\setminus B} |w_r|^2 |\nabla F|^2_{g_r}\, dv_{g_r}\right)^{\frac{1}{4}}$$

$$+ \lambda_1(U_r, g_r) \int\limits_{U_r\setminus B} |w_r|^2 F\, dv_{g_r} + C \oint\limits_{\partial B} |\nabla w_r|_{g_r} w_r F\, d\sigma.$$

$$\le (\lambda_1(U_r, g_r))^{\frac{1}{2}} \left(\int\limits_{U_r\setminus B} |w_r|^2 |\nabla F|^2_{g_r}\, dv_{g_r}\right)^{\frac{1}{2}}$$

$$+ \lambda_1(U_r, g_r) \int\limits_{U_r\setminus B} |w_r|^2 F\, dv_{g_r} + C \oint\limits_{\partial B} |\nabla w_r|_{g_r} w_r F\, d\sigma.$$

By uniform convergence of w_r to zero with uniform bound of ∇w_r on compact sets and (4), we have

$$\oint_{\partial B} |\nabla w_r|_{g_r} w_r F \, d\sigma \to 0.$$

Using (8) and (17), we conclude that

$$\lim_{r\to 0} \int_{U_r \setminus B} |\nabla w_r|^2_{g_r} F \, dv_{g_r} = 0.$$

In the same way as above, there holds

$$\int_{U_r} |x|^3 |\nabla w_r|^2 \, dx \to O(1).$$

Clearly we have

$$|\langle R_{x_0}(X, \nabla w_r)X, \nabla w_r\rangle_g| \leq C\,|x|^2\,|\nabla w_r|^2,$$

and also since ϕ_1 is radial, $\nabla\phi_1 = \frac{X}{|X|}\phi_1'$ namely $\langle R_{x_0}(X, \nabla\phi_1)X, \nabla\phi_1\rangle = 0$ and since $w_r(x) \to \phi_1(x)$ in B, we infer that as above

$$\int_{U_r\cap B} \langle R_{x_0}(X, \nabla w_r)X, \nabla w_r\rangle_g \, dx \longrightarrow 0, \quad \int_{U_r\setminus B} \langle R_{x_0}(X, \nabla w_r)X, \nabla w_r\rangle_g \, dx \longrightarrow 0.$$

Finally we obtain that

$$\lambda_1(U_r, g_r) = \int_{U_r} |\nabla w_r|^2 \, dx - \frac{\rho_r^2}{6} \int_B Ric_{x_0}(X, X)\,|\nabla\phi_1|^2 \, dx + O(r)\rho_r^2. \tag{18}$$

From the FK isoperimetric inequality (1) one has

$$\int_{U_r} |\nabla w_r|^2 \, dx \geq \left(\frac{|B|}{|U_r|}\right)^{\frac{2}{n+1}} \lambda_1(B) \int_{U_r} |w_r|^2 \, dx.$$

We use (4) (recall that $\int_{U_r} w_r^2 \, dv_{g_r} = 1$) and (17) to have

$$\begin{aligned}\int_{U_r} |w_r|^2 \, dx &= 1 + \frac{r^2}{6} \int_{U_r} Ric_{x_0}(X, X)\,|w_r|^2 \, dx + O\left(\rho_r^3 \int_{U_r} |x|^3 |w_r|^2 \, dx\right)\\ &= 1 + \frac{r^2}{6} \int_B Ric_{x_0}(X, X)\,\phi_1^2 \, dx + O(r)\rho_r^2.\end{aligned}$$

By (10), there holds

$$\frac{\left|\Omega_r' \triangle \rho_r B\right|}{\left|\Omega_r'\right|} \to 0$$

and by Lemma 2.2 we get the following expansion

$$|\Omega'_r| = |\rho_r B| \left(1 + \frac{1}{6} \int\limits_{\mathbb{R}^{n+1}} Ric_{x_0}(Y,Y) \frac{1}{|\Omega'_r|} 1_{\rho_r B}\, dy + O(r)\rho_r^2 \right).$$

