Sharp local estimates for the Szegö-Weinberger profile in Riemannian manifolds

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Abstract

We study the local Szegö-Weinberger profile in a geodesic ball $B_g(y_0, r_0)$ centered at a point y_0 in a Riemannian manifold $(\mathcal{M}, g)$. This profile is obtained by maximizing the first nontrivial Neumann eigenvalue μ_2 of the Laplace-Beltrami Operator Δ_g on $\mathcal{M}$ among subdomains of $B_g(y_0, r_0)$ with fixed volume. We derive a sharp asymptotic bounds of this profile in terms of the scalar curvature of $\mathcal{M}$ at y_0. As a corollary, we deduce a local comparison principle depending only on the scalar curvature. Our study is related to previous results on the profile corresponding to the minimization of the first Dirichlet eigenvalue of Δ_g, but additional difficulties arise due to the fact that μ_2 is degenerate in the unit ball in $\mathbb{R}^N$ and geodesic balls do not yield the optimal lower bound in the asymptotics we obtain.

1 Introduction

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold of dimension N, $N \geq 2$. For a bounded regular domain $\Omega \subset \mathcal{M}$ we consider the Neumann eigenvalue problem

$$\Delta_g f + \mu f = 0 \quad \text{in } \Omega, \qquad \langle \nabla f, \eta \rangle_g = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Delta_g f = div_g(\nabla f)$ is the Laplace-Beltrami operator on $\mathcal{M}$ and η is the outer unit normal to $\partial\Omega$. The set of eigenvalues, counted with multiplicities, in the above eigenvalue problem is given as an increasing sequence

$$0 = \mu_1(\Omega, g) < \mu_2(\Omega, g) \leq \cdots + \infty.$$

By results of Szegö [13] and Weinberger [14], balls maximize μ_2 among domains having fixed volume in $\mathcal{M} = \mathbb{R}^N$. More precisely, in [13] this was proved for the planar case $N = 2$, whereas in [14] the case $N \geq 3$ was considered. As remarked in [4] and [2], this result extends to the case of the N-dimensional hyperbolic space. Moreover, the same conclusion holds for domains contained in a hemisphere [2] and – under further restrictions on the domain – also in rank-1 symmetric spaces [1].

The aim of the present paper is to study the geometric variational problem of maximizing $\mu_2(\Omega, g)$ among domains with fixed volume *locally* in a general complete Riemannian manifold $(\mathcal{M}, g)$. In order to state our results, we need to introduce some notations. For a subset $\Omega \subset \mathcal{M}$, we let $|\Omega|_g$ denote the volume of Ω with respect to the metric g. For $0 < v < |\mathcal{M}|_g$, we define the *Szegö-Weinberger profile* of $\mathcal{M}$ as

$$SW_{\mathcal{M}}(v, g) := \sup_{\Omega \subset \mathcal{M},\, |\Omega|_g = v} \mu_2(\Omega, g).$$

Here and in the following, we assume without further mention that only regular bounded domains $\Omega \subset \mathcal{M}$ are considered. For open subsets $\mathcal{A} \subset \mathcal{M}$ and $0 < v < |\mathcal{A}|_g$, we also define

$$SW_{\mathcal{A}}(v, g) := \sup_{\Omega \subset \mathcal{A},\, |\Omega|_g = v} \mu_2(\Omega, g),$$

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assuming again without further mention that only regular bounded domains $\Omega \subset \mathcal{A}$ are considered. By Weinberger's result in [14], we then have

$$SW_{\mathbb{R}^N}(v) = \left(\frac{|B|}{v}\right)^{\frac{2}{N}} \mu_2(B).$$

where B denotes the unit ball in $\mathbb{R}^N$. The eigenvalue $\mu_2(B)$ has multiplicity N with corresponding eigenfunctions $x \mapsto \varphi(|x|)\frac{x_i}{|x|}$, $i = 1, \dots, N$, where φ can be expressed in terms of a rescaled Bessel function of the first kind and satisfies $\varphi(0) = \varphi'(1) = 0$. For matters of convenience, we normalize φ such that

$$\int_0^1 \varphi^2(t)t^{N-1}dt = \frac{1}{|B|}, \tag{2}$$

see Section 2 below. We are interested in the local effect of curvature terms on the Szegö-Weinberger profile. For this we study the profile in a small geodesic ball $B_g(y_0, r)$ of $\mathcal{M}$ centered at a point $y_0 \in \mathcal{M}$ with radius r. In our main result, we obtain the following optimal two-sided local bound.

Theorem 1.1 *Let $\mathcal{M}$ be a complete N-dimensional Riemannian manifold with $N \geq 2$, and let S denote the scalar curvature function on $\mathcal{M}$. Moreover, let $y_0 \in \mathcal{M}$, and let*

$$\begin{aligned}\gamma_N &= \frac{2\mu_2(B) + (N+2)(N-2) - (N+2)|B|\varphi^2(1)}{6N(N+2)\mu_2(B)} \\ &= \frac{1}{3N(N+2)} + \frac{N-2}{6N\mu_2(B)} - \frac{1}{3N(\mu_2(B) - N + 1)}\end{aligned} \tag{3}$$

Then we have:

(i) As $v \to 0$,

$$\frac{SW_{\mathcal{M}}(v)}{SW_{\mathbb{R}^N}(v)} \geq 1 - \gamma_N S(y_0)\Big(\frac{v}{|B|}\Big)^{\frac{2}{N}} + o(v^{\frac{2}{N}}). \tag{4}$$

(ii) For every $y_0 \in \mathcal{M}$ and every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$1 - (\gamma_N\, S(y_0) + \varepsilon)\left(\frac{v}{|B|}\right)^{\frac{2}{N}} \leq \frac{SW_{B_g(y_0,r_\varepsilon)}(v)}{SW_{\mathbb{R}^N}(v)} \leq 1 - (\gamma_N\, S(y_0) - \varepsilon)\left(\frac{v}{|B|}\right)^{\frac{2}{N}} \tag{5}$$

for $v \in \big(0\,,\, |B_g(y_0, r_\varepsilon)|_g\big)$.

(iii) $\gamma_N < 0$ for all $N \geq 2$, and $\gamma_N \to 0$ as $N \to \infty$.

Some remarks are in order. The right hand side of (4) can be replaced by

$$1 - \gamma_N\Big[\sup_{y_0\in\mathcal{M}} S(y_0)\Big]\Big(\frac{v}{|B|}\Big)^{\frac{2}{N}} + o(v^{\frac{2}{N}})$$

if the supremum of the scalar curvature is attained on $\mathcal{M}$, e.g. if $\mathcal{M}$ is compact. The equality (3) is derived from an integral identity for Bessel functions which gives $|B|\varphi^2(1) = \frac{2\mu_2(B)}{\mu_2(B)-N+1}$, see Lemma 2.1 below. The coefficient γ_N is uniquely determined by the two-sided estimate (5) and therefore sharp. To determine the sign of γ_N for large N, fine estimates on $\mu_2(B)$ are needed. In Lemma 2.1 below, we will prove that

$$N + 1 \leq \mu_2(B) < \begin{cases} N+2, & \text{for } N = 2, 3, 4, \\ N + 1 + \dfrac{2}{N-2} & \text{for } N \geq 5. \end{cases} \tag{6}$$

From this we then deduce part (iii) of Theorem 1.1. The bounds in (6) might not be new, but we could not find any suitable bound in the literature. It seems natural to deduce bounds on $\mu_2(B)$ from the fact that $\sqrt{\mu_2(B)}$ is the first positive zero of the derivative of the function $t \mapsto t^{(2-N)/2} J_{N/2}(t)$, where $J_{N/2}$ is the Bessel function of the first kind of order $N/2$. However, to obtain the upper bound in (6), we use the variational characterization of $\mu_2(B)$ instead. See Section 2 below for details.

As a consequence of Theorem 1.1, we readily deduce the following local isochoric comparison principle related to the Szegö-Weinberger profile.

Corollary 1.2 *Let $(\mathcal{M}_1, g_1)$, $(\mathcal{M}_2, g_2)$ be two N-dimensional complete Riemannian manifolds, $N \geq 2$ with scalar curvature functions S_1, S_2 respectively. Let $y_1 \in \mathcal{M}_1$ and $y_2 \in \mathcal{M}_2$ such that $S_1(y_1) < S_2(y_2)$. Then there exists $r > 0$ such that*

$$SW_{B_{g_1}(y_1,r)}(v) < SW_{B_{g_2}(y_2,r)}(v) \tag{7}$$

for any $v \in (0, \min\{|B_{g_1}(y_1, r)|_{g_1}, |B_{g_2}(y_2, r)|_{g_2}\})$.

We emphasize that in the special case where $(\mathcal{M}_2, g_2)$ is a space form of constant curvature, the right hand side in (7) may be replaced with $\mu_2(E, g_2)$, where E is any geodesic ball of volume v in $\mathcal{M}_2$. This follows from the local expansion of μ_2 in small geodesic balls in these manifolds, see Remark 3.3(ii) below.

Corollary 1.2 should be seen in comparison with the results in [6, 7, 9] concerning the isoperimetric profile $I_{\mathcal{M}}$ and the Faber-Krahn profile $FK_{\mathcal{M}}$ of $\mathcal{M}$. More precisely, set

$$I_{\mathcal{M}}(v, g) := \inf_{\Omega \subset \mathcal{M},\, |\Omega|_g = v} |\partial\Omega|_g$$

and

$$FK_{\mathcal{M}}(v, g) := \inf_{\Omega \subset \mathcal{M},\, |\Omega|_g = v} \lambda_1(\Omega, g),$$

with $\lambda_1(\Omega, g)$ being the first Dirichlet eigenvalue of $-\Delta_g$ in Ω. Let $y \in \mathcal{M}$ and $k \in \mathbb{R}$ be such that $S(y) < (N-1)N\, k$, where $S(y)$ denotes the scalar curvature of $\mathcal{M}$ at y. Furthermore, let $(\mathbb{M}^N, g_k)$ denote the space form of constant sectional curvature k. Then there exists $r_y > 0$ such that for any $v \in \Big(0\,,\, |B_g(y, r_y)|_g\Big)$ and any geodesic ball E of volume v in $(\mathbb{M}^N, g_k)$, we have

$$I_{B_g(y,r_y)}(v, g) > |\partial E|_{g_k}, \tag{8}$$

$$FK_{B_g(y,r_y)}(v, g) > \lambda_1(E, g_k). \tag{9}$$

Inequality (8) was established by Druet [7], and (9) was derived independently by Druet [6] and the first author [9]. The first ingredient in the proof of (9) is the following expansion of $\lambda_1(B_g(y, r), g)$ when $r \to 0$:

$$\lambda_1(B_g(y, r), g) = \frac{\lambda_1(B)}{r^2} - \frac{S(y_0)}{6} + O(r) \tag{10}$$

