REGIONAL FRACTIONAL LAPLACIANS: BOUNDARY REGULARITY

MOUHAMED MOUSTAPHA FALL

Abstract. We study boundary regularity for solutions to a class of equations involving the so called regional fractional Lapacians $(-\Delta)^s_\Omega$, with $\Omega \subset \mathbb{R}^{\mathbb{N}}$. Recall that the regional fractional Laplacians are generated by symmetric stable processes which are not allowed to jump outside Ω. We consider weak solutions to the equation $(-\Delta)^s_\Omega w(x) = p.v. \int_\Omega \frac{w(x)-w(y)}{|x-y|^{N+2s}}\, dy = f(x)$, for $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$, subject to zero Neumann or Dirichlet boundary conditions. The boundary conditions are defined by considering w as well as the test functions in the fractional Sobolev spaces $H^s(\Omega)$ or $H^s_0(\Omega)$ respectively. While the interior regularity is well understood for these problems, little is known in the boundary regularity, mainly for the Neumann problem. Under optimal regularity assumptions on Ω and provided $f \in L^p(\Omega)$, we show that $w \in C^{2s-N/p}(\overline{\Omega})$ in the case of zero Neumann boundary conditions. As a consequence for $2s - N/p > 1$, $w \in C^{1,2s-\frac{N}{p}-1}(\overline{\Omega})$. As what concerned the Dirichlet problem, we obtain $w/\delta^{2s-1} \in C^{1-N/p}(\overline{\Omega})$, provided $p > N$ and $s \in (1/2, 1)$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. To prove these results, we first classify all solutions, with some growth control, when Ω is a half-space and the right hand side is zero. We then carry over a fine blow up and some compactness arguments to get the results.

1. Introduction

We consider Ω an open subset of $\mathbb{R}^N$ with Lipschitz boundary. The present paper is concerned with boundary regularity of solutions to some equations involving nonlocal operators generated by symmetric stable processes describing motions of random particles in a region Ω which are only allowed to jump inside Ω but are either reflected in Ω or killed when they reach the boundary $\partial\Omega$. In the Brownian case these phenomenon can be described by the Laplace operator subject to Neumann or Dirichlet boundary conditions. A natural generalization in the fractional setting was considered by Bogdan, Burdzy and Chen in [4], where they constructed censored processes in Ω. The generated infinitesimal operators, denoted by $(-\Delta)^s_\Omega$, are obtained by limiting the integral in the fractional Laplacian in the region Ω. Recall that the fractional Laplacian of a function $w \in C^2_c(\mathbb{R}^N)$ is given by

$$(-\Delta)^s w(x) = c_{N,s} p.v. \int_{\mathbb{R}^N} \frac{w(x)-w(y)}{|x-y|^{N+2s}}\, dx, \tag{1.1}$$

where $s \in (0,1)$ and $c_{N,s} = \frac{s4^s\Gamma(\frac{N}{2}+s)}{\pi^{N/2}\Gamma(1-s)}$. The regional fractional Laplacian operator we are interested in is defined as

$$(-\Delta)^s_\Omega w(x) = c_{N,s} p.v. \int_{\Omega} \frac{w(x)-w(y)}{|x-y|^{N+2s}}\, dx. \tag{1.2}$$

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We mention that the meaning of boundary conditions has to be made more precise since in some case, depending on the fractional power s, it can be meaningless. However by considering some natural Sobolev spaces these problems can be defined in a natural manner. Indeed, let $f \in L^1_{loc}(\mathbb{R}^N)$ and $s \in (0,1)$. We consider functions $u \in H^s(\Omega)$ satisfying

$$\frac{c_{N,s}}{2}\int_{\Omega\times\Omega}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy=\int_\Omega f(x)\varphi(x)\,dx \qquad \text{for all } \varphi\in C^1_c(\overline{\Omega}) \quad (1.3)$$

and $v \in H^s_0(\Omega)$ satisfying

$$\frac{c_{N,s}}{2}\int_{\Omega\times\Omega}\frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dxdy=\int_\Omega f(x)\varphi(x)\,dx \qquad \text{for all } \varphi\in C^1_c(\Omega). \quad (1.4)$$

Here and in the following $H^s(\Omega)$ denotes the usual fractional Sobolev space and $H^s_0(\Omega)$ is the closure of $C^1_c(\Omega)$ with respect to the $H^s(\Omega)$-norm. The above two problems have an interesting intersection. Indeed, since Ω is Lipschitz, if $s \in (0,1/2]$ then $H^s(\Omega) = H^s_0(\Omega)$, see e.g. [19, Theorem 1.4.2.4]. However these two spaces are different for $s > 1/2$. As a consequence (1.3) and (1.4) are equivalent for $s \in (0,1/2]$.
We notice that for $s \in (0,1)$, $f \in L^2(\Omega)$, with $\int_\Omega f(x)\,dx = 0$, and Ω bounded then (1.3) has a unique solution among the set of functions $u \in H^s(\Omega)$ satisfying $\int_\Omega u(x)\,dx = 0$. On other hand, provided $s \in (1/2,1)$ and Ω is bounded, the Dirichlet problem (1.4) has unique solution in $H^s_0(\Omega)$ for all $f \in L^2(\Omega)$.
The Dirichlet forms associated to the two variational problems (1.3) and (1.4) generate some fractional order operators, which we denote by $(-\Delta)^s_{\overline{\Omega}}$ and $(-\Delta)^s_\Omega$, and are given by (1.2) when acting on smooth functions. Problems involving these fractional order operators have been intensively studied in the recent years both in the analytic and probabilistic point of view, [1, 4, 21–23]. We quote in particular the paper [21] which contains useful results and integration by parts formula related to $(-\Delta)^s_{\overline{\Omega}}$.
In the recent literature the terminology used for these operators are, respectively, *Reflected or Regional fractional Laplacian* and *Censored fractional Laplacian* [4, 5, 21]. Recall that their counterparts in the local case ($s = 1$) are, respectively, the Poisson problems with Neumann and Dirichlet boundary conditions. In fact, thanks to the normalization constant $c_{N,s}$, as $s \to 1$ equations (1.5) and (1.6) tend precisely to the classical Poisson problem with Neumann and Dirichlet boundary conditions, respectively, see also Section 8 below.
Here and in the following, whenever necessary, we will simply write

$$(-\Delta)^s_{\overline{\Omega}}u = f \qquad \text{in } \Omega, \tag{1.5}$$

for the variational equation (1.3) and

$$\begin{cases}(-\Delta)^s_\Omega v = f & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega,\end{cases} \tag{1.6}$$