Moreover since $\rho_r^{n+1}|U_r| = |\Omega'_r|$ and

$$\int\limits_{\rho_r B} Ric_{x_0}(Y,Y)\, dy = \frac{\rho_r^2 |\rho_r B|}{n+3} \mathbf{S}(x_0),$$

we get, using also the second equation in (3),

$$|U_r| \le |B| + \frac{\rho_r^2 |B|}{6(n+3)} \mathbf{S}(x_0) + cr^3,$$

for some constant $c > 0$ independent on r. Consequently we have

$$\int\limits_{U_r} |\nabla w_r|^2\, dx \ge \left(1 - \frac{2\rho_r^2}{6(n+3)(n+1)} \mathbf{S}(x_0)\right) \left(1 + \frac{\rho_r^2}{6} \int\limits_B Ric_{x_0}(X,X)\, \phi_1^2\, dx\right) \lambda_1(B) - cr^3.$$

Putting this in (18), we therefore obtain that

$$\begin{aligned} \lambda_1(U_r, g_r) \ge& \left(1 - \frac{2\rho_r^2}{6(n+3)(n+1)} \mathbf{S}(x_0)\right) \lambda_1(B) \\ &- \frac{\rho_r^2}{6} \int\limits_B Ric_{x_0}(X,X)\, \left(|\nabla \phi_1|^2 - \lambda_1(B)\phi_1^2\right)\, dx - cr^3. \end{aligned}$$

Using the fact that ϕ_1 is radial, one has

$$\int\limits_B Ric_{x_0}(X,X)\, \left(|\nabla \phi_1|^2 - \lambda_1(B)\phi_1^2\right)\, dx = |B|\mathbf{S}(x_0) \int\limits_0^1 t^{n+2} \left((\phi'_1)^2 - \lambda_1(B)\phi_1^2 \right)\, dt.$$

Recall that ϕ_1 solves

$$\phi''_1 + \frac{n}{t} \phi'_1 + \lambda_1(B)\phi_1 = 0, \quad \phi_1(1) = 0. \tag{19}$$

Multiply the first equation above by $t^{n+2}\phi_1$ and integrate by parts to have

$$\begin{aligned} \int\limits_0^1 t^{n+2} \left((\phi'_1)^2 - \lambda_1(B)\phi_1^2 \right)\, dt &= -2 \int\limits_0^1 t^{n+1} \phi'_1 \phi_1\, dt \\ &= (n+1) \int\limits_0^1 t^n \phi_1^2\, dt \\ &= \frac{1}{|B|} \int\limits_B \phi_1^2\, dx \\ &= \frac{1}{|B|}. \end{aligned}$$

Collecting these and using (8) we arrive to

$$\left(1-\beta_n\,\rho_r^2\,\mathbf{S}(x_0)-cr^3\right)\right)\,\lambda_1(B)\leq\lambda_1(U_r,g_r)<\left(1-(\beta_n\,\mathbf{S}(x_0)+\varepsilon_0)\,\rho_r^2\right)\,\lambda_1(B),$$

namely $0<\varepsilon_0<cr\to 0$, as $r\to 0$, which is the contradiction. □

4 Proof of Theorems 1.2 and 1.3

In order to prove the theorems, we need to recall the *Cheeger constants*, $h_D(\Omega,g)$ and $h_N(\Omega,g)$ relative to the Dirichlet and Neumann eigenvalue problems respectively,

$$h_D(\Omega,g)=\inf_{\omega\subset\subset\Omega}\frac{|\partial\omega|_g}{|\omega|_g};\quad h_N(\Omega,g)=\inf_{\omega\subset\Omega}\frac{|\partial\omega\cap\Omega|_g}{\min\{|\omega|_g,\,|\Omega\backslash\omega|_g\}}.$$

Here the range of ω is the set of smooth sub-domains of Ω. For example in $\mathbb{R}^{n+1}$, for any $r>0$, the Cheeger constant of rB is given by

$$h_D(rB)=\frac{n+1}{r}.$$

Referring to [2] (see also [9]) for a proof, we have the following Cheeger's inequalities:

$$\lambda_1(\Omega,g)\geq\left(\frac{h_D(\Omega,g)}{2}\right)^2,\quad \mu_2(\Omega,g)\geq\left(\frac{h_N(\Omega,g)}{2}\right)^2.$$

Similarly we define the Cheeger isoperimetric profiles of $\mathcal{M}$ as

$$\mathcal{H}^*_{\mathcal{M}}(v,g):=\inf_{\Omega\subset\mathcal{M},\,|\Omega|_g=v}h_*(\Omega,g),$$

with $*=D,N$.