This expansion had already been obtained by Chavel in [4, Chapter 8]. In the proof of Theorem 1.1, we need to derive a corresponding expansion for $\mu_2(B_g(y, r), g)$. This is more difficult since $\mu_2(B)$ is degenerate with multiplicity N and the corresponding eigenfunctions are nonradial. As a consequence, an anisotropic curvature term appears in the corresponding expansion. More precisely, we have

$$\mu_2(B_g(y_0, r), g) = \frac{\mu_2(B)}{r^2} + \alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0) + o(1) \qquad \text{as } r \to 0 \tag{11}$$

with suitable constants $\alpha_N^\pm$ and $R_{min}(y_0) = \inf\{Ric_{y_0}(A, A) \,:\, A \in T_{y_0}\mathcal{M},\ |A| = 1\}$, see Proposition 3.1 below. In order to obtain an expansion depending only on the scalar curvature, we need to consider suitable geodesic ellipsoids with small eccentricity. This is a crucial step in the proof of Theorem 1.1, since – in contrast to the Faber-Krahn profile – geodesic balls do not give rise to optimal two-sided bounds. As a further tool, we need a quantitative version of the Szegö-Weinberger inequality, which has been obtained very recently in the euclidean case by Brasco

and Pratelli [3]. In the proof of Theorem 1.1 we combine these tools with variants of ideas in [9] and [1, 2, 14] to control error terms and to construct suitable test functions for the variational characterization of μ_2, see Section 5 below.
We like to mention that [9] also contains a statement about the local expansion of a profile related to *minimizing* μ_2 among domains of fixed volume relative to an open set, see [9, Theorem 1.3]. However, the proof of this statement is not correct since it relies on a comparison with a relative isoperimetric profile which does not correspond to the Neumann boundary conditions in (1) but rather to mixed boundary conditions.
Theorem 1.1 gives a first hint that critical domains for μ_2 which are nearly balls, if they exists, might be located near critical points of the scalar curvature of $\mathcal{M}$ (at least in the twodimensional case). Here, roughly speaking, by a critical domain we mean a domain where μ_2 is critical with respect to volume preserving perturbations. Pacard and Sicbaldi [12] showed that close to nondegenerate critical points of the scalar curvature there exist small critical domains for the first Dirichlet eigenvalue of Δ_g. The corresponding problem for the Neumann eigenvalue μ_2 seems much more difficult. We note that Zanger [15] derived a Hadamard type formula (in the spirit of [10, p. 522]) for a Neumann eigenvalue which depends smoothly on domain variations. However, due to possible degeneracy, μ_2 might not depend smoothly on domain variations, and therefore it is not clear how critical domains should be defined. On the other hand, in [8] a notion of critical domains for higher Dirichlet eigenvalues, which may also be degenerate, is derived via analytic perturbation theory. It therefore seems natural – but far from obvious – to develop and analyze a similar notion for μ_2. We wish to address this problem in future work.

The paper is organized as follows. In Section 2 we collect properties of $\mu_2(B)$ and the function φ appearing in the definition of the corresponding eigenfunctions. In particular, we prove the bounds (6) and part (iii) of Theorem 1.1 in Lemma 2.1 below. We close this section by recalling some basic notations from Riemannian geometry. In Section 3 we provide an expansion of $\mu_2(B_g(y_0, r))$ as $r \to 0$. In Section 4 we calculate a corresponding expansion for suitably chosen geodesic ellipsoids with small eccentricity. As shown by Corollary 4.2, these ellipsoids are suitable test domains to derive Part (i) of Theorem 1.1, and from this the lower bound in Part (ii) follows. Section 5 is devoted to collect all tools needed for the proof of the upper bound in Theorem 1.1(ii). In particular, we use the above-mentioned stability estimate of Brasco and Pratelli [3] in this section, see Lemma 5.2. Arguing by contradiction, we then complete the proof of Theorem 1.1 in Section 6.

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2 Preliminaries and Notations

We denote by B the unit ball in $\mathbb{R}^N$. Moreover, for a smooth bounded domain Ω of a complete Riemannian manifold $(\mathcal{M}, g)$, we write $\mu_2 = \mu_2(\Omega, g)$ for the first nontrivial eigenvalue of (1). The variational characterization of $\mu_2(\Omega, g)$ is given by

$$\mu_2(\Omega, g) = \inf\left\{ \frac{\int_\Omega |\nabla u|_g^2 dv_g}{\int_\Omega u^2 dv_g} \;:\; u \in H^1(\Omega) \setminus \{0\},\ \int_\Omega u\, dv_g = 0 \right\}, \tag{12}$$

where v_g denotes the volume element of the metric g. We recall that the minimizers of this minimization problem are precisely the eigenfunctions corresponding to $\mu_2(\Omega, g)$. If $\mathcal{M} = \mathbb{R}^N$ and g is the euclidean metric, we simply write $\mu_2(\Omega)$ in place of $\mu_2(\Omega, g)$. As noted already, $\mu_2(B)$ is of multiplicity N with corresponding eigenfunctions given by $\varphi(|x|)\frac{x_i}{|x|}$, $i = 1, \dots, N$ with

$$\varphi'' + \frac{N-1}{t}\varphi' + \left(\mu_2(B) - \frac{N-1}{t^2}\right)\varphi = 0, \quad t \in (0,1), \quad \varphi(0) = \varphi'(1) = 0. \tag{13}$$

Throughout this paper, we assume the normalization (2), which equivalently yields

$$\int_B \varphi^2(|x|)\, dx = N \qquad \text{and} \qquad \int_B \varphi^2(|x|) \left(\frac{x^i}{|x|}\right)^2 dx = 1 \quad \text{for } i = 1, \dots, N. \tag{14}$$

The function φ and the eigenvalue $\mu_2(B)$ are obtained via $J_{N/2}$, the Bessel function of the first kind of order $N/2$. Indeed, $\sqrt{\mu_2(B)}$ is the first positive zero of the derivative of $t \mapsto t^{(2-N)/2} J_{N/2}(t)$, and φ is a scalar multiple of the function

$$t \mapsto g(t) = t^{(2-N)/2} J_{N/2}(\sqrt{\mu_2(B)}t). \tag{15}$$

More precisely, by (2) we have

$$\varphi(t) = \frac{g(t)}{\sqrt{|B| \int_0^1 g^2 t^{N-1} dt}}. \tag{16}$$

Lemma 2.1 *We have:*

(i) $\mu_2(B) \geq N+1$ *for every dimension* $N \in \mathbb{N}$, *and*

$$\mu_2(B) < \begin{cases} N+2, & \text{for } N = 2,3,4, \\ \dfrac{N(N-1)}{N-2} & \text{for } N \geq 5. \end{cases} \tag{17}$$

(ii) $|B|\varphi^2(1) = \frac{2\mu_2(B)}{\mu_2(B)-N+1}$ *for every* $N \in \mathbb{N}$.

(iii) $\gamma_N < 0$ *for all* $N \geq 2$.

(iv) $\gamma_N \to 0$ *as* $N \to \infty$.

Proof. Set $\mu_2 := \mu_2(B)$. We start with the proof of (ii). Since $\sqrt{\mu_2}$ is the first zero of the derivative of the function $t \mapsto t^{(2-N)/2} J_{N/2}(t)$, we infer that

$$(J'_{N/2})^2(\sqrt{\mu_2}) = \frac{(N-2)^2}{4} \frac{1}{\mu_2} J^2_{N/2}(\sqrt{\mu_2}). \tag{18}$$

Moreover, for the function g defined in (15) we have by [11, p.129, formula (5.14.5)]

$$\int_0^1 t^{N-1} g^2 dt = \int_0^1 t J^2_{N/2}(\sqrt{\mu_2}t) dt = \frac{1}{2}\Big[(J'_{N/2})^2(\sqrt{\mu_2}) + \left(1 - \frac{N^2}{4\mu_2}\right) J^2_{N/2}(\sqrt{\mu_2})\Big]. \tag{19}$$

Inserting (18) in (19) yields

$$\int_0^1 t^{N-1} g^2 dt = \frac{\mu_2 - N + 1}{2\mu_2} J^2_{N/2}(\sqrt{\mu_2}). \tag{20}$$

In particular, this implies $\mu_2 > N-1$. Moreover, since $g(1) = J_{N/2}(\sqrt{\mu_2})$, we conclude by (16) that

$$|B|\varphi^2(1) = \frac{g^2(1)}{\int_0^1 t^{N-1} g^2 dt} = \frac{2\mu_2}{\mu_2 - N + 1},$$

as claimed.
We now turn to (i), and we first prove that $\mu_2 \geq N+1$. Let $u, v : B \to \mathbb{R}$ be given by $u(x) = x_1$ and $v(x) = \frac{x_1}{|x|}\varphi(|x|)$, so that $-\Delta v = \mu_2 v$ in B and $\partial_\eta v = 0$ on ∂B. Hence we find

$$\begin{aligned} |B|\varphi(1) &= \int_{\partial B} \frac{x_1^2}{|x|}\varphi(|x|)\, d\sigma = \int_{\partial B} uv\, d\sigma = \int_{\partial B} \Big(v\partial_\nu u - u\partial_\nu v\Big)\, d\sigma \\ &= \int_B \Big(v\Delta u - u\Delta v\Big)\, dx = \mu_2 \int_B uv\, dx = \frac{\mu_2}{N} \int_B |x|\varphi(|x|)\, dx \\ &= \mu_2|B| \int_0^1 t^N \varphi(t)\, dt \leq \mu_2|B|\Big(\int_0^1 t^{N+1}\, dt\Big)^{\frac{1}{2}}\Big(\int_0^1 t^{N-1}\varphi^2(t)\, dt\Big)^{\frac{1}{2}} = \frac{\mu_2\sqrt{|B|}}{\sqrt{N+2}}, \end{aligned}$$

using Hölder's inequality and (2) in the last two steps. Using (ii) we therefore get

$$\mu_2^2 - (N-1)\mu_2 - 2(N+2) \geq 0,$$

and this gives

$$\mu_2 \geq \frac{1}{2}\Big(N-1+\sqrt{(N-1)^2+8(N+2)}\Big) \geq N+1.$$

To prove (17), we consider the functions $x \mapsto u_s(x) = x_1|x|^s$ for $s > -\frac{N}{2}$, so that $u \in H^1(B)$ and $\int_B u_s\,dx = 0$. Since these functions are not eigenfunctions corresponding μ_2, the variational characterization (12) gives

$$\mu_2 < \frac{\int_B |\nabla u_s|^2\,dx}{\int_B u_s^2\,dx} = \frac{\int_B \Big(|x|^{2s} + (s^2+2s)x_1^2|x|^{2(s-1)}\Big)\,dx}{\int_B x_1^2|x|^{2s}\,dx} = \frac{(N+s^2+2s)\int_B |x|^{2s}\,dx}{\int_B |x|^{2s+2}\,dx}$$
$$= \frac{(N+s^2+2s)(N+2s+2)}{N+2s}.$$