for (1.4). According to the above discussions, we shall consider (1.6) only for $s \in (1/2,1)$, and we note that it makes sense to invoke the trace of $v \in H^s_0(\Omega)$ on $\partial\Omega$ thanks to the trace theorem, see e.g. [13].
The aim of this paper is to study boundary regularity for solutions u and v to (1.5) and (1.6). The interior regularity of both problems are well understood and can be naturally deduced from the one of the fractional Laplacian. Indeed, both $u1_\Omega$ and $v1_\Omega$ solve a problem of the form

$$(-\Delta)^s w + Vw = f \qquad \text{in } \Omega, \tag{1.7}$$

with $V \in C^\infty(\Omega)$. On the other hand in the case of the Dirichlet problem, boundary Harnack inequalities and Green function estimates are well studied for $s > 1/2$, see [4, 5, 10, 11]. In fact, provided Ω is of class $C^{1,1}$ and $f \in L^\infty(\Omega)$, it is known that any solution v to (1.6) satisfies

$$|v(x)| \leq C\delta(x)^{2s-1} \quad \text{for } x \in \Omega, \tag{1.8}$$

where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$. However, the boundary regularity for problem (1.5) is less understood and the only paper, beside the present one, dealing with this appeared few days ago. Indeed, Audrito, Felipe-Navarro and Ros-Oton showed recently in [2] that if Ω is of class C^1 and $f \in L^p(\Omega)$, $p > \frac{N}{2s}$, then $u \in C^\alpha(\overline{\Omega})$, for some $\alpha > 0$. Moreover for $s \in (1/2, 1)$ and $2s - N/p > 2s - 1$ they obtain $u \in C^{2s-1+\alpha}(\overline{\Omega})$, for some $\alpha > 0$. We would like to mention that the authors in [2] considered an other interesting fractional order equation with exterior Neumann condition.

We prove in this paper that, under optimal regularity assumptions on Ω and f, the solutions to (1.5) and (1.6) are Hölder continuous up to the boundary. Our result for solutions to (1.5), improves those obtained in [2], since under the same assumptions, we obtain that $u \in C^{\min(2s-N/p,1-\varepsilon)}(\overline{\Omega})$, for all $s, \varepsilon \in (0, 1)$. Moreover for $2s > 1$, we also obtain Hölder estimates, up to the boundary, of the gradient of both solutions to (1.5) and (1.6), provided $2s - N/p > 1$. We then derive further qualitative properties of u and v for $s > 1/2$. Indeed, we show that the normal derivative of u vanishes on $\partial\Omega$ and a weighted normal derivative of v is proportional to v/δ^{2s-1} on $\partial\Omega$. This latter fact turns out to be useful in order to obtain monotonicity (in the normal direction) of the solutions near the boundary in the spirit of the Hopf boundary point lemma for classical elliptic equations. In the same vain, we improve (1.8) to a Hölder regularity of v/δ^{2s-1} up to the boundary. We mention that in the case of the fractional Laplacian with zero exterior Dirichlet data such type of regularity, of the ratio between the solution and the power of the distance function, has been first obtained by Ros-Oton and Serra [27] followed by [16, 20, 24, 25].

We notice that prior to the recent paper [2] and the present paper, even the continuity of solutions to (1.5) up to the boundary was an open question.

Our first main result for regional (or reflected) fractional Laplacian is the following.

Theorem 1.1. *Let $s_0 \in (0, 1)$, $s \in [s_0, 1)$ and Ω be an open subset of $\mathbb{R}^N$ of class C^1 and $f \in L^p(\Omega)$, for some $p > \frac{N}{2s_0}$. Let $u \in H^s_{loc}(\overline{\Omega}) \cap L^2(\Omega)$ be a solution to (1.5). Then for every $\varepsilon \in (0, 1)$ and for every $\Omega' \subset\subset \overline{\Omega}$ there exists $C > 0$ depending only on $N, s_0, \Omega, \varepsilon, \Omega'$ and p such that*

$$\|u\|_{C^{\min(2s-N/p,1-\varepsilon)}(\Omega')} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

In the case of higher order regularity, we obtain the

Theorem 1.2. *Let $s \in (1/2, 1)$ and Ω be an open subset of $\mathbb{R}^N$ of class $C^{1,\beta}$, with $\beta > 0$. Let $f \in L^p(\Omega)$ and $u \in H^s_{loc}(\overline{\Omega}) \cap L^2(\Omega)$ be a solution to (1.5). If $2s - N/p > 1$, then $u \in C^{1,\min(2s-\frac{N}{p}-1,\beta)}_{loc}(\overline{\Omega})$ and for every $\Omega' \subset\subset \overline{\Omega}$ there exists $C > 0$ depending only on $N, s, \Omega, \beta, \Omega'$ and p such that*

$$\|\nabla u\|_{C^{\min(2s-\frac{N}{p}-1,\beta)}(\Omega')} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

Moreover the normal derivative of u vanishes on $\partial\Omega$. More precisely, for all $\sigma \in \partial\Omega$,

$$\lim_{t\searrow 0} \frac{u(\sigma + t\nu(\sigma)) - u(\sigma)}{t} = 0, \tag{1.9}$$

where ν is the unit interior normal vectorfield of $\partial\Omega$.

We observe that, Theorem 1.1 and Theorem 1.2 provide similar regularity properties as the interior regularity of weak solutions to equations involving nonlocal operators of class C^β, which states that $u \in C^{1,\min(2s-N/p-1,\beta)}_{loc}(\Omega)$, see [15]. This is, in fact, the case in the local setting ($s = 1$) with zero Neumann boundary condition on a $C^{1,\beta}$ domain. Such high boundary regularity does not in general hold in the Dirichlet problem. The reason for this in the fractional setting can be seen, intuitively on the one hand, from the fact that the variational problem (1.3) allow for a larger class of test functions. On the other hand, one can look at harmonic functions, with locally finite energy, in the half-space $\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$ with respect to the regional fractional Laplacians and the fractional Laplacian. In the Neumann case, they are affine functions (of the form $a \cdot x' + b$) as long as their pointwise growth is strictly smaller than $2s$. However in the Dirichlet case they are proportional to x_N^{2s-1}, while in the case of the fractional Laplacian with zero exterior data, they are proportional to $(x_N)^s_+$.
We expect (1.9) to hold in the case $s = 1/2$, but then it might be necessary to show first the continuity of gradient of u up to the boundary which should hold provided $f \in C^\alpha(\overline{\Omega})$ and $\beta = 2s - 1 + \alpha$, for some $\alpha > 0$.

We now turn to solutions to the Censored fractional Laplacian. As mentioned earlier, it is the same as the reflected one in the case $s \in (0, 1/2]$, provided Ω has Lipschitz boundary.