As mentioned earlier Theorem 1.2 is a direct consequence of the expansion of the isoperimetric profile near zero for compact Riemannian manifolds that we recall here, see [6,12]. Letting

$$I_{\mathcal{M}}(v,g)=\inf_{E\subset\mathcal{M},\,|E|_g=v}|\partial E|_g,$$

for any $\delta>0$ there exists $v_\delta>0$ such that for any $v\in(0,v_\delta)$

$$\left(1-(\beta_n\,\mathbf{S}_m+\delta)\left(\frac{v}{|B|}\right)^{\frac{2}{n+1}}\right)I_{\mathbb{R}^{n+1}}(v)\leq I_{\mathcal{M}}(v,g)$$
$$\leq\left(1-(\beta_n\,\mathbf{S}_m-\delta)\left(\frac{v}{|B|}\right)^{\frac{2}{n+1}}\right)I_{\mathbb{R}^{n+1}}(v),\tag{20}$$

where $I_{\mathbb{R}^{n+1}}(v)=(n+1)|B|^{\frac{1}{n+1}}v^{\frac{n}{n+1}}$ and $\mathbf{S}_m$ is the maximum of the scalar curvature of $\mathcal{M}$ while $\beta_n=\frac{1}{2(n+3)(n+1)}$.

Notice that if r is sufficiently small, the Cheeger constant of geodesic balls can be estimated as

$$h_D(B_g(x_0,r),g)=\inf_{\omega\subset\subset B_g(x_0,r)}\frac{|\partial\omega|_g}{|\omega|_g}\leq\lim_{\rho\to 1^-}\frac{\left|\partial B_g(x_0,\rho r)\right|_g}{\left|B_g(x_0,\rho r)\right|_g}.$$

We choose x_0 to be the maximum point of the scalar curvature,

$$\mathbf{S}_m = \mathbf{S}(x_0),$$

then by simple computations using Lemma 2.2, we have

$$h_D(B_g(x_0,r),g) \leq \left(1 - \beta_n \left(\frac{|B_g(x_0,r)|_g}{|B|}\right)^{\frac{2}{n+1}} \mathbf{S}_m + O\left(|B_g(x_0,r)|_g^{\frac{3}{n+1}}\right)\right) \mathcal{H}^D_{\mathbb{R}^{n+1}}(|B_g(x_0,r)|_g), \tag{21}$$

where the Cheeger isoperimetric profile of $\mathbb{R}^{n+1}$ is given by

$$\mathcal{H}^D_{\mathbb{R}^{n+1}}(v) = (n+1)\left(\frac{|B|}{v}\right)^{\frac{1}{n+1}}.$$

Clearly by Cheeger's inequality (Dirichlet), Theorem 1.2 is a consequence of the following

Theorem 4.1 *Let $(\mathcal{M}^{n+1}, g)$ be a compact Riemannian manifold, $n \geq 2$. Near $v = 0$ ($v > 0$), there holds*

$$\mathcal{H}^D_{\mathcal{M}}(v,g) = \left(1 + O(v^{\frac{3}{n+1}})\right) \frac{I_{\mathcal{M}}(v,g)}{I_{\mathbb{R}^{n+1}}(v)} \mathcal{H}^D_{\mathbb{R}^{n+1}}(v).$$

Observe that according to (21), the proof of the above theorem is contained in the

Proposition 4.2 *For any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $r \in (0, r_\varepsilon)$ there holds*

$$\mathcal{H}^D_{\mathcal{M}}(|rB|,g) \geq \left(1 - (\beta_n \mathbf{S}_m + \varepsilon)\, r^2\right) \mathcal{H}^D_{\mathbb{R}^{n+1}}(|rB|),$$

where $\beta_n = \frac{1}{2(n+3)(n+1)}$ and $\mathbf{S}_m$ is the maximum of the scalar curvature of $\mathcal{M}$.

Proof We proceed here also by contradiction by supposing that there exists $\varepsilon_0 > 0$ and a sequence $r \searrow 0$ such that

$$\mathcal{H}^D_{\mathcal{M}}(|rB|,g) < \left(1 - (\beta_n \mathbf{S}_m + \varepsilon_0)\, r^2\right) \mathcal{H}^D_{\mathbb{R}^{n+1}}(|rB|),$$

so there exist $\Omega_r \subset \mathcal{M}$ domains with $|\Omega_r|_g = |rB|$ such that

$$h_D(\Omega_r, g) < \left(1 - (\beta_n \mathbf{S}_m + \varepsilon_0)\, r^2\right) \mathcal{H}^D_{\mathbb{R}^{n+1}}(|rB|),$$

this imply the existence of domains $\omega_r \subset\subset \Omega_r$ such that

$$\frac{|\partial\omega_r|_g}{|\omega_r|_g} < \left(1 - (\beta_n \mathbf{S}_m + \varepsilon_0)\, r^2\right) \mathcal{H}^D_{\mathbb{R}^{n+1}}(|rB|). \tag{22}$$