If $N \in \{2,3,4\}$, we may take $s=0$ and obtain the first inequality in (17). If $N \geq 5$, we may take $s=-1$ and obtain the second inequality in (17). This finishes the proof of (i).
To prove (iii), we consider the function

$$\sigma_N : (N-1,\infty) \to \mathbb{R}, \qquad \sigma_N(t) = \frac{1}{3N(N+2)} + \frac{N-2}{6Nt} - \frac{1}{3N(t-N+1)}$$

and recall from (3) that $\gamma_N = \sigma_N(\mu_2)$. We note that

$$\sigma_N'(t) = \frac{2t^2-(N-2)(t-N+1)^2}{6N(t-N+1)^2t^2} = \frac{(4-N)t^2+(N-2)(N-1)[2t-(N-1)]}{6N(t-N+1)^2t^2}. \tag{21}$$

Hence $\sigma_N'(t) > 0$ in $(N-1,\infty)$ if $N \in \{2,3,4\}$, and therefore (17) implies that

$$\gamma_N = \sigma_N(\mu_2) < \sigma_N(N+2) = \frac{N-4}{18N(N+2)} \leq 0 \qquad \text{if } N \in \{2,3,4\}.$$

If $N \geq 5$, the zeros of the numerator in (21) are given by $\tau_N^\pm = \frac{N-1}{N-4}[N-2 \pm \sqrt{2(N-2)}]$, whereas $\tau_N^- < N-1 < \frac{N(N-1)}{N-2} < \tau_N^+$. Moreover, $\sigma_N'(t) > 0$ on $(N-1,\tau_N^+)$, and thus (17) implies that

$$\gamma_N = \sigma(\mu_2) < \sigma_N(\frac{N(N-1)}{N-2}) = \frac{4-N}{3N^2(N+2)(N-1)} < 0 \qquad \text{if } N \geq 5.$$

This ends the proof of (iii).
(iv) simply follows from the definition of γ_N and the fact that $\gamma_N \geq N+1$, as shown in (ii). ■

Let $(\mathcal{M},g)$ be a complete Riemannian manifold of dimension N. We fix $y_0 \in \mathcal{M}$ and consider an orthonormal basis $E_1,\dots,E_N$ of $T_{y_0}\mathcal{M}$. In the sequel, it will be convenient to use the (somewhat sloppy) notation

$$X := x^iE_i \in T_{y_0}\mathcal{M} \qquad \text{for } x \in \mathbb{R}^N.$$

Here and in the following, we sum over repeated upper and lower indices as usual. We consider the geodesic coordinate system

$$\mathbb{R}^N \ni x \mapsto \Psi(x) := \operatorname{Exp}_{y_0}(X) \tag{22}$$

A geodesic ball in $\mathcal{M}$ centered at y_0 with radius $r>0$ is defined as $B_g(y_0,r) = \Psi(rB)$. The map Ψ induces coordinate vector fields $Y_i := \Psi_*\frac{\partial}{\partial x^i}$, which are pointwise given by

$$Y_i(x) = d\operatorname{Exp}_{y_0}(X)E_i \in T_{\Psi(x)}\mathcal{M}, \qquad i=1,\dots,N.$$

As usual, we write the metric in local coordinates by setting

$$g_{ij}(x) = \langle Y_i(x), Y_j(x)\rangle_g \qquad \text{for } x \in \mathbb{R}^N.$$

The following local expansions are well known and can be found e.g. in [5, §II.8].

Lemma 2.2 *In the above notations, for any $i, j = 1, ..., N$, we have*

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\langle R_{y_0}(X, E_i)X, E_j\rangle_g + O(|x|^3) \quad \text{and} \quad dv_g(x) = \Big(1 - \frac{1}{6}\,Ric_{y_0}(X, X) + O(|x|^3)\Big)dx.$$

Here dx is the volume element of $\mathbb{R}^N$, dv_g is the volume element of $\mathcal{M}$,

$$R_{y_0} : T_{y_0}\mathcal{M} \times T_{y_0}\mathcal{M} \times T_{y_0}\mathcal{M} \to T_{y_0}\mathcal{M}$$

is the Riemannian curvature tensor at y_0 and

$$Ric_{y_0} : T_{y_0}\mathcal{M} \times T_{y_0}\mathcal{M} \to \mathbb{R}, \qquad Ric_{y_0}(X, Y) = -\sum_{i=1}^{N}\langle R_{y_0}(X, E_i)Y, E_i\rangle$$

is the Ricci tensor at y_0. Moreover, the volume expansion of metric balls is given by

$$|B_g(y_0, r)|_g = r^N\,|B|\,\left(1 - \frac{1}{6(N+2)}\,r^2 S(y_0) + O(r^4)\right); \tag{23}$$

where S is the scalar curvature function on $\mathcal{M}$.

Here and in the following, once y_0 is fixed, we also write $\langle\cdot,\cdot\rangle$ in place of $\langle\cdot,\cdot\rangle_g$ to denote the scalar product on $T_{y_0}\mathcal{M}$ induced by the metric g. It will turn out useful to put

$$R_{ijkl} := \langle R_{y_0}(E_i, E_j)E_k, E_l\rangle \quad \text{and} \quad R_{ij} := Ric_{y_0}(E_i, E_j) \qquad \text{for } i, j = 1, \ldots, N. \tag{24}$$

The scalar curvature of $\mathcal{M}$ at y_0 is given by $S(y_0) = \sum\limits_{i=1}^{N} R_{ii}$. We point out that the orthonormal basis E_i, $i = 1, \ldots, N$ can be chosen such that

$$R_{ij} = 0 \qquad \text{for } i \neq j, \tag{25}$$

and we will fix such a choice from now on. We finally note that the euclidean scalar product of $x, y \in \mathbb{R}^N$ will simply be denoted by $x \cdot y$.

3 Expansion of μ_2 for small geodesic balls

The main goal of the this section is the derivation of the the following expansion for μ_2 on small geodesic balls centered at y_0.

Proposition 3.1 *For $r > 0$ we have*

$$\mu_2(B_g(y_0, r), g) = \frac{\mu_2(B)}{r^2} + \alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0) + o(1),$$

where

$$\alpha_N^- = \frac{1}{6}\left(\frac{|B|\varphi^2(1)}{N+2} - 1\right), \quad \alpha_N^+ = \frac{1}{6}\left(\frac{|B|\varphi^2(1)}{N+2} + 1\right), \quad R_{min}(y_0) = \inf_{A \in T_{y_0}\mathcal{M}, |A|=1} Ric_{y_0}(A, A)$$

and $o(1) \to 0$ as $r \to 0$.

Proof. Let $r > 0$ be smaller than the injectivity radius of $\mathcal{M}$ at y_0, so that $B_g(y_0, r)$ is a regular domain. Let $u_r \in C^3(\overline{B_g(y_0, r)})$ be an eigenfunction corresponding to the eigenvalue problem

$$\Delta_g u_r + \mu_2(B_g(y_0, r), g)\,u_r = 0 \quad \text{in } B_g(y_0, r), \qquad \langle\nabla u_r, \eta_r\rangle_g = 0 \quad \text{on } \partial B_g(y_0, r),$$

where η_r denotes the outer unit normal on $\partial B_g(y_0, r)$. Replacing u_r by a scalar multiple if necessary, we may assume that u_r is a minimizer of the minimization problem

$$\mu_2(B_g(y_0, r), g) = \inf\left\{\int_{B_g(y_0,r)} |\nabla f|_g^2\,dv_g : f \in H^1(B_g(y_0, r)), \int_{B_g(y_0,r)} f^2\,dv_g = 1, \int_{B_g(y_0,r)} f\,dv_g = 0\right\}.$$

Via the exponential map, we pull back the problem to the unit ball $B \subset \mathbb{R}^N$. For this we consider the pull back metric of g under the map $B \to \mathcal{M}$, $x \mapsto \Psi(rx)$, rescaled with the factor $\frac{1}{r^2}$. Denoting this metric on B by g_r, we then have, in euclidean coordinates,

$$[g_r]_{ij}(x) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{g_r}\Big|_x = \langle Y_i(\Psi(rx)), Y_j(\Psi(rx)) \rangle_g = g_{ij}(rx),$$

so that

$$[g_r]_{ij}(x) = \delta_{ij} + \frac{r^2}{3} \langle R_{y_0}(X, E_i)X, E_j \rangle + O(r^3) \tag{26}$$

and

$$g_r^{ij}(x) = \delta^{ij} - \frac{r^2}{3} \langle R_{y_0}(X, E_i)X, E_j \rangle + O(r^3) \tag{27}$$

uniformly for $x \in \overline{B}$ as a consequence of Lemma 2.2. Here, as usual, $(g_r^{ij})_{ij}$ denotes the inverse of the matrix $([g_r]_{ij})_{ij}$. Setting $|g_r| = \det([g_r]_{ij})_{ij}$, we also have $\sqrt{|g_r|}(x) = 1 - \frac{r^2}{6} Ric_{y_0}(X, X) + O(r^3)$ for $x \in \overline{B}$ by Lemma 2.2. Since this expansion is valid in the sense of C^1-functions on $\overline{B}$, we have

$$\frac{\partial}{\partial x^i} \sqrt{|g_r|} = -\frac{r^2}{3} Ric_{y_0}(X, E_i) + O(r^3) \qquad \text{for } i = 1, \dots, N. \tag{28}$$

We now consider the rescaled eigenfunction

$$\Phi_r : \overline{B} \to \mathbb{R}, \qquad \Phi_r(x) = r^{\frac{N}{2}} u_r(\Psi(rx))$$

which satisfies

$$\Delta_{g_r} \Phi_r + \mu_2(B, g_r)\, \Phi_r = 0 \quad \text{in } B, \qquad \langle \nabla \Phi_r, \eta \rangle_{g_r} = 0 \quad \text{on } \partial B,$$

with

$$\Delta_{g_r} \Phi_r = \frac{1}{\sqrt{|g_r|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g_r|} g_r^{ij} \frac{\partial \Phi_r}{\partial x^j} \right) \qquad \text{and} \qquad \mu_2(B, g_r) = r^2 \mu_2(B_g(y_0, r), g).$$

Moreover, $\int_B \Phi_r^2 \, dv_{g_r} = 1$ and $\int_B \Phi_r \, dv_{g_r} = 0$ with $dv_{g_r} = \sqrt{|g_r|} dx$. Since g_r converges to the Euclidean metric in $\overline{B}$, it is easy to see from the variational characterization of μ_2 that $\mu_2(B, g_r) \to \mu_2(B)$. Moreover, by using standard elliptic regularity theory and compact Sobolev embeddings, one may show that, along a sequence $r_k \to 0$, we have $\Phi_{r_k} \to \Phi$ in $H^1(B)$ for some function $\Phi \in C^2_{loc}(B) \cap C^1(\overline{B})$ satisfying

$$\Delta \Phi + \mu_2(B)\, \Phi = 0 \quad \text{in } B, \qquad \langle \nabla \Phi, \eta \rangle = 0 \quad \text{on } \partial B, \qquad \int_B \Phi^2 \, dx = 1 \quad \text{and} \qquad \int_B \Phi \, dx = 0.$$