Theorem 1.3. *Let $s_0 \in (1/2, 1)$, $s \in [s_0, 1)$ and Ω be an open subset of $\mathbb{R}^N$ of class C^1 and $f \in L^p(\Omega)$ for some $p > \frac{N}{2s_0}$. Let $u \in H^s_0(\Omega)$ be a solution to (1.6). Then for every $\varepsilon \in (0, 2s_0 - 1)$ and $\Omega' \subset\subset \overline{\Omega}$, there exists $C > 0$ depending only on $N, s_0, \Omega, \varepsilon, \Omega'$ and p such that*

$$\|u\|_{C^{\min(2s-N/p,2s-1-\varepsilon)}(\Omega')} \le C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

In the case of higher order regularity we obtain the

Theorem 1.4. *Let $s \in (1/2, 1)$ and Ω be an open subset of $\mathbb{R}^N$ of class $C^{1,\beta}$, $\beta > 0$ and $\Omega' \subset\subset \overline{\Omega}$. Let $f \in L^p(\Omega)$ and $v \in H^s_0(\Omega)$ be a solution to (1.5). Then*

(i) *if $2s - N/p > 2s - 1$, there exists $C > 0$ depending only on $N, s, \Omega, \beta, \Omega'$ and p such that*

$$\|u\|_{C^{2s-1}(\Omega')} + \|v/\delta^{2s-1}\|_{C^{\min(1-N/p,\beta)}(\Omega')} \le C \left(\|v\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right). \tag{1.10}$$

(ii) *if $2s - N/p > 1$ and $\beta > 2s - 1$, there exists $C > 0$ depending only on $N, s, \Omega, \beta, \Omega'$ and p such that*

$$\|\delta^{2-2s}\nabla v\|_{C^{2s-\frac{N}{p}-1}(\Omega')} \le C \left(\|v\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right) \tag{1.11}$$

and

$$\delta^{2-2s}\nabla v = (2s-1)\frac{v}{\delta^{2s-1}}\nu \qquad \text{on } \partial\Omega, \tag{1.12}$$

where ν is the interior normal vectorfield of $\partial\Omega$ and δ coincides with dist$(x, \partial\Omega)$ in a neighbourhood of $\partial\Omega$, Lipscitz continuous and positive in the complement, in Ω, of this neighbourhood.

We recall, in Theorem 1.4, that for $\sigma \in \partial\Omega$, we define $\frac{v}{\delta^{2s-1}}(\sigma) = \lim_{\substack{x\to\sigma \\ x\in\Omega}} \frac{v}{\delta^{2s-1}}(x)$ and $(\delta^{2-2s}\nabla v)(\sigma) = \lim_{\substack{x\to\sigma \\ x\in\Omega}} \delta^{2-2s}(x)\nabla v(x)$. Next, we observe that if Ω is of class $C^{1,1}$ then the extension of δ to the signed distance function to $\partial\Omega$ is of class $C^{1,1}$ in a neighbourhood of $\partial\Omega$ and its gradient coincides with the unit interior normal ν on $\partial\Omega$. Therefore (1.10), (1.11) and (1.12) imply, for $2s - N/p > 1$, that

$$|\nabla \left(v/\delta^{2s-1}\right)| \leq C\delta^{2s-\frac{N}{p}-2} \left(\|v\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right) \qquad \text{in } \Omega.$$

We notice that Theorem 1.4-(i), in the case of the fractional Laplacian with zero exterior data, was first obtained by Grubb in [20] in the case of C^∞ domains and in [16] in the case of $C^{1,\beta}$ domains and even with a larger class of right hand sides containing $L^p(\Omega)$. Now (1.11) and (1.12) was recently obtained for the fractional Laplacian in [14].

We point out that the above results in the present paper provide new insights to the Poisson problem involving these regional nonlocal operators and, as such, we believe that they might motivate the study of several problems involving these operators in the spirit of the fractional Laplacian.

Remark 1.5. *We note that starting $p > \frac{N}{2s_0}$ and $\beta < 2s_0 - 1$, the constant C in Theorem 1.2 and Theorem 1.4-(i) can be chosen to remain bounded as $s \to 1$, while for Theorem 1.4-(ii), we need to assume that $\beta = 1$. We refer to Remark 7.2 below for more details.*

To prove these results, we first make a full classification of all solutions, with some growth control when Ω is a half-space and $f \equiv 0$. We then carry out a blow up and some compactness arguments to obtain Hölder continuity up to the boundary. We note that the classification of the solutions on the half-space was left as a challenging open problem in [2]. To achieve it, we use several ingredients of independent interest, including some reflection principles, Hardy-type inequalities and the Caffarelli-Silvestre extension [7]. The blow up argument is partly inspired by the work of Serra in [28] and [15], where we study optimal regularity estimates involving a class of nonlocal operators extending the classical operators in divergence form, see also [16, 24] for other uses.
The main difficulties reside on the fact that the operators depend on the domain. We use local parameterization that flatten the boundary and thus allows to work on a fixed domain during the blow up process. However, in this case the resulting operator falls in a class of nonlocal operators in divergence form (see e.g. [15]). To deal with these nontranslation invariant kernels, we use a freezing argument of the kernels in the spirit of [15]. We also mention that, except in dimension 1 where we have the exact expression of the potential V in (1.7), we do not use (1.7) in our argument because of some scaling issues. Recall that $V(x) \sim \delta(x)^{-2s}$ and thus scales as $(-\Delta)^s$.

The paper is organized as follows. Section 3 contains the a priori estimates in the half-space. The Liouville-type results are proven in Section 4. The blow up argument leading to boundary regularity on curved domains for the regional and the censored fractional Laplacians are given in Section 5 and Section 6, respectively. We finally collect the proofs of the main results in Section 7.