Letting $\delta > 0$, by (20), there exists $r_\delta > 0$ such that

$$\frac{|\partial\omega_r|_g}{|\omega_r|_g} \geq \left(1 - (\beta_n \mathbf{S}_m + \delta)\left(\frac{|\omega_r|_g}{|B|}\right)^{\frac{2}{n+1}}\right) \mathcal{H}^D_{\mathbb{R}^{n+1}}(|\omega_r|_g), \tag{23}$$

for any $r \in (0, r_\delta)$. Recalling that

$$\mathcal{H}^D_{\mathbb{R}^{n+1}}(|\omega_r|_g) = (n+1)\left(\frac{|B|}{|\omega_r|_g}\right)^{\frac{1}{n+1}}.$$

Therefore (23) with (22) imply, if r is small, that

$$\frac{|\omega_r|_g}{|rB|} \geq (1 - cr)\,, \tag{24}$$

for some constant $c > 0$.

Since $|\omega_r|_g \leq |\Omega_r|_g = |rB|$, we get from (24) that, as $r \to 0$,

$$\frac{|\omega_r|_g}{|rB|} \to 1. \tag{25}$$

Now note that

$$\mathcal{H}^D_{\mathbb{R}^{n+1}}(|\omega_r|_g) \geq \mathcal{H}^D_{\mathbb{R}^{n+1}}(|\Omega_r|_g).$$

Hence by the assumption (22) and using (23) we get

$$-\,(\beta_n\,\mathbf{S}_m + \varepsilon_0)\; r^2 \geq -\,(\beta_n\,\mathbf{S}_m\; + \delta)\;\left(\frac{|\omega_r|_g}{|B|}\right)^{\frac{2}{n+1}}.$$

Therefore dividing by r^2 and using (25) we get $0 < \varepsilon_0 \leq \delta + O(r)$ which is the contradiction. □

As a consequence, we have the following Cheeger isoperimetric comparison:

Corollary 4.3 *Let $(\mathcal{M}^{n+1}, g)$ be a compact Riemannian manifold, $n \geq 2$. Assume that the scalar curvature satisfies $\mathbf{S}_m < (n+1)n\,k$ for some $k \in \mathbb{R}$. Then there exists $v_0 > 0$ such that for any $v \in (0, v_0)$, and any E a ball of volume v in $(\mathbb{M}^{n+1}, g_k)$, there holds*

$$\mathcal{H}^D_{\mathcal{M}}(v, g) > h_D(E, g_k).$$

Here also $(\mathbb{M}^{n+1}, g_k)$ is the space of constant sectional curvature k.

In the same way Theorem 1.3 is a direct consequence of the expansion of the relative isoperimetric profile $I^N_{\mathcal{M}_1}$ near zero for a smooth bounded domain $\mathcal{M}_1$ of a Riemannian manifold $(\mathcal{M}^{n+1}, g)$ (not necessary compact). Letting

$$I^N_{\mathcal{M}_1}(v, g) = \inf_{E \subset \mathcal{M}_1,\, |E|_g = v} |\partial E \cap \mathcal{M}_1|_g,$$

one has, see [7], for any $\delta > 0$ there exists $v_\delta > 0$ such that for any $v \in (0, v_\delta)$

$$\begin{aligned}\left(1 - (\alpha_n\,\mathbf{H}_m + \delta)\,\left(\frac{v}{|B_+|}\right)^{\frac{1}{n+1}}\right) I^N_{\mathbb{R}^{n+1}_+}(v) &\leq I^N_{\mathcal{M}_1}(v, g)\\ &\leq \left(1 - (\alpha_n\,\mathbf{H}_m - \delta)\,\left(\frac{v}{|B_+|}\right)^{\frac{1}{n+1}}\right) I^N_{\mathbb{R}^{n+1}_+}(v),\end{aligned}$$

where $I^N_{\mathbb{R}^{n+1}_+}(v) = (n+1)\,|B_+|^{\frac{1}{n+1}}\, v^{\frac{n}{n+1}}$ and $\mathbf{H}_m$ is the maximum of the mean curvature of $\partial\mathcal{M}_1$ while $\alpha_n = \frac{n}{(n+1)(n+2)}\frac{|B^n|}{|B_+|}$. Here B^n is a unit ball of $\mathbb{R}^n$ and B_+ is a half ball of $\mathbb{R}^{n+1}$.