Hence there exists $a = (a_1, \dots, a_N) = (a^1, \dots, a^N) \in \mathbb{R}^N$ with $|a| = 1$ and such that

$$\Phi(x) = \varphi(|x|) \frac{a \cdot x}{|x|} \qquad \text{for } x \in \overline{B}.$$

For matters of convenience, we will continue to write r instead of r_k in the following. By integration by parts, using $\langle \nabla \Phi_r, \eta \rangle_{g_r} = 0$ and $dv_{g_r} = \sqrt{|g_r|} dx$, we have

$$\mu_2(B, g_r) \int_B \Phi \Phi_r dv_{g_r} = - \int_B \Phi \Delta_{g_r} \Phi_r dv_{g_r} = \int_B \sqrt{|g_r|} g_r^{ij} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Phi_r}{\partial x^j} \, dx.$$

In the following, it will be convenient to use the notation

$$\tilde{\nabla} h = \sum_{i=1}^N \frac{\partial h}{\partial x^i} E_i : \overline{B} \to T_{y_0}\mathcal{M} \qquad \text{for a } C^1\text{-function } h \text{ defined on } \overline{B}.$$

With this notation we find, using (27) and (28) and integrating by parts again,

$$\int_B \sqrt{|g_r|} g_r^{ij} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Phi_r}{\partial x^j}\, dx = \int_B \sqrt{|g_r|}\Big(\nabla \Phi_r \cdot \nabla \Phi - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla \Phi_r)X, \tilde\nabla \Phi\rangle\Big) dx + O(r^3) \quad (29)$$

$$= -\int_B \sqrt{|g_r|}(\Delta \Phi)\Phi_r\, dx - \int_B \Phi_r \nabla \sqrt{|g_r|}\cdot \nabla \Phi\, dx - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla \Phi_r)X, \tilde\nabla \Phi\rangle\, dx + O(r^3)$$

$$= \mu_2(B)\int_B \Phi \Phi_r\, dv_{g_r} + \frac{r^2}{3}\int_B \Phi_r\, Ric_{y_0}(X, \tilde\nabla \Phi)\, dx - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla \Phi_r)X, \tilde\nabla \Phi\rangle\, dx + O(r^3).$$

Therefore, since $\int_B \Phi\Phi_r\, dv_{g_r} \to 1$ and $\Phi_r \to \Phi$ in $H^1(B)$ as $r \to 0$, we obtain

$$\mu_2(B, g_r) = \mu_2(B) + \frac{r^2}{3}\int_B \Phi\, Ric_{y_0}(X, \tilde\nabla \Phi)\, dx - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla \Phi)X, \tilde\nabla \Phi\rangle\, dx + o(r^2). \quad (30)$$

Noticing that

$$\tilde\nabla \Phi(x) = \frac{1}{|x|^2}\Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)\langle A, X\rangle X + \frac{\varphi(|x|)}{|x|}A \qquad \text{with } A := a^i E_i \in T_{y_0}\mathcal{M}, \quad (31)$$

we find

$$\int_B \langle R_{y_0}(X, \tilde\nabla \Phi)X, \tilde\nabla \Phi\rangle\, dx = \int_B \frac{\varphi^2(|x|)}{|x|^2}\langle R_{y_0}(X, A)X, A\rangle\, dx = \langle R_{y_0}(E_i, A)E_j, A\rangle \int_B \frac{\varphi^2(|x|)}{|x|^2} x^i x^j\, dx$$

$$= \langle R_{y_0}(E_i, A)E_j, A\rangle \int_0^1 \varphi^2 t^{N-1}\, dt \int_{\partial B} x^i x^j\, d\sigma = -Ric_{y_0}(A, A). \quad (32)$$

Here we used the identity $\int_{\partial B} x^i x^j\, d\sigma = \delta^{ij}|B|$ and the normalization (2) in the last step. Moreover, we compute via integration by parts, using (25),

$$2\int_B \Phi\, Ric_{y_0}(X, \tilde\nabla \Phi)\, dx = 2R_{ij}\int_B x^i \frac{\partial \Phi}{\partial x^j}\Phi\, dx = R_{ii}\int_B x^i \frac{\partial \Phi^2}{\partial x^i}\, dx$$

$$= -\Big(\sum_{i=1}^N R_{ii}\Big)\int_B \Phi^2\, dx + R_{ii}\int_{\partial B} [x^i]^2 \Phi^2\, d\sigma$$

$$= -S(y_0)\int_B \frac{(a\cdot x)^2}{|x|^2}\varphi^2(|x|)\, dx + \varphi^2(1)R_{ii}\int_{\partial B} [x^i]^2 (a\cdot x)^2\, d\sigma.$$

Recalling (14) and using the identities

$$\int_{\partial B} [x^1]^4 d\sigma = 3\int_{\partial B} [x^i]^2[x^j]^2 d\sigma = \frac{3|B|}{N+2} \text{ and } \int_{\partial B} x^i x^j [x^k]^2 d\sigma = 0 \quad \text{for } i, j, k = 1, \dots, N,\ i \neq j,$$

we find that

$$2\int_B \Phi\, Ric_{y_0}(X, \tilde\nabla \Phi)\, dx = -S(y_0) + \varphi^2(1)R_{ii}a_k a_l \int_{\partial B} [x^i]^2 x^k x^l\, d\sigma$$

$$= -S(y_0) + \varphi^2(1)R_{ii}[a_k]^2 \int_{\partial B} [x^i]^2[x^k]^2\, d\sigma = -S(y_0) + \frac{\varphi^2(1)|B|}{N+2}\Big(S(y_0) + 2R_{kk}[a^k]^2\Big)$$

$$= \Big(\frac{\varphi^2(1)|B|}{N+2} - 1\Big) S(y_0) + 2\frac{\varphi^2(1)|B|}{N+2} Ric_{y_0}(A, A). \quad (33)$$

Combining (30), (31) and (32), we get

$$\mu_2(B, g_r) = \mu_2(B) + \frac{r^2}{6}\Big(\frac{|B|\varphi^2(1)}{N+2} - 1\Big)S(y_0) + \frac{r^2}{3}\Big(\frac{|B|\varphi^2(1)}{N+2} + 1\Big)Ric_{y_0}(A, A) + o(r^2)$$

and therefore

$$\mu_2(B_g(y_0, r), g) = \frac{\mu_2(B, g_r)}{r^2} = \frac{\mu_2(B)}{r^2} + \alpha_N^- S(y_0) + 2\alpha_N^+ Ric_{y_0}(A, A) + o(1). \quad (34)$$

We now need to recall that – more precisely – here we have passed to a sequence $r = r_k \to 0$. Nevertheless, the argument implies that

$$\mu_2(B_g(y_0,r),g) \geq \frac{\mu_2(B)}{r^2} + \alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0) + o(1) \qquad \text{as } r \to 0. \tag{35}$$

Indeed, if - arguing by contradiction - there is a sequence $r_k \to 0$ such that

$$\limsup_{k\to\infty} \Big[\mu_2(B_g(y_0,r_k),g) - \frac{\mu_2(B)}{r_k^2}\Big] < \alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0), \tag{36}$$

then by the above argument there exists a subsequence along which the expansion (34) holds with some $A \in T_{y_0}\mathcal{M}$ with $|A| = 1$, thus contradicting (36). By (35), the proof of Proposition 3.1 is finished once we have shown that

$$\mu_2(B_g(y_0,r),g) \leq \frac{\mu_2(B)}{r^2} + \alpha_N^- S(y_0) + 2\alpha_N^+ Ric_{y_0}(A,A) + o(1) \tag{37}$$

for all $A \in T_{y_0}\mathcal{M}$ with $|A| = 1$. So now consider $a = (a^1,\dots,a^N) \in \mathbb{R}^N$ arbitrary with $|a| = 1$, and let $A = a^i E_i \in T_{y_0}\mathcal{M}$. We define

$$\tilde{\Phi} : \overline{B} \to \mathbb{R}, \qquad \tilde{\Phi}(x) = \varphi(|x|)\frac{a \cdot x}{|x|}$$

and

$$c_r := \frac{1}{|B|_{g_r}} \int_B \tilde{\Phi} dv_{g_r}.$$

Then, by Lemma 2.2,

$$c_r = \Big(\frac{1}{|B|} + O(r^2)\Big)\Big(\int_B \tilde{\Phi}(x)[1 - \frac{1}{6}Ric_{y_0}(X,X)]dx + O(r^3)\Big) = \Big(\frac{1}{|B|} + O(r^2)\Big)O(r^3) = O(r^3),$$

since the function $x \mapsto \tilde{\Phi}(x)[1 - \frac{1}{6}Ric_{y_0}(X,X)]$ is odd with respect to reflection at the origin. Hence, using the variational characterization of $\mu_2(B,g_r)$, we find that

$$\mu_2(B,g_r) \leq \frac{\int_B |\nabla(\tilde{\Phi} - c_r)|^2_{g_r} \, dv_{g_r}}{\int_B (\tilde{\Phi} - c_r)^2 \, dv_{g_r}} = \frac{\int_B |\nabla\tilde{\Phi}|^2_{g_r} \, dv_{g_r}}{\int_B \tilde{\Phi}^2 + O(r^3) \, dv_{g_r}} = \frac{\int_B |\nabla\tilde{\Phi}|^2_{g_r} \, dv_{g_r} + O(r^3)}{\int_B \tilde{\Phi}^2 \, dv_{g_r}}$$

and therefore

$$\mu_2(B,g_r)\int_B \tilde{\Phi}^2 \, dv_{g_r} \leq \int_{B_g(y_0,r)} |\nabla\tilde{\Phi}|^2_{g_r} \, dv_{g_r} + O(r^3) = \int_B \sqrt{|g_r|} g_r^{ij} \frac{\partial\tilde{\Phi}}{\partial x^i}\frac{\partial\tilde{\Phi}}{\partial x^j} \, dx + O(r^3).$$

It is by now straightforward that the same estimates as above – starting from (29) – hold with both Φ_r and Φ replaced by $\tilde{\Phi}$. We thus obtain (37), as required. ■

Corollary 3.2 *We have*

$$\mu_2(B_g(y_0,r),g) = \left(1 - \beta(y_0)\left(\frac{v}{|B|}\right)^{\frac{2}{N}} + o\left(\frac{v}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(v) \tag{38}$$

as $v = |B_g(y_0,r)|_g \to 0$ *with*

$$\beta(y_0) = \frac{\mu_2(B)S(y_0) - 3N(N+2)\left(\alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0)\right)}{3N(N+2)\mu_2(B)}.$$