2. Notations

For E a Lipschitz open subset of $\mathbb{R}^N$ and a weight function a, we use the standard notations for weighted Lebesgue spaces $L^p(E; a(x)) = \{u : E \to \mathbb{R}^M : \int_E |u(x)|^p a(x)dx < \infty\}$. See

e.g. [19], the fractional Sobolev space $H^s(E)$ is given by the set of measurable functions u such that

$$\|u\|^2_{H^s(E)} := \|u\|^2_{L^2(E)} + [u]^2_{H^s(E)} < \infty,$$

where $[u]^2_{H^s(E)} := \int_E \int_E \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty$. We denote by $H^s_0(E)$ the closure of $C^\infty_c(E)$ with respect to the norm $\|\cdot\|_{H^s(E)}$. We say that $u \in H^s_{loc}(E)$ (resp. $H^s_{loc}(\overline{E})$) if for all $\varphi \in C^1_c(E)$ (resp. $\varphi \in C^1_c(\overline{E})$), we have $\varphi u \in H^s(E)$. We define the Hilbert space

$$\mathcal{H}^s_0(E) = \{u \in H^s(\mathbb{R}^N) : \quad u = 0 \text{ on } \mathbb{R}^N \setminus E\},$$

endowed with the norm $\|u\|_{H^s(\mathbb{R}^N)}$. An interesting characterization of $\mathcal{H}^s_0(E)$, see [19], is that

$$\mathcal{H}^s_0(E) = \left\{ u \in H^s(\mathbb{R}^N) : \int_E \frac{u^2(x)}{\delta^{2s}_E(x)} dx < \infty, \quad u = 0 \text{ on } \mathbb{R}^N \setminus E \right\}, \tag{2.1}$$

where $\delta_E(x) := \text{dist}(x, \partial E)$.
We denote by $B_r(x_0)$, the ball of $\mathbb{R}^N$ centred at x_0 with radius r and $B_r = B_r(0)$. Moreover $B'_r(z) = \partial\mathbb{R}^N_+ \cap B_r(z)$ for $z \in \partial\mathbb{R}^N_+$ and $B'_r = B'_r(0)$. Moreover $B^+_r(z) = B_r(z) \cap \mathbb{R}^N_+$.
For the Hölder and Lipschitz seminorm of u, we write

$$[u]_{C^{0,\alpha}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha},$$

for $\alpha \in (0, 1]$. If there is no ambiguity, when $\alpha \in (0, 1)$, we will write $[u]_{C^\alpha(\Omega)}$ instead of $[u]_{C^{0,\alpha}(\Omega)}$. If $m \in \mathbb{N}$ and $\alpha \in (0, 1)$, the Hölder space $\|u\|_{C^{m,\alpha}(\Omega)}$ is given by the set of functions in $C^m(\Omega)$ such that

$$\|u\|_{C^{m+\alpha}(\Omega)} := \|u\|_{C^{m,\alpha}(\Omega)} = \sup_{\gamma \in \mathbb{N}^N, |\gamma| \leq m} \|\partial^\gamma u\|_{L^\infty(\Omega)} + \sup_{\gamma \in \mathbb{N}^N, |\gamma| = m} \|\partial^\gamma u\|_{C^\alpha(\Omega)} < \infty.$$

3. A priori Hölder regularity estimate on the half-space

We introduce the reflection function with respect to the e_N direction given by

$$\mathcal{R} : \mathbb{R}^N \to \mathbb{R}^N, \qquad \mathcal{R}(x) = x - 2e_N \cdot x = (x', -x_N).$$

Here and in the following, we define

$$\mathcal{C}_e := \{\varphi \in C^\infty_c(\mathbb{R}^N) : \varphi(x) = \varphi(\mathcal{R}(x)), \qquad \text{for all } x \in \mathbb{R}^N \}$$

and

$$H^s_{loc,e}(\mathbb{R}^N) := \{u \in H^s_{loc}(\mathbb{R}^N) : u(x', x_N) = u(\mathcal{R}(x)), \qquad \text{for all } x \in \mathbb{R}^N\}.$$

Moreover for $u \in H^s_{loc}(\overline{\mathbb{R}^N_+})$, we define

$$\overline{u}(x', x_N) := \begin{cases} u(x', x_N) & \text{for } x_N \geq 0, \\ u(x', -x_N) & \text{for } x_N < 0. \end{cases} \tag{3.1}$$

We define the kernels $\overline{K}, K^+_s : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty]$ by

$$K^+_s(x, y) := c_{N,s} 1_{\mathbb{R}^N_+ \times \mathbb{R}^N_+}(x, y)|x - y|^{-N-2s}, \tag{3.2}$$

$$\overline{K}(x, y) := 1_{x_N y_N > 0}(x, y) \frac{c_{N,s}}{2|x-y|^{N+2s}} + 1_{x_N y_N < 0}(x, y) \frac{c_{N,s}}{2(|x' - y'|^2 + (x_N + y_N)^2)^{\frac{N+2s}{2}}}. \tag{3.3}$$

We observe that $\overline{K}$ is an even extension of the kernel K^+_s with respect to the e_N direction. On the other hand, the kernel $\overline{K}$ is *not* strongly elliptic but has some nice properties.

(i) For every $x, y \in \mathbb{R}^N$,

$$\overline{K}(x,y) = \overline{K}(y,x) = \overline{K}(\mathcal{R}(x), y). \tag{3.4}$$

(ii) Since $(x_N + y_N)^2 \leq (x_N - y_N)^2$ for $x_N y_N \leq 0$, we have that

$$\overline{K}(x,y) \geq \frac{c_{N,s}}{2|x-y|^{N+2s}} \qquad \text{for all } x, y \in \mathbb{R}^N. \tag{3.5}$$

(iii) For a function $\eta : \mathbb{R}^N \times \mathbb{R}^N$ and measurable sets $A, B \subset \mathbb{R}^N$, we have that

$$\begin{aligned} &\int_{A\times B} \eta(x,y)\overline{K}(x,y)\,dxdy \\ &= \int_{A\times B} \eta(x,y)K_s^+(x,y)\,dxdy + \int_{\mathcal{R}(A)\times\mathcal{R}(B)} \eta(\mathcal{R}(x),\mathcal{R}(y))K_s^+(x,y)\,dxdy \\ &+ \int_{A\times\mathcal{R}(B)} \eta(x,\mathcal{R}(y))K_s^+(x,y)\,dxdy + \int_{\mathcal{R}(A)\times B} \eta(\mathcal{R}(x),y)K_s^+(x,y)\,dxdy. \end{aligned} \tag{3.6}$$

As a consequence if $A = \mathcal{R}(A)$ and $B = \mathcal{R}(B)$ then, for every $v \in \mathcal{H}_e^s$,

$$\int_{A\times B} (v(x)-v(y))^2\overline{K}(x,y)\,dxdy = 2\int_{A\times B} (v(x)-v(y))^2 K_s^+(x,y)\,dxdy. \tag{3.7}$$

We start with the following crucial result for getting a priori Hölder estimate below.