Likewise the proof as the above proposition carries over for the following result.

Proposition 4.4 *For any $\varepsilon > 0$, there exists $v_\varepsilon > 0$ such that for any $v \in (0, v_\varepsilon)$ there holds*

$$\mathcal{H}^N_{\mathcal{M}_1}(v, g) \geq \left(1 - (\alpha_n \boldsymbol{H}_m + \varepsilon)\left(\frac{v}{|B_+|}\right)^{\frac{1}{n+1}}\right)\mathcal{H}^N_{\mathbb{R}^{n+1}_+}(v),$$

where $\alpha_n = \frac{n}{(n+1)(n+2)}\frac{|B^n|}{|B_+|}$, $\boldsymbol{H}_m$ is the maximum of the mean curvature of $\partial\mathcal{M}_1$ and

$$\mathcal{H}^N_{\mathbb{R}^{n+1}_+}(v) = (n+1)\left(\frac{|B_+|}{v}\right)^{\frac{1}{n+1}}.$$

Theorem 1.3 follows at once by applying the Cheeger inequality (Neumann).

Acknowledgments We would like to thank Professor Frank Pacard for taking our attention to these questions. We also thank Professor Andrea Malchiodi and Professor Frank Pacard for helpful discussions. Part of this work was done when the author was visiting the I.H.P. (Institut Henri Poincaré) on the Trimester "Ricci curvature and Ricci flow" during the period April 15th–May 15th, 2008, we are grateful for there kind hospitality. The author has been supported by M.U.R.S.T within the PRIN 2006 Variational Methods and Nonlinear Differential Equations.

References

1. Aubin, T., Li, Y.Y.: On the best Sobolev inequality. J. Math. Pures Appl. **78**, 353–387 (1999)
2. Berger, M., Gauduchon, P., Mazet, E.: Le spectre dune variété riemanniene. Lecture Notes in Mathematics, vol. 194. Springer, New York
3. Chavel, I.: Eigenvalues in Riemannian Geometry. Academic Press, New York (1984)
4. Cheeger, J.: A lower bound for the smallest eigenvalue of the Laplacian. In: Gunning, R.C. (ed.) Problems in Analysis, A Symposium in Honor of Salomon Bochner, pp. 195–199. Princeton University Press, New Jersey (1970)
5. Druet, O.: Asymptotic expansion of the Faber–Krahn prole of a compact Riemannian manifold. C. R. Acad. Sci. Paris, Ser. I **346**, 1163–1167 (2008)
6. Druet, O.: Sharp local isoperimetric inequalities involving the scalar curvature. Proc. Am. Math. Soc. **130**(8), 2351–2361 (2002)
7. Fall, M.M.: Area minimizing regions with small volume in Riemannian manifolds with boundary. Pac. J. (to appear, SISSA preprint 2008)
8. Fusco, N., Maggi, F., Pratelli, A.: Stability estimates for certain Faber–Krahn and isocapacitary inequalities, accepted on Annali Scuola Normale Superiore di Pisa (preprint cvgmt.sns.it)
9. Gallot, S.: Minoration sur le λ_1 des variétés riemanniennes. Seminaire N. Bourbaki, exp. N^o **569**, 132–148 (1980–1981)
10. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order Grundlehren, vol. 224, 2nd edn. Springer, Berlin (1983)
11. Lindqvist, P.: On non-linear Rayleigh quotients. Potential Anal **2**, 199–218 (1993)
12. Nardulli, S.: The isoperimetric profile of a compact Riemannian manifold for small volumes. University of Paris Sud (Orsay) (2006)
13. Pacard, F., Sicbaldi, P.: Extremal domains for the first eigenvalue of the Laplace-Beltrami operator. Annales de l'Institut Fourier (2007)
14. Pacard, F., Xu, X.: Constant mean curvature spheres in Riemannian manifolds. Manuscripta Math (2005, preprint)
15. Pólya, G.G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. Annals of Math. Studies, vol. 27. Princeton University Press, New Jersey (1951)