Proof. By the volume expansion (23) of geodesic balls we have

$$\Big(\frac{v}{r^N|B|}\Big)^{\frac{2}{N}} = \Big(\frac{|B_g(y_0,r)|_g}{r^N|B|}\Big)^{\frac{2}{N}} = 1 - \frac{1}{3N(N+2)}S(y_0)r^2 + o(r^2)$$
$$= 1 - \frac{1}{3N(N+2)}S(y_0)\Big(\frac{v}{|B|}\Big)^{\frac{2}{N}} + o\Big(\frac{v}{|B|}\Big)^{\frac{2}{N}}$$

as $v = |B_g(y_0,r)|_g \to 0$. Together with Lemma 3.1 this yields

$$\mu_2(B_g(y_0,r),g) = \frac{\mu_2(B)}{r^2} + \alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0) + o(1)$$
$$= \left(\left(\frac{v}{r^N|B|}\right)^{\frac{2}{N}} + \frac{\alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0)}{\mu_2(B)}\left(\frac{v}{|B|}\right)^{\frac{2}{N}} + o\left(\frac{v}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(v)$$
$$= \left(1 - \left(\frac{1}{3N(N+2)}S(y_0) - \frac{\alpha_N^- S(y_0) + 2\alpha_N^+ R_{min}(y_0)}{\mu_2(B)}\right)\left(\frac{v}{|B|}\right)^{\frac{2}{N}} + o\left(\frac{v}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(v)$$
$$= \left(1 - \beta(y_0)\left(\frac{v}{|B|}\right)^{\frac{2}{N}} + o\left(\frac{v}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(v)$$

as $v = |B_g(y_0,r)|_g \to 0$. ■

Remark 3.3 (i) Since

$$R_{min}(y_0) \le \frac{S(y_0)}{N}, \tag{39}$$

and

$$\alpha_N^- + \frac{2\alpha_N^+}{N} = \frac{|B|\varphi^2(1) - (N-2)}{6N}, \tag{40}$$

Proposition 3.1 and Corollary 3.2 yield

$$\mu_2(B_g(y_0,r),g) \le \frac{\mu_2(B)}{r^2} + (\alpha_N^- + \frac{2\alpha_N^+}{N})S(y_0) + o(1) \tag{41}$$
$$= \left(1 - \gamma_N\left(\frac{v}{|B|}\right)^{\frac{2}{N}} S(y_0) + o\left(\frac{v}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(v)$$

as $v = |B_g(y_0,r)|_g \to 0$ (and therefore $r \to 0$) with γ_N as in (3). Notice that when $N = 2$, equality holds in (39) and (41). Therefore the two-dimensional version of (38) is

$$\mu_2(B_g(y_0,r),g) = \left(1 - \gamma_2\frac{v}{|B|}S(y_0) + o\left(\frac{v}{|B|}\right)\right) SW_{\mathbb{R}^2}(v).$$

(ii) Denote by $(\mathbb{M}^N, g_k)$ a space of constant sectional curvature k. Then equality holds in (39) because $Ric = (N-1)k\, g_k$ on $\mathbb{M}^N$. In particular if E is a ball in $(\mathbb{M}^N, g_k)$ with small volume, one has that

$$\mu_2(E,g_k) = \left(1 - \gamma_N\left(\frac{|E|_{g_k}}{|B|}\right)^{\frac{2}{N}} N(N-1)\,k + o\left(\frac{|E|_{g_k}}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(|E|_{g_k}). \tag{42}$$

4 Expansion of μ_2 for small geodesic ellipsoids

As before we fix $y_0 \in \mathcal{M}$, and we continue to assume that the orthonormal basis $E_1,\dots,E_N$ of $T_{y_0}\mathcal{M}$ is chosen such that (25) holds. In the following, we consider

$$\nu_N = \frac{2\mu_2(B) + N|B|\varphi^2(1)}{N+2} > 0,$$

and we let

$$b_i = b^i := \frac{\alpha_N^+}{\nu_N}(R_{ii} - \frac{S(y_0)}{N}) \qquad \text{for } i = 1, \dots, N, \tag{43}$$

where α_N^+ is defined in Proposition 3.1. The reason for this choice will become clear later. We note that $\sum\limits_{i=1}^N b_i = 0$ since $S(y_0) = \sum\limits_{i=1}^N R_{ii}$. For $r > 0$, we now consider the geodesic ellipsoids $E(y_0, r) := F_r(B) \subset \mathcal{M}$, where

$$F_r : B \to \mathcal{M}, \qquad F_r(x) = \mathrm{Exp}_{y_0}\big(r(1 + r^2 b_i)x^i E_i\big).$$

The special choice of the values b_i gives rise to the following asymptotic expansion where the local geometry only enters via the scalar curvature at y_0.

Proposition 4.1 *As $r \to 0$, we have*

$$\mu_2(E(y_0, r), g) = \frac{\mu_2(B)}{r^2} + (\alpha_N^- + \frac{2\alpha_N^+}{N})S(y_0) + o(1), \tag{44}$$

with $\alpha_N^\pm$ as in Proposition 3.1 and

$$|E(y_0, r)|_g = |B_g(y_0, r)|_g + O(r^{N+4}) = r^N |B| \Big(1 - \frac{1}{6(N+2)} r^2 S(y_0) + O(r^4)\Big). \tag{45}$$

Proof. We consider the pull back metric h_r on B of g under the map F_r rescaled with the factor $\frac{1}{r^2}$. Then we have

$$\begin{aligned}[h_r]_{i,j}(x) &= (1 + r^2 b_i)(1 + r^2 b_j)[g_r]_{ij}((1 + r^2 b_k)x^k e_k) = [g_r]_{ij}(x) + r^2(b_i + b_j)\delta_{ij} + O(r^4) \quad (46)\\ &= \delta_{ij} + r^2\Big(\frac{1}{3}\langle R_{y_0}(X, E_i)X, E_j\rangle + (b_i + b_j)\delta_{ij}\Big) + O(r^3)\end{aligned}$$

uniformly in $x \in B$. Setting $|h_r| = \det([h_r]_{ij})_{ij}$, we deduce the expansion

$$|h_r|(x) = |g_r|(x) + 2r^2 \sum_{i=1}^N b_i + O(r^4) = |g_r|(x) + O(r^4) \qquad \text{for } x \in B.$$

This implies that

$$|E(y_0, r)|_g = r^N |B|_{h_r} = r^N \Big(|B|_{g_r} + O(r^4)\Big) = |B_g(y_0, r)|_g + O(r^{N+4}),$$

as claimed in (45).
We now turn to (44). We first note that $\mu_2(B, h_r) = r^2 \mu_2(E(y_0, r), g)$; therefore (44) is equivalent to

$$\mu_2(B, h_r) = \mu_2(B) + r^2(\alpha_N^- + \frac{2\alpha_N^+}{N})S(y_0) + o(r^2). \tag{47}$$

Let Φ_r be an eigenfunction for $\mu_2(B, h_r)$, normalized such that $\int_B \Phi_r^2 \, dv_{h_r} = 1$ with $dv_{h_r} = \sqrt{|h_r|}dx$. Then we have

$$\Delta_{h_r}\Phi_r + \mu_2(B, h_r)\,\Phi_r = 0 \quad \text{in } B, \qquad \langle \nabla \Phi_r, \eta\rangle_{h_r} = 0 \quad \text{on } \partial B,$$

where

$$\Delta_{h_r}\Phi_r = \frac{1}{\sqrt{|h_r|}} \frac{\partial}{\partial x^i}\left(\sqrt{|h_r|} h_r^{ij} \frac{\partial \Phi_r}{\partial x^j}\right).$$

Since h_r converges to the Euclidean metric in B, the variational characterization of μ_2 implies that $\mu_2(B, h_r) \to \mu_2(B)$. Moreover, as in the proof of Proposition 3.1 we have $\Phi_{r_k} \to \Phi$ in $H^1(B)$ along a sequence $r_k \to 0$ with some function $\Phi \in C^2_{loc}(B) \cap C^1(\overline{B})$ satisfying

$$\Delta \Phi + \mu_2(B)\,\Phi = 0 \quad \text{in } B, \qquad \langle \nabla \Phi, \eta\rangle = 0 \quad \text{on } \partial B, \qquad \int_B \Phi^2 \, dx = 1 \quad \text{and} \quad \int_B \Phi \, dx = 0.$$

Hence there exists a vector $a = (a_1, \dots, a_N) = (a^1, \dots, a^N) \in \mathbb{R}^N$ with $|a| = 1$ and such that

$$\Phi(x) = \varphi(|x|)\frac{a \cdot x}{|x|} \qquad \text{for } x \in \overline{B}.$$

For matters of convenience, we will continue to write r instead of r_k in the following. By multiple integration by parts, using (46) and (28), we have

$$\begin{aligned}
\mu_2(B, h_r) \int_B \Phi\Phi_r dv_{h_r} &= -\int_B \Phi\Delta_{h_r}\Phi_r dv_{h_r} = \int_B \sqrt{|h_r|}h_r^{ij}\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi_r}{\partial x^j}\,dx \\
&= \int_B \sqrt{|h_r|}\Big[\nabla\Phi_r\nabla\Phi - r^2\Big(\frac{1}{3}\langle R_{y_0}(X, \tilde\nabla\Phi_r)X, \tilde\nabla\Phi\rangle + 2b^i\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi_r}{\partial x^i}\Big)\Big]\,dx + O(r^3) \\
&= -\int_B \sqrt{|h_r|}(\Delta\Phi)\Phi_r\,dx - \int_B \Phi_r\nabla\sqrt{|h_r|}\cdot\nabla\Phi\,dx - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla\Phi_r)X, \tilde\nabla\Phi\rangle\,dx \\
&\quad - 2r^2\int_B b^i\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi_r}{\partial x^i}\,dx + O(r^3) \\
&= \mu_2(B)\int_B \Phi\Phi_r\,dv_{h_r} + \frac{r^2}{3}\int_B Ric_{y_0}(\tilde\nabla\Phi, X)\Phi_r\,dx - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla\Phi_r)X, \tilde\nabla\Phi\rangle\,dx \\
&\quad - 2r^2\int_B b^i\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi_r}{\partial x^i}\,dx + O(r^3).
\end{aligned}$$