Lemma 3.1. *Suppose that $u \in H^s_{loc}(\overline{\mathbb{R}^N_+}) \cap L^1(\mathbb{R}^N_+; (1+|x|^{N+2s})^{-1})$ and $f \in L^\infty(\mathbb{R}^N)$ solves*

$$(-\Delta)^s_{\overline{\mathbb{R}^N_+}} u = f \qquad \text{in } \mathbb{R}^N_+$$

i.e. for all $\varphi \in H^s_{loc}(\overline{\mathbb{R}^N_+})$ with Supp$(\varphi) \subset\subset \mathbb{R}^N$

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (u(x)-u(y))(\varphi(x)-\varphi(y))K_s^+(x,y)\,dxdy = \int_{\mathbb{R}^N} f(x)\varphi(x)\,dx. \tag{3.8}$$

Then $\overline{u} \in H^s_{loc,e}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N; (1+|x|^{N+2s})^{-1})$ and for all $\varphi \in H^s_{loc,e}(\mathbb{R}^N)$, with Supp$(\varphi) \subset\subset \mathbb{R}^N$

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy = 2\int_{\mathbb{R}^N} f(x)\varphi(x)\,dx.$$

Proof. The fact that $\overline{u} \in H^s_{loc,e}(\mathbb{R}^N)$ follows from (3.7). Moreover by a change of variable, we have $\overline{u} \in L^1(\mathbb{R}^N; (1+|x|^{N+2s})^{-1})$. We let $\varphi \in \mathcal{C}_e$ and put $\eta(x,y) = (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))$. Then for all $x, y \in \mathbb{R}^N$,

$$\eta(x,y) = \eta(y,x) = \eta(\mathcal{R}(x), y).$$

Using (3.6) and the fact that $\mathbb{R}^N = \mathcal{R}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \eta(x,y)\overline{K}(x,y)\,dxdy = 2\int_{\mathbb{R}^N\times\mathbb{R}^N} (u(x)-u(y))(\varphi(x)-\varphi(y))K_s^+(x,y)\,dxdy.$$

The proof is thus complete thanks (3.8). □

In our next result we will show Hölder continuity of $\overline{u}$ (solution to (3.8)) on B_1. The argument is based on the De Giorgi iteration techniques. In view of the recent work in the literature see e.g. [12], it suffices to prove that $\overline{u}$ belongs to a fractional De Giorgi class as defined in [12]. The proof is similar as in [12], we shall only focus our attention to the fact that $\overline{K}$ is not controlled from above by $|x-y|^{-N-2s}$. But using the symmetry properties of the tests functions and the solution, we can reach our goal.

We fix the notations as in the aforementioned paper. For a real number a, we write $a_+ = \max(a,0)$ and $a_- = \max(-a,0)$. For a measurable real valued function v and a real number k,

$$A_v^+(k) := \{x \in \mathbb{R}^N \,:\, v > k\}, \qquad A_v^+(k,x_0,r) = A_v^+(k) \cap B_r(x_0).$$

Proposition 3.2. *Let $\overline{u} \in H^s_{loc,e}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N;(1+|x|^{N+2s})^{-1})$ and $f \in L^\infty(\mathbb{R}^N)$ satisfy, for all $\varphi \in H^s_{loc,e}(\mathbb{R}^N)$, Supp$(\varphi) \subset\subset \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy = \int_{\mathbb{R}^N} f(x)\varphi(x)\,dx. \tag{3.9}$$

Let $k \in \mathbb{R}$, $r_0 > 0$ and $x_0 \in \mathbb{R}^N$. Then for $0 < \rho < \tau \le 1$, we have

$$\begin{aligned}
&\frac{1}{2}[w_\pm]_{H^s(B_\rho(x_0))} + \frac{1}{2}\int_{B_\rho(x_0)} w_\pm(x) \int_{B_{r_0}(x)} \frac{w_\mp(y)}{|x-y|^{N+2s}}\,dydx \\
&\le \frac{C(N)}{(\tau-\rho)^2(1-s)}\|w_\pm\|^2_{L^2(B_\tau(x_0))} + \frac{16}{(\tau-\rho)^{N+2s}}\|w_\pm\|_{L^1(B_\tau(x_0))}\int_{|y|\ge\rho}\frac{w_\pm(y)}{|y-x_0|^{N+2s}}\,dy \\
&+ \frac{2}{c_{N,s}}\|f\|^2_{L^\infty(\mathbb{R}^N)}A^\pm(k,x_0,\tau).
\end{aligned}$$

where $w_\pm = (\overline{u}-k)_\pm$.

Proof. We follow very closely the argument in [12, Proof of Proposition 8.5]. We let $\eta \in \mathcal{C}_e$ be such that $0 \le \eta \le 1$, with Supp$(\eta) = \overline{B_{\frac{\rho+\tau}{2}}}$ and $\eta = 1$ on B_ρ. We use $\varphi = \eta^2 w_+ \in H^s_{loc,e}(\mathbb{R}^N)$ as a test function in (3.9) to get

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy \le \int_{\mathbb{R}^N} |f(x)|\eta^2(x)w_+(x)\,dx.$$

As a consequence

$$\begin{aligned}
&\int_{B_\tau\times B_\tau} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy \\
&+ 2\int_{B_\tau\times\mathbb{R}^N\setminus B_\tau} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy \le \int_{A^+(k,\frac{\tau+\rho}{2})} |f(x)|\eta^2(x)w_+(x)\,dx.
\end{aligned} \tag{3.10}$$

Now from the argument in [12, Section 8], we have

$$\begin{aligned}
&\int_{B_\tau\times B_\tau} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy \\
&\qquad \ge \frac{1}{2}\int_{B_\rho\times B_\rho} (w_+x)-w_+(y))^2\overline{K}(x,y)\,dxdy + \frac{1}{2}\int_{B_\rho} w_+(x)\int_{B_\tau} w_-(y)\overline{K}(x,y)\,dydx \\
&\qquad - 2\int_{B_\tau\times B_\tau} \max(w_+(x),w_+(y))^2(\eta(x)-\eta(y))^2\overline{K}(x,y)\,dxdy.
\end{aligned}$$

Hence, by (3.5), we have

$$\begin{aligned}
&\int_{B_\tau\times B_\tau} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy \\
&\qquad \ge \frac{c_{N,s}}{2}[w_+]_{H^s(B_\rho)} + \frac{c_{N,s}}{2}\int_{B_\rho} w_+(x)\int_{B_\tau} w_-(y)|x-y|^{-N-2s}\,dydx \\
&\qquad - 2\int_{B_\tau\times B_\tau} \max(w_+(x),w_+(y))^2(\eta(x)-\eta(y))^2\overline{K}(x,y)\,dxdy.
\end{aligned} \tag{3.11}$$

Now by (3.6), we get

$$\begin{aligned}
&\int_{B_\tau\times B_\tau} \max(w_+(x),w_+(y))^2(\eta(x)-\eta(y))^2\overline{K}(x,y)\,dxdy\\
&=4\int_{B_\tau\times B_\tau} \max(w_+(x),w_+(y))^2(\eta(x)-\eta(y))^2K_+(x,y)\,dxdy\\
&\le 8c_{N,s}\|\nabla\eta\|^2_{L^\infty(B_{\frac{\rho+\tau}{2}})}\|w_+\|^2_{L^2(B_\tau)}\sup_x\int_{|x-y|<2\tau}|x-y|^{-N-2s+2}\,dy\\
&\le \frac{c_{N,s}C(N)}{(\tau-\rho)^2(1-s)}\|w_+\|^2_{L^2(B_\tau)}.
\end{aligned}$$

Using this in (3.11), we find that

$$\frac{1}{c_{N,s}}\int_{B_\tau\times B_\tau}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy \tag{3.12}$$

$$\ge \frac{1}{2}[w_+]_{H^s(B_\rho)}+\frac{1}{2}\int_{B_\rho}w_+(x)\int_{B_\tau}w_-(y)|x-y|^{-N-2s}\,dydx-\frac{C(N)R^{2(1-s)}}{(\tau-\rho)^2(1-s)}\|w_+\|^2_{L^2(B_\tau)}. \tag{3.13}$$

Next, following once more [12, Section 8], for every $r_0>0$, we also have that

$$\begin{aligned}
&\int_{B_\tau\times\mathbb{R}^N\setminus B_\tau}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy\\
&\ge\int_{B_\rho}w_+(x)\int_{B_{r_0}(x)\setminus B_\tau}w_-(y)\overline{K}(x,y)\,dydx\\
&-2\int_{A^+_{\overline{u}}(k,\frac{\tau+\rho}{2})}w_+(x)\int_{\{\overline{u}(y)>\overline{u}(x)\}\setminus B_\tau}(\overline{u}(y)-\overline{u}(x))\overline{K}(x,y)\,dy\,dx\\
&\ge\int_{B_\rho}w_+(x)\int_{B_{r_0}(x)\setminus B_\tau}w_-(y)\overline{K}(x,y)\,dydx-2\int_{A^+_{\overline{u}}(k,\frac{\tau+\rho}{2})}w_+(x)\int_{\mathbb{R}^N\setminus B_\tau}w_+(y)\overline{K}(x,y)\,dy\,dx.
\end{aligned}$$

This then implies that

$$\begin{aligned}
&\int_{B_\tau\times\mathbb{R}^N\setminus B_\tau}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy\\
&\ge\int_{B_\rho}w_+(x)\int_{B_{r_0}(x)\setminus B_\tau}w_-(y)\overline{K}(x,y)\,dydx-2\int_{B_{\frac{\tau+\rho}{2}}}w_+(x)\int_{\mathbb{R}^N\setminus B_\tau}w_+(y)\overline{K}(x,y)\,dy\,dx.
\end{aligned}$$

Moreover from (3.5) and (3.6), we get,

$$\begin{aligned}
&\int_{B_\tau\times\mathbb{R}^N\setminus B_\tau}(\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy\\
&\ge c_{N,s}\int_{B_\rho}w_+(x)\int_{B_{r_0}(x)\setminus B_\tau}\frac{w_-(y)}{|x-y|^{N+2s}}\,dydx-8\int_{B_{\frac{\tau+\rho}{2}}}w_+(x)\int_{\mathbb{R}^N\setminus B_\tau}w_+(y)K_+(x,y)\,dy\,dx\\
&\ge c_{N,s}\int_{B_\rho}w_+(x)\int_{B_{r_0}(x)\setminus B_\tau}\frac{w_-(y)}{|x-y|^{N+2s}}\,dydx-8c_{N,s}\int_{B_{\frac{\tau+\rho}{2}}}w_+(x)\int_{\mathbb{R}^N\setminus B_\tau}\frac{w_+(y)}{|x-y|^{N+2s}}\,dy\,dx.
\end{aligned}$$

Using that $|x-y| \geq \frac{\tau-\rho}{2}|y|$ we then deduce that

$$\int_{B_\tau \times \mathbb{R}^N \setminus B_\tau} (\overline{u}(x)-\overline{u}(y))(\varphi(x)-\varphi(y))\overline{K}(x,y)\,dxdy$$
$$\geq c_{N,s}\int_{B_\rho} w_+(x)\int_{B_{r_0}(x)\setminus B_\tau}\frac{w_-(y)}{|x-y|^{N+2s}}\,dydx - \frac{8c_{N,s}}{(\tau-\rho)^{N+2s}}\|w_+\|_{L^1(B_\tau)}\int_{|y|\geq\tau}\frac{w_+(y)}{|y|^{N+2s}}\,dy. \tag{3.14}$$

Now, using Young's inequality, we can estimate

$$\int_{A^+(k,\frac{\tau+\rho}{2})} |f(x)|\eta^2(x)w_+(x)\,dx \leq 2\|f\|^2_{L^\infty(\mathbb{R}^N)}A^+(k,\frac{\tau+\rho}{2}) + \frac{2}{(\tau-\rho)^2}\|w_+\|^2_{L^2(B_\tau)}.$$

Combining the above estimate with (3.14), (3.10) and (3.13), we conclude that

$$\frac{c_{N,s}}{2}[w_+]_{H^s(B_\rho)} + \frac{c_{N,s}}{2}\int_{B_\rho} w_+(x)\int_{B_{r_0}(x)}\frac{w_-(y)}{|x-y|^{N+2s}}\,dydx$$
$$\leq \frac{c_{N,s}C(N)}{(\tau-\rho)^2(1-s)}\|w_+\|^2_{L^2(B_\tau)} + \frac{16c_{N,s}}{(\tau-\rho)^{N+2s}}\|w_+\|_{L^1(B_\tau)}\int_{|y|\geq\tau}\frac{w_+(y)}{|y|^{N+2s}}\,dy$$
$$+ 2\|f\|^2_{L^\infty(\mathbb{R}^N)}A^+(k,\tau).$$

Next, testing the equation with $\eta^2 w_-$, we get the same estimates as above, with w_- in the place of w_+ and vice versa. Now by the translation invariance of the problem we get the result. □

Our next result provides Hölder continuity up to the boundary for harmonic functions with respect to $(-\Delta)^s_{\mathbb{R}^N_+}$.