Since $\int_B \Phi\Phi_r\,dv_{h_r} \to 1$ and $\Phi_r \to \Phi$ in $H^1(B)$ as $r \to 0$, we may use the calculations in the proof of Proposition 3.1 starting from (30) to obtain

$$\begin{aligned}
\mu_2(B, h_r) &= \mu_2(B) + \frac{r^2}{3}\int_B Ric_{y_0}(\tilde\nabla\Phi, X)\Phi_r\,dx - \frac{r^2}{3}\int_B \langle R_{y_0}(X, \tilde\nabla\Phi_r)X, \tilde\nabla\Phi\rangle\,dx \\
&\quad - 2r^2\int_B b^i\frac{\partial\Phi}{\partial x^i}\frac{\partial\Phi_r}{\partial x^i}\,dx + o(r^2) \\
&= \mu_2(B) + r^2\Big(\alpha_N^- S(y_0) + 2r^2\alpha_N^+ Ric_{y_0}(A, A) - 2\int_B b^i\Big(\frac{\partial\Phi}{\partial x^i}\Big)^2\,dx\Big) + o(r^2) \qquad (48)
\end{aligned}$$

with $A := a^iE_i \in T_{y_0}\mathcal{M}$. It remains to compute $\int_B b^i\Big(\frac{\partial\Phi}{\partial x^i}\Big)^2\,dx$.
We have $\frac{\partial\Phi}{\partial x^i} = \Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)\frac{x^i}{|x|^2}a_jx^j + a_i\frac{\varphi(|x|)}{|x|}$ and thus

$$\Big(\frac{\partial\Phi}{\partial x^i}\Big)^2 = \frac{1}{|x|^4}\Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)^2 a_ja_k[x^i]^2x^jx^k + a_i^2\frac{\varphi^2(|x|)}{|x|^2} + 2\frac{\varphi(|x|)}{|x|^3}\Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)a_ia_jx^ix^j$$

for $x \in B$ and $i = 1, \dots, N$. Noting the oddness of some of the integrands and passing to polar coordinates, we therefore obtain

$$\begin{aligned}
\int_B b^i\Big(\frac{\partial\Phi}{\partial x^i}\Big)^2\,dx &= b_ia_j^2\int_B \frac{1}{|x|^4}\Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)^2[x^i]^2[x^j]^2\,dx + b^ia_i^2\int_B \frac{\varphi^2(|x|)}{|x|^2}\,dx \\
&\quad + 2b^ia_i^2\int_B \frac{\varphi(|x|)}{|x|^3}\Big(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\Big)[x^i]^2 \\
&= b_ia_j^2\int_0^1 t^{N-1}\Big(\varphi'(t) - \frac{\varphi(t)}{t}\Big)^2\,dt\int_{\partial B}[x^i]^2[x^j]^2\,d\sigma + b^ia_i^2|\partial B|\int_0^1 t^{N-3}\varphi^2\,dt \\
&\quad + 2b^ia_i^2\int_0^1 t^{N-2}\varphi(t)\Big(\varphi'(t) - \frac{\varphi(t)}{t}\Big)\,dt\int_{\partial B}[x^i]^2\,d\sigma. \qquad (49)
\end{aligned}$$

Put $d_N := \int_0^1 t^{N-3}\varphi^2\,dt$. By (13), we have

$$\int_0^1 t^{N-1}(\varphi')^2(t)\,dt = \frac{\mu_2(B)}{|B|} - (N-1)d_N \qquad \text{and} \qquad \int_0^1 t^{N-2}\varphi'(t)\varphi(t)\,dt = \frac{\varphi^2(1)}{2} - \frac{N-2}{2}d_N$$

hence

$$\int_0^1 t^{N-1}\Big(\varphi'(t)-\frac{\varphi(t)}{t}\Big)^2\,dt=\frac{\mu_2(B)}{|B|}-\varphi^2(1)\qquad\text{and}\qquad\int_0^1 t^{N-2}\varphi(t)\Big(\varphi'(t)-\frac{\varphi(t)}{t}\Big)\,dt=\frac{1}{2}\Big(\varphi^2(1)-Nd_N\Big).$$

Inserting this in (49), we get

$$\int_B b^i\Big(\frac{\partial\Phi}{\partial x^i}\Big)^2\,dx=b_ia_j^2\Big(\frac{\mu_2(B)}{|B|}-\varphi^2(1)\Big)\int_{\partial B}[x^i]^2[x^j]^2\,d\sigma+b^ia_i^2|\partial B|d_N+b^ia_i^2\Big(\varphi^2(1)-Nd_N\Big)\int_{\partial B}[x^i]^2\,d\sigma.$$

Recalling furthermore the identities

$$\int_{\partial B}[x^i]^4\,d\sigma=3\int_{\partial B}[x^i]^2[x^j]^2\,d\sigma=\frac{3}{N+2}|B|,\qquad\int_{\partial B}(x^i)^2\,d\sigma=\frac{|\partial B|}{N}=|B|$$

for $i,j=1,\dots,N$, $i\neq j$ and also that $\sum\limits_{i=1}^{N}b_i=0$, we obtain

$$b_i\int_{\partial B}[x^i]^2[x^j]^2\,d\sigma=\frac{2|B|}{N+2}b^j\qquad\text{for } j=1,\dots,N$$

and thus

$$\begin{aligned}\int_B b^i\Big(\frac{\partial\Phi}{\partial x^i}\Big)^2\,dx&=b^ja_j^2\Big(\frac{\mu_2(B)}{|B|}-\varphi^2(1)\Big)\frac{2|B|}{N+2}+b^ia_i^2N|B|d_N+b^ia_i^2\Big(\varphi^2(1)-Nd_N\Big)|B|\\&=b^ia_i^2\,\frac{2\mu_2(B)+N|B|\varphi^2(1)}{N+2}=\nu_N\,b^ia_i^2.\end{aligned}$$

Inserting this in (48), we obtain

$$\begin{aligned}\mu_2(B,h_r)=\;&\mu_2(B)+r^2\Big[\alpha_N^-S(y_0)+2(\alpha_N^+Ric_{y_0}(A,A)-\nu_Nb^ia_i^2)\Big]+o(r^2)\\=\;&\mu_2(B)+r^2\Big[(\alpha_N^-+\frac{2a_N^+}{N})S(y_0)+2(\alpha_N^+(Ric_{y_0}(A,A)-\frac{S(y_0)}{N})-\nu_Nb^ia_i^2)\Big]+o(r^2),\end{aligned}$$

where

$$\alpha_N^+(Ric_{y_0}(A,A)-\frac{S(y_0)}{N})-\nu_Nb^ia_i^2=[a^i]^2\Big(\alpha_N^+(R_{ii}-\frac{S(y_0)}{N})-\nu_Nb_i\Big)=0$$

by our choice of the $b_i=b^i$ in (43). This shows (47), as required. ■

Corollary 4.2 *We have*

$$\mu_2(E(y_0,r),g)=\left(1-\gamma_N\left(\frac{v}{|B|}\right)^{\frac{2}{N}}+o\left(\frac{v}{|B|}\right)^{\frac{2}{N}}\right)SW_{\mathbb{R}^N}(v),\tag{50}$$

as $v=|E(y_0,r)|_g\to 0$ with γ_N as in (3).

Proof. This follows readily by combining (40), (44) and (45). ■

5 A local upper bound for μ_2

We fix $r_0>0$ less than the convexity radius of $\mathcal{M}$ at y_0, so that r_0 is also less than the injectivity radius of $\mathcal{M}$ at y_0. As in [14], we consider the function

$$G:\mathbb{R}\to\mathbb{R},\qquad G(t)=\begin{cases}\varphi(t)&\text{if } t\le 1,\\ \varphi(1)&\text{if } t>1,\end{cases}$$

where φ is the function defined in Section 2. Throughout this section, we consider a sequence of numbers $r_k \in (0, \frac{r_0}{3})$ such that $r_k \to 0$ as $k \to \infty$, and we suppose that we are given regular domains $\Omega_{r_k} \subset B_g(y_0, r_k)$, $k \in \mathbb{N}$. In order to keep the notation as simple as possible, we will write r instead of r_k in the following. By [1, Theorem 3], there exists a point $p_r \in B_g(y_0, r)$ such that

$$\int_{\Omega_r} \frac{G(|\mathrm{Exp}_{p_r}^{-1}(q)|_g)}{|\mathrm{Exp}_{p_r}^{-1}(q)|_g} \mathrm{Exp}_{p_r}^{-1}(q)\, dv_g = 0. \tag{51}$$

Moreover, there exists a unique $\rho_r \in (0, r)$ such that that $|\Omega_r|_g = |B_g(p_r, \rho_r)|_g$. We have that, for every $r > 0$ small, $B_g(p_r, \rho_r) \subset B_g(y_0, 2r)$ and also $\Omega_r \subset B_g(p_r, 2r)$. Now we need to extend some of the notations introduced in Section 2. For this we let

$$y \mapsto E_i^y \in T_y\mathcal{M}, \qquad i = 1, \dots, N$$

denote a smooth orthonormal frame on $B_g(y_0, r_0)$, and we define

$$\Psi_r : \mathbb{R}^N \to \mathcal{M}, \qquad \Psi_r(x) = \mathrm{Exp}_{p_r}(x^i E_i^y).$$

We also define

$$B^r := \frac{2r}{\rho_r} B \qquad \text{and} \qquad U_r := \frac{1}{\rho_r}\, \Psi_r^{-1}(\Omega_r) \subset B^r, \tag{52}$$

and we consider the pull back metric of g under the map $B^r \to \mathcal{M},\ x \mapsto \Psi_r(\rho_r x)$, rescaled with the factor $\frac{1}{\rho_r^2}$. We denote this metric on B^r by g_r, and we point out that this definition differs from the notation used in the proof of Proposition 3.1. By (51), it is plain that

$$\int_{U_r} \frac{G(|x|)}{|x|} x^i dv_{g_r} = 0 \qquad \text{for } i = 1, \dots, N. \tag{53}$$

We also write

$$R^r_{ijkl} := \langle R_y(E_i^{p_r}, E_j^{p_r})E_k^{p_r}, E_l^{p_r}\rangle \quad \text{and} \quad R^r_{ij} := Ric_{p_r}(E_i^{p_r}, E_j^{p_r})$$

for $i, j, k, l = 1, \dots, N$. To be consistent with the notation introduced in the end of Section 2, we also write

$$R_{ijkl} := \langle R_{y_0}(E_i^{y_0}, E_j^{y_0})E_k^{y_0}, E_l^{y_0}\rangle \quad \text{and} \quad R_{ij} := Ric_{y_0}(E_i^{y_0}, E_j^{y_0}).$$

Since $\mathrm{dist}(p_r, y_0) = O(r)$, we then have

$$R^r_{ijkl} = R_{ijkl} + O(r) \quad \text{and} \quad R^r_{ij} = R_{ij} + O(r) \qquad \text{for } i, j, k, l = 1, \dots, N. \tag{54}$$

By Lemma 2.2 we also have

$$\begin{aligned} (g_r)_{ij}(x) &= \delta_{ij} + \tfrac{\rho_r^2}{3} R^r_{kilj} x^k x^l + |x|^3 O(\rho_r^3); \\ dv_{g_r}(x) = \sqrt{|g_r(x)|}\, dx &= \Big(1 - \tfrac{\rho_r^2}{6} R^r_{lk} x^l x^k + |x|^3 O(\rho_r^3)\Big)\, dx, \end{aligned} \tag{55}$$

uniformly on B^r, where $|g_r|$ is the determinant of g_r, so in particular

$$(g_r)_{ij}(x) = \delta_{ij} + O(r^2) \quad \text{and} \quad dv_{g_r}(x) = (1 + O(r^2))dx \qquad \text{uniformly on } B^r. \tag{56}$$