Theorem 3.3. *Let $f \in L^\infty(\mathbb{R}^N)$ and suppose that $u \in H^s_{loc}(\overline{\mathbb{R}^N_+}) \cap L^1(\mathbb{R}^N_+;(1+|x|^{N+2s})^{-1})$ solves*

$$(-\Delta)^s_{\mathbb{R}^N_+}u = f \qquad \text{on } \mathbb{R}^N_+.$$

Then there exist $\alpha_0 = \alpha_0(N,s) > 0$ and $C = C(N,s) > 0$ such that

$$\|u\|_{C^{\alpha_0}\left(\overline{B_1^+}\right)} \leq C\left(\|u\|_{L^2(B_2^+)} + \int_{\{|y|\geq 1/2\}\cap\mathbb{R}^N_+}\frac{|u(y)|}{|y|^{N+2s}}dy + \|f\|_{L^\infty(\mathbb{R}^N)}\right).$$

Proof. By Lemma 3.1, $\overline{u} = u\circ\mathcal{R} \in H^s_{loc,e}(\mathbb{R}^N)\cap L^1(\mathbb{R}^N;(1+|x|^{N+2s})^{-1})$. Moreover by Proposition 3.2, $\overline{u}$ belongs to the De Giorgi class in the sense of [12]. We can therefore apply [12, Theorem 6.2 and Theorem 6.4] to get

$$\|\overline{u}\|_{C^{\alpha_0}(B_1)} \leq \left(\|\overline{u}\|_{L^2(B_2)} + \int_{\{|y|\geq 1/2\}}\frac{|\overline{u}(y)|}{|y|^{N+2s}}dy + \|f\|_{L^\infty(\mathbb{R}^N)}\right).$$

This completes the proof from the definition of $\overline{u}$.

□

4. Liouville theorem

The main result of the present section is the following classification result of functions $u \in H^s_{loc}(\overline{\mathbb{R}^N_+})$ solving

$$(-\Delta)^s_{\mathbb{R}^N_+} u = 0 \qquad \text{in } \mathbb{R}^N_+ \tag{4.1}$$

and having a some growth control smaller than $2s$. The argument we develop below will provides also a Liouville-type result in the case of the Censored fractional Laplacian $(-\Delta)^s_{\mathbb{R}^N_+}$. Its proof requires several preliminary results.

Theorem 4.1. *Let u be a solution to (4.1) and suppose that, for some constants $C > 0$ and $\varepsilon \in (0, 2s)$,*

$$\|u\|_{L^2(B_R)} \le C R^{\frac{N}{2}+\varepsilon} \qquad \text{for all } R \ge 1. \tag{4.2}$$

(i) *If $2s \le 1$ then u is a constant function on $\mathbb{R}^N_+$.*
(ii) *If $2s > 1$ then $u(x', x_N) = a + c \cdot x'$ for all $x = (x', x_N) \in \mathbb{R}^N_+$, for some constants $a \in \mathbb{R}$ and $c \in \mathbb{R}^{N-1}$.*

We remark that in the case $2s = 1$, the growth assumption (4.2) is not necessary. In fact we will show that any solution $u \in H^{1/2}_{loc}(\overline{\mathbb{R}^N_+}) \cap L^1(\mathbb{R}^N_+; (1 + |x|^{N+1})^{-1})$ to (4.1) is a constant function.

The following result provides a higher order regularity of solutions in the tangential direction. This takes advantages on the translation invariant of the problem the in the tangential direction which allows to estimate the incremental quotient of u given by $\frac{u(x+he)-u(x)}{|h|^\alpha}$, $e \in \mathbb{R}^{N-1}$, as in [8]. In the nonlocal setting, one should cut off the solution at each step in order to deal with the tail.

Proposition 4.2. *We consider $u \in H^s_{loc}(\overline{\mathbb{R}^N_+}) \cap L^1(\mathbb{R}^N_+; (1 + |x|^{N+2s})^{-1})$ satisfying*

$$(-\Delta)^s_{\mathbb{R}^N_+} u = 0 \quad \text{in} \quad \mathbb{R}^N_+. \tag{4.3}$$

Then

$$\|\nabla^2_{x'} u\|_{L^\infty(B_1^+)} \le C \left(\|u\|_{L^2(B_2^+)} + \int_{\{|y| \ge 1/2\} \cap \mathbb{R}^N_+} |u(y)||y|^{-N-2s}\, dy \right).$$

Proof. For simplicity, we assume that $\|u\|_{L^2(B_2^+)} + \int_{\{|y|\ge 1/2\}\cap\mathbb{R}^N_+} |u(y)||y|^{-N-2s}\, dy \le 1$. Hence by Theorem 3.3, we have $\|u\|_{C^\alpha(B_2)} \le C$. Let $r_i := 1 + 2^{-i}$ and $\phi_i \in C^\infty_c(B_{r_i})$ such that $\phi_i = 1$ on $B_{r_{i+1}}$. Let $v_i = \phi_i u$ which satisfies

$$(-\Delta)^s_{\mathbb{R}^N_+} v_i(x) = f_i(x) \qquad \text{for all } x \in B^+_{r_{i+2}},$$

where

$$f_i(x) := \phi_{i+2}(x) \int_{|y| \ge r_i} v_i(y) 1_{\mathbb{R}^N_+}(y) |x-y|^{-N-2s}\, dy.$$

It clearly satisfies $\|f_i\|_{C^2(\mathbb{R}^N)} \le C$. In view of Theorem 3.3, we get

$$\|v_i\|_{C^\alpha(\mathbb{R}^N_+)} \le C. \tag{4.4}$$

For $e \in \mathbb{R}^{N-1}$ and $|e| = 1$ and $h \in \mathbb{R}$, we define

$$u^{h,e}(x) := \frac{u(x+he) - u(x)}{|h|^\alpha},$$

Then we can consider $w_{i,h,\alpha}(x) = \frac{v_i(x+he)-v_i(x)}{|h|^\alpha}$. Then by (4.4), for $h \in B_{r_{i+5}}$,

$$\|w_{i,h,\alpha}\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

Moreover, by the translation invariant of the problem in the tangential direction,

$$(-\Delta)^s_{\mathbb{R}^N_+} w_{i,h,\alpha} = g_i \qquad \text{for all } x \in B^+_{r_{i+7}},$$

with

$$g_i(x) = f_i^{h,e} + \phi_{i+5}(x) \int_{|y|\geq r_{i+5}} v_i^{h,e}(y) 1_{\mathbb{R}^N_+}(y)|x-y|^{-N-2s}\, dy.$$

Therefore, by Theorem 3.3,

$$\|w_{i,h,\alpha}\|_{C^\alpha(B^+_{r_i+8})} \leq C.$$

Hence, see [8, Section 5.3], we have that u in $C^{2\alpha}$ in the direction of e if $2\alpha < 1$ while u in $C^{0,1}$ in the direction of e if $2\alpha > 1$. Iterating this procedure a finite number of times, we will find an integer n such that $u \in C^{0,1}(B_{r_n})$ and

$$\|\nabla_{x'} u\|_{L^\infty(B_{r_n})} \leq C.$$

By the same argument as above considering $\partial_{x_i} u$, with $i = 1, \ldots, N-1$, in the place of u, we will get the Lipschitz bound of $\partial_{x_i} u$. This then gives the result. □

Proposition 4.3. *We consider $u \in H^s_{loc}(\overline{\mathbb{R}^N_+})$ satisfying*

$$(-\Delta)^s_{\mathbb{R}^N_+} u = 0 \quad \text{in} \quad \mathbb{R}^N_+ \tag{4.5}$$

and

$$\|u\|^2_{L^2(B_R)} \leq CR^{N+2\varepsilon} \qquad \text{for all } R \geq 1,$$

for some $C > 0$ and $\varepsilon < \min(1, 2s)$. Then u is one dimension. More precisely, it is a function on $\mathbb{R}_+$.