Observe that

$$|U_r|_{g_r} = {\rho_r}^{-N} |\Omega_r|_g = {\rho_r}^{-N} |B_g(p_r, \rho_r)|_g = |B|_{g_r} \qquad \text{and} \qquad \mu_2(U_r, g_r) = \frac{\mu_2(\Omega_r,\, g)}{\rho_r^2}. \tag{57}$$

Moreover, since $U_r \subset B^r$ and $B \subset B^r$, we infer from (56) that

$$|U_r| = (1 + O(r^2))|U_r|_{g_r} = (1 + O(r^2))|B|_{g_r} = (1 + O(r^2))|B|. \tag{58}$$

Setting

$$f_i : \mathbb{R}^N \to \mathbb{R}, \qquad f_i(x) = \frac{G(|x|)}{|x|} x^i,$$

we find that $\int_{U_r} f_i dv_{g_r} = 0$ for $i = 1, \dots, N$ by (53), and hence the variational characterization of μ_2 yields

$$\mu_2(U_r, g_r) \leq \frac{\sum_{i=1}^N \int_{U_r} |\nabla f_i|^2_{g_r} dv_{g_r}}{\sum_{i=1}^N \int_{U_r} f_i^2 dv_{g_r}}. \tag{59}$$

We also note that

$$\frac{\partial f_i}{\partial x^k} = \frac{G'}{|x|^2} x^i x^k + \frac{G}{|x|}[\delta_{ik} - \frac{x^i x^k}{|x|^2}] \qquad \text{for every } i, k = 1, \dots, N \tag{60}$$

and, by direct calculation as in [14],

$$\sum_{i=1}^N f_i^2 = G^2, \qquad \sum_{i=1}^N |\nabla f_i|^2 = (G')^2 + (N-1)\frac{G^2}{|x|^2}. \tag{61}$$

Here and in the following, we simply write G instead of $G(|\cdot|)$ or $G(|x|)$ and G' instead of $G'(|\cdot|)$ or $G'(|x|)$ if the meaning is clear from the context. In particular, using (61), (14) and recalling that φ and G coincide in $[0, 1]$, we observe that

$$\begin{aligned} \int_B \Big((G')^2 + (N-1)\frac{G^2}{|x|^2}\Big)\, dx = \sum_{i=1}^N \int_B |\nabla f_i|^2\, dx = \mu_2(B) \sum_{i=1}^N \int_B |f_i|^2\, dx \\ = \mu_2(B) \int_B \varphi^2(|x|)\, dx = N\mu_2(B). \end{aligned} \tag{62}$$

Lemma 5.1 *In the above setting, we have*

$$\mu_2(U_r, g_r) \leq \frac{\int_{U_r} \left((G')^2 + (N-1)\frac{G^2}{|x|^2}\right) dv_{g_r} + \frac{\rho_r^2}{3}\int_{U_r} \frac{G^2}{|x|^2} R^r_{lk} x^l x^k\, dv_{g_r}}{\int_{U_r} G^2\, dv_{g_r}} + O(r\rho_r^2). \tag{63}$$

as $r \to 0$. Moreover,

$$(1 + O(r^2))\mu_2(U_r, g_r) \leq \mu_2(U_r) \tag{64}$$

and

$$\int_{U_r} G^2\, dv_{g_r} \geq N - \frac{|B|\rho_r^2}{6} S(y_0) \int_0^1 \varphi^2 t^{N+1} dt + O(r\rho_r^2). \tag{65}$$

Proof. We start by proving (65). Clearly

$$\int_{U_r} G^2\, dv_{g_r} = \int_B G^2\, dv_{g_r} + \int_{U_r \setminus (U_r \cap B)} G^2\, dv_{g_r} - \int_{B \setminus (U_r \cap B)} G^2\, dv_{g_r}.$$

Using (57), the fact that G is non-decreasing and that $G = \varphi(1)$ on $U_r \setminus (U_r \cap B)$ we get

$$\int_{U_r} G^2\, dv_{g_r} \geq \int_B G^2\, dv_{g_r} + \varphi(1)^2 \left(|U_r \setminus (U_r \cap B)|_{g_r} - |B \setminus (U_r \cap B)|_{g_r}\right) = \int_B G^2\, dv_{g_r}.$$

Now by (55) and (14) we have

$$\begin{aligned} \int_B G^2\, dv_{g_r} &= \int_B \varphi^2(|x|)\, dx - \frac{\rho_r^2}{6} \int_0^1 \varphi^2 t^{N+1} dt \int_{\partial B} R^r_{lk} x^l x^k d\sigma + O(\rho_r^3) \\ &= N - \frac{|B|\rho_r^2}{6} S(y_0) \int_0^1 \varphi^2 t^{N+1} dt + O(r\rho_r^2). \end{aligned}$$

From this we conclude

$$\int_{U_r} G^2\, dv_{g_r} \geq N - \frac{|B|\rho_r^2}{6} S(y_0) \int_0^1 \varphi^2 t^{N+1} dt + O(r\rho_r^2),$$

so that (65) holds. Moreover, by (55) we have

$$|\nabla f_i|^2_{g_r} = |\nabla f_i|^2 - \frac{\rho_r^2}{3} R^r_{jklm} x^j \frac{\partial f_i}{\partial x^k} x^l \frac{\partial f_i}{\partial x^m} + O\left(\rho_r^3 |x|^3 |\nabla f_i|^2\right),$$

in B^r. Using furthermore that, by general properties of the Riemannian curvature tensor, $R^r_{jklm} x^j x^k = 0$ for every l, m and $R^r_{jklm} x^l x^m = 0$ for every j, k, we obtain

$$R^r_{jklm} x^j \frac{\partial f_i}{\partial x^k} x^l \frac{\partial f_i}{\partial x^m} = \frac{G^2}{|x|^2} R^r_{jili} x^j x^l$$

by (60). Therefore, summing over i and using (61), we find that

$$\sum_{i=1}^N |\nabla f_i|^2_{g_r} = (G')^2 + (N-1)\frac{G^2}{|x|^2} + \frac{\rho_r^2 G^2}{3|x|^2} R^r_{jl} x^j x^l + O\left(\rho_r^3 |x|^3 \left((G')^2 + (N-1)\frac{G^2}{|x|^2}\right)\right). \quad (66)$$

To estimate the last term in (66), we first note that, since $G' \equiv 0$ in $U_r \setminus (U_r \cap B) \subset B^r$,

$$\begin{aligned}\rho_r^3 \int_{U_r \setminus (U_r \cap B)} |x|^3 \left((G')^2 + (N-1)\frac{G^2}{|x|^2}\right) dv_{g_r} &= \rho_r^3 G(1)^2 \int_{U_r \setminus (U_r \cap B)} \frac{|x|^3}{|x|^2} dv_{g_r} \\ &\leq r\rho_r^2 G(1)^2 |U|_{g_r} = O(r\rho_r^2).\end{aligned}$$

Moreover, since $|x| \leq 1$ in B we have

$$\rho_r^3 \int_B |x|^3 \left((G')^2 + (N-1)\frac{G^2}{|x|^2}\right) dv_{g_r} = O(\rho_r^3) = O(r\rho_r^2).$$

Hence we deduce that

$$\int_{U_r} O\left(\rho_r^3 |x|^3 \left((G')^2 + (N-1)\frac{G^2}{|x|^2}\right)\right) dv_{g_r} = O(r\rho_r^2). \quad (67)$$

Now (63) follows immediately from (59), (61), (65), (66) and (67). Combining (63) and (65), we also deduce that

$$\mu_2(U_r, g_r) \leq C + O(r^2) \qquad \text{as } r \to 0 \text{ with a constant } C > 0. \quad (68)$$

Now to prove (64), we consider a normalized eigenfunction h_r corresponding to $\mu_2(U_r)$, i.e. $h_r \in H^1(U_r)$ satisfies

$$\int_{U_r} h_r^2\, dx = 1, \qquad \int_{U_r} h_r\, dx = 0 \qquad \text{and} \qquad \int_{U_r} |\nabla h_r|^2\, dx = \mu_2(U_r).$$

By (56) and (58) we then have

$$\Big|\frac{1}{|U_r|_{g_r}} \int_{U_r} h_r\, dv_{g_r}\Big| = O(r^2)\frac{1}{|U_r|_{g_r}} \int_{U_r} |h_r|\, dx \leq O(r^2)\frac{\sqrt{|U_r|}}{|U_r|_{g_r}} = O(r^2),$$

With $c_r = \frac{1}{|U_r|_{g_r}} \int_{U_r} h_r\, dv_{g_r}$ we therefore deduce

$$\int_{U_r} (h_r - c_r)^2\, dv_{g_r} = \int_{U_r} (h_r + O(r^2))^2 (1 + O(r^2))\, dx = 1 + O(r^2).$$

Therefore the variational characterization of $\mu_2(U_r, g_r)$ yields

$$\mu_2(U_r, g_r) \leq \frac{1}{1 + O(r^2)} \int_{U_r} |\nabla h_r|^2_{g_r}\, dv_{g_r} = (1+O(r^2)) \int_{U_r} |\nabla h_r|^2 (1+O(r^2))\, dx = (1+O(r^2))\mu_2(U_r),$$

and (64) follows. ■

The following lemma controls the symmetric distance between B and U_r with the help of a recent stability estimate of Brasco and Pratelli [3] for μ_2 in the euclidean setting.

Lemma 5.2 *Assume that* $\mu_2(U_r, g_r) \geq \mu_2(B)(1 + o(1))$ *as* $r \to 0$ *for the family of domains* U_r *defined in* (52)*, and let* $U_r \triangle B = (U_r \cup B) \setminus (U_r \cap B)$*. Then*

$$|U_r \triangle B| \to 0 \qquad \text{as } r \to 0. \tag{69}$$

PROOF. We consider the rescaled set $U_r' = (1+\delta(r))U_r$, where $\delta(r)$ is chosen such that $|U_r'| = |B|$. Then $\delta(r) = O(r^2)$ by (58). By (64) and by assumption, we see that

$$\mu_2(U_r') = (1 + \delta(r))^{-2}\mu_2(U_r) \geq (1 + O(r^2))\mu_2(U_r, g_r) \geq \mu_2(B)(1 + o(1)) \qquad \text{as } r \to 0,$$

whereas $\mu_2(U_r') \leq \mu(B)$ by Weinberger's result [14]. By [3, Theorem 4.1], there exist points $x_r \in \mathbb{R}^N$ such that

$$|U_r' \triangle B(x_r)|^2 \leq C(\mu_2(B) - \mu_2(U_r')) \to 0 \qquad \text{as } r \to 0 \tag{70}$$

with some constant $C > 0$, where $B(x_r)$ stands for the ball in $\mathbb{R}^N$ centered at x_r with radius 1. Since $\delta(r) = O(r^2)$, it is easy to see that

$$\lim_{r\to0} |U_r \triangle B(x_r)| = \lim_{r\to0} |U_r' \triangle B(x_r)| = 0. \tag{71}$$

Consequently, (69) follows once we have shown that $x_r \to 0$ as $r \to 0$. So we suppose by contradiction that, after passing to a subsequence, $\inf_r |x_r| > 0$ and $\frac{x_r}{|x_r|} \to x_0$ as $r \to 0$ for some $x_0 \in \mathbb{R}^N$ with $|x_0| = 1$. From (53), (56) and (71), we then infer that

$$\int_B G(|x + x_r|)\frac{x + x_r}{|x + x_r|} \cdot x_0 \, dx = \int_{B(x_r)} G(|x|)\frac{x}{|x|} \cdot x_0 \, dx = \int_{U_r} G(|x|)\frac{x}{|x|} \cdot x_0 \, dx + o(1) \to 0$$

as $r \to 0$. If $|x_r| \to \infty$ for a subsequence, it would follow by the definition of G that

$$\int_B \frac{x + x_r}{|x + x_r|} \cdot x_0 \, dx \to 0 \qquad \text{as } r \to 0,$$

whereas, on the other hand, $\frac{x+x_r}{|x+x_r|} \to x_0$ uniformly on B. This is impossible, so we conclude that the sequence x_r is bounded and therefore, along a subsequence, $x_r \to \tilde{x} \neq 0$ as $r \to 0$ for some $\tilde{x} \in \mathbb{R}^N \setminus \{0\}$. Using (53), (56) and (71) similarly as before, we now infer that

$$\int_B G(|x + \tilde{x}|)\frac{x + \tilde{x}}{|x + \tilde{x}|} \cdot \tilde{x} \, dx = 0 \tag{72}$$

Let $D := \{x \in B \; : \; x \cdot \tilde{x} > 0\}$, and let $\sigma : B \to B$ denote the reflection at the hyperplane $\{x \in \mathbb{R}^N \; : \; x \cdot \tilde{x} = 0\}$ given by $\sigma(x) = x - 2x \cdot \frac{\tilde{x}}{|\tilde{x}|^2}\tilde{x}$. Elementary geometric considerations show that

$$|x + \tilde{x}| > |\sigma(x) + \tilde{x}| \qquad \text{and} \qquad \frac{x + \tilde{x}}{|x + \tilde{x}|} \cdot \tilde{x} > \Big|\frac{\sigma(x) + \tilde{x}}{|\sigma(x) + \tilde{x}|} \cdot \tilde{x}\Big| \qquad \text{for } x \in D \setminus \mathbb{R}\tilde{x}.$$

Since $G(|x|)$ is nondecreasing in $|x|$ and positive for $x \neq 0$, we conclude by a change of variable that

$$\int_B G(|x + \tilde{x}|)\frac{x + \tilde{x}}{|x + \tilde{x}|} \cdot \tilde{x} \, dx = \int_D \Big[G(|x + \tilde{x}|)\frac{x + \tilde{x}}{|x + \tilde{x}|} \cdot \tilde{x} + G(|\sigma(x) + \tilde{x}|)\frac{\sigma(x) + \tilde{x}}{|\sigma(x) + \tilde{x}|} \cdot \tilde{x}\Big] \, dx > 0,$$

contradicting (72). The contradiction shows that $x_r \to 0$ as $r \to 0$, which, as remarked before, yields the claim. ■

Lemma 5.3 *Assume that*

$$|U_r \triangle B| \to 0 \qquad \text{as } r \to 0 \tag{73}$$

for the family of domains U_r *defined in* (52)*. Then*

$$\mu_2(U_r, g_r) \leq \left(1 - \frac{N - 2 - |B|\varphi(1)^2}{6N\mu_2(B)}\rho_r^2 \, S(y_0) + o(\rho_r^2)\right) \mu_2(B) \qquad \text{as } r \to 0. \tag{74}$$

Proof. We shall estimate the terms in (63) to reach the upper bound (74). First note that, by (73), (54) and (14),

$$\int_{U_r}\frac{G^2}{|x|^2}R^r_{lk}x^lx^k dv_{g_r}=\int_B\frac{G^2}{|x|^2}R^r_{lk}x^lx^k dv_{g_r}+o(1)=R_{kk}\int_B\varphi^2(|x|)\frac{[x^k]^2}{|x|^2}\,dx+o(1)$$
$$=S(y_0)+o(1). \tag{75}$$

Since, as noted in [14, p. 636], the mapping $|x|\mapsto(G')^2(|x|)+(N-1)\frac{G(|x|)^2}{|x|^2}$ is non-increasing, we have by (57)

$$\int_{U_r}\left((G')^2+(N-1)\frac{G^2}{|x|^2}\right)dv_{g_r}=\int_B\dots dv_{g_r}+\int_{U_r\setminus(U_r\cap B)}\dots dv_{g_r}-\int_{B\setminus(U_r\cap B)}\dots dv_{g_r}$$
$$\le\int_B\left((G')^2+(N-1)\frac{G^2}{|x|^2}\right)dv_{g_r}. \tag{76}$$

Moreover, using (55) and (62), we compute

$$\int_B\left((G')^2+(N-1)\frac{G^2}{|x|^2}\right)dv_{g_r}=\int_B\left((G')^2+(N-1)\frac{G^2}{|x|^2}\right)\left(1-\frac{\rho_r^2}{6}R^r_{lk}x^lx^k+|x|^3O(\rho_r^3)\right)dx$$
$$=\int_B\left((G')^2+(N-1)\frac{G^2}{|x|^2}\right)dx-\frac{\rho_r^2}{6}\int_{\partial B}R^r_{lk}x^lx^k\,d\sigma\int_0^1\left((\varphi')^2+(N-1)\frac{\varphi^2}{t^2}\right)t^{N+1}dt+O(\rho_r^3)$$
$$=N\mu_2(B)-\frac{|B|\rho_r^2}{6}S(y_0)\int_0^1\left((\varphi')^2+(N-1)\frac{\varphi^2}{t^2}\right)t^{N+1}dt+o(\rho_r^2).$$

Notice that, by (13),

$$\int_0^1\left((\varphi')^2+(N-1)\frac{\varphi^2}{t^2}\right)t^{N+1}dt=\frac{1}{|B|}\int_B\varphi^2(|x|)dx-\varphi(1)^2+\mu_2(B)\int_0^1\varphi^2t^{N+1}dt$$
$$=\frac{N}{|B|}-\varphi(1)^2+\mu_2(B)\int_0^1\varphi^2t^{N+1}dt$$

The two equalities above and (76) yield

$$\int_{U_r}\left((G')^2+(N-1)\frac{G^2}{|x|^2}\right)dv_{g_r}\le N\mu_2(B)-\frac{\rho_r^2N}{6}S(y_0)+\frac{|B|\varphi(1)^2}{6}\rho_r^2S(y_0)$$
$$-\frac{|B|\rho_r^2}{6}\mu_2(B)S(y_0)\int_0^1\varphi^2t^{N+1}dt+o(\rho_r^2).$$

Combining this with (65) and (75), we obtain

$$\mu_2(U_r,g_r)\le\mu_2(B)-\frac{N-2-|B|\varphi(1)^2}{6N}\rho_r^2S(y_0)+o(\rho_r^2)$$

and the proof is complete. ■

6 Proof of the main result

In this section we complete the proof of Theorem 1.1. Part (i) follows immediately from Corollary 4.2, and the lower bound in Part (ii) is a direct consequence of Part (i). Hence it remains to prove the upper bound in Part (ii). For this we assume by contradiction that there exists $\varepsilon_0>0$ and sequences of numbers $r_k>0$ and $v_{r_k}\in(0\,,\,|B_g(y_0,r_k)|_g)$ such that $r_k\to0$ as $k\to\infty$ and

$$SW_{B_g(y_0,r_k)}(v_{r_k})>\left(1-(\gamma_NS(y_0)-\varepsilon_0)\left(\frac{v_{r_k}}{|B|}\right)^{\frac{2}{N}}\right)SW_{\mathbb{R}^N}(v_{r_k}).$$

Then there exist regular domains $\Omega_{r_k} \subset B_g(y_0, r_k)$ with $|\Omega_{r_k}|_g = v_{r_k}$ and such that

$$\mu_2(\Omega_{r_k}, g) > \left(1 - (\gamma_N S(y_0) - \varepsilon_0)\left(\frac{v_{r_k}}{|B|}\right)^{\frac{2}{N}}\right) SW_{\mathbb{R}^N}(v_{r_k}). \tag{77}$$

As in Section 5, we write r instead of r_k in the following. We obtain $p_r \in B_g(y_0, r)$ such that (51) holds and we define ρ_r, g_r and U_r accordingly as above. It is easy to see from (77) and the scale invariance of $\Omega \mapsto |\Omega|_g^{\frac{2}{N}} \mu_2(\Omega, g)$ that

$$|U_r|_{g_r}^{\frac{2}{N}} \mu_2(U_r, g_r) > \left(1 - (\gamma_N S(y_0) - \varepsilon_0)\left(\frac{|B_g(p_r, \rho_r)|_g}{|B|}\right)^{\frac{2}{N}}\right) |B|^{\frac{2}{N}} \mu_2(B). \tag{78}$$

By (23) and (57), we also find

$$\Big(\frac{|B|}{|U_r|_{g_r}}\Big)^{\frac{2}{N}} = \Big(\frac{\rho_r^N |B|}{|B_g(p_r, \rho_r)|_g}\Big)^{\frac{2}{N}} = 1 + \frac{1}{3N(N+2)} S(y_0)\rho_r^2 + o(\rho_r^2)$$

and

$$\Big(\frac{|B_g(p_r, \rho_r)|_g}{|B|}\Big)^{\frac{2}{N}} = \rho_r^2 + o(\rho_r^2).$$

Combining this with (78), we obtain

$$\begin{aligned} \mu_2(U_r, g_r) &> \left(1 + \Big(\big[\frac{1}{3N(N+2)} - \gamma_N\big] S(y_0) + \varepsilon_0\Big)\rho_r^2 + o(\rho_r^2)\right) \mu_2(B) \\ &= \left(1 - \Big(\frac{N - 2 - |B|\varphi(1)^2}{6N\mu_2(B)} S(y_0) - \varepsilon_0\Big)\rho_r^2 + o(\rho_r^2)\right) \mu_2(B) \end{aligned} \tag{79}$$

and in particular $\mu_2(U_r, g_r) \geq \mu_2(B)(1 + o(1))$ as $r \to 0$. From this, we can apply Lemma 5.2 to get that

$$|U_r \triangle B| \to 0 \quad \text{as } r \to 0.$$

Therefore by Lemma 5.3 we get

$$\mu_2(U_r, g_r) \leq \left(1 - \frac{N - 2 - |B|\varphi(1)^2}{6N\mu_2(B)} S(y_0)\, \rho_r^2 + o(\rho_r^2)\right) \mu_2(B) \qquad \text{as } r \to 0,$$

and this contradicts (79). Hence Theorem 1.1 is proved.

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