Proof. By scaling, we have that

$$\|\nabla_{x'} u\|_{L^\infty(B^+_R)} \leq C \left(R^{\varepsilon-1} \|u\|_{L^2(B^+_{2R})} + R^{2s-1} \int_{\{|y|\geq 2R\}\cap\mathbb{R}^N_+} |u(y)||y|^{-N-2s}\, dy \right).$$

We now estimate

$$\begin{aligned}
\int_{\{|y|\geq R\}\cap\mathbb{R}^N_+} \frac{|u_R|}{|y|^{N+2s}} dy &\leq \sum_{i=0}^\infty \int_{\{R2^{i+1}\geq|y|\geq 2^i R\}\cap\mathbb{R}^N_+} \frac{|u_R(y)|}{|y|^{N+2s}} dy \\
&\leq \sum_{i=0}^\infty (2^i R)^{-N-2s} \|u\|_{L^1(B_{2^{i+1}R})} \leq C \sum_{i=0}^\infty (2^i R)^{-N-2s} (2^{i+1}R)^{N+\varepsilon} \\
&\leq CR^{-2s+\varepsilon} \sum_{i=0}^\infty (2^i)^{-2s+\varepsilon} \leq CR^{-2s+\varepsilon}.
\end{aligned}$$

We then deduce that $\|\nabla_{x'} u\|_{L^\infty(B^+_R)} \leq R^{\varepsilon-1}$. Letting $R \to \infty$, we see that $\nabla_{x'} u = 0$ on $\mathbb{R}^N_+$. □

4.1. **The one dimensional problem.** This section is devoted to the classification of solutions $u \in H^s_{loc}([0,\infty)) \cap L^1_s(\mathbb{R}_+)$ satisfying some growth control and solving the equation

$$(-\Delta)^s_{\mathbb{R}_+} u = 0 \qquad \text{on } \mathbb{R}_+, \tag{4.6}$$

where, here and in the following, we define

$$L^1_s(A) := L^1(A; (1+|x|)^{-1-2s}) \qquad \text{for } A \text{ a measurable subset of } \mathbb{R}.$$

Let $v \in C^2_{loc}(0,\infty) \cap L^1_s(\mathbb{R}_+)$ and define $\widetilde{v} = v 1_{(0,\infty)}$. A direct computation shows that, for $x \in \mathbb{R}_+$,

$$(-\Delta)^s \widetilde{v}(x) = (-\Delta)^s_{\mathbb{R}_+} v(x) + v(x) \int_{\mathbb{R}_-} \frac{c_{1,s}}{|x-y|^{1+2s}}\, dy = (-\Delta)^s_{\mathbb{R}_+} \widetilde{v}(x) + a_s x^{-2s} \widetilde{v}(x), \tag{4.7}$$

where $(-\Delta)^s$ is the standard fractional Laplacian (see (1.1)) and $a_s := \frac{c_{1,s}}{2s}$. Indeed, by a change of variable, we deduce that for $x \in \mathbb{R}_+$,

$$\int_{\mathbb{R}_-} \frac{1}{|x-y|^{1+2s}}\, dy = x^{-2s} \int_{-\infty}^0 |1-r|^{-1-2s}\, dr = \frac{x^{-2s}}{2s}. \tag{4.8}$$

As a consequence, if u solves (4.6) then we find that $\widetilde{u} = u 1_{(0,\infty)}$ solves

$$(-\Delta)^s \widetilde{u} - a_s x^{-2s} \widetilde{u} = 0 \qquad \text{on } \mathbb{R}_+. \tag{4.9}$$

The above discussion shows that to study (4.6) we can consider (4.9). Now the study of the latter problem leads us to consider optimal Hardy-type inequalities and allows us to consider the local version of the problem via the extension on the upper half-space $\mathbb{R}^2_+$.
We define

$$\mathbb{R}^2_+ := \{z = (x,t) \in \mathbb{R}^2 \,:\, t > 0\}, \qquad \partial \mathbb{R}^2_+ =: \mathbb{R}.$$

We recall the Poisson kernel with respect to the extended operator $\mathrm{div}(t^{1-2s}\cdot)$ on $\mathbb{R}^2_+$, see e.g. [7], given by

$$P_s(z) := b_s \frac{t^{2s}}{|z|^{1+2s}} = b_s \frac{t^{2s}}{|(x,t)|^{1+2s}},$$

where b_s is a normalization constant such that $\int_{\mathbb{R}} P_s(t,x)\, dx = 1$ for all $t > 0$. We recall the following result [7].

Lemma 4.4. *Let $u \in L^1_s(\mathbb{R})$ and define*

$$w(x,t) = b_s t^{2s} \int_{\mathbb{R}} \frac{u(y)}{(|x-y|^2 + t^2)^{(1+2s)/2}}\, dy.$$

Then

$$div(t^{1-2s} w) = 0 \qquad on\ \mathbb{R}^2_+.$$

(i) If $u \in H^s_{loc}(\mathbb{R})$, then $w \in H^1_{loc}(\overline{\mathbb{R}^2_+}; t^{1-2s})$. Moreover for every $\Phi \in C^\infty_c(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2_+} t^{1-2s} \nabla w(x,t) \cdot \nabla \Phi(x,t)\, dxdt = \overline{\kappa}_s \frac{c_{1,s}}{2} \int_{\mathbb{R}^2} \frac{(u(x)-u(y))(\Phi(x,0)-\Phi(y,0))}{|x-y|^{1+2s}}\, dxdy,$$

for some constant $\overline{\kappa}_s > 0$.

(ii) If $u \in C(a,b)$ then

$$\lim_{t\to 0} w(x,t) = u(x) \qquad for\ all\ x \in (a,b).$$

(iii) If $u \in C^2(a,b)$ then

$$-\lim_{t\to 0} t^{1-2s} \partial_t w(t,x) = \overline{\kappa}_s (-\Delta)^s u(x) \qquad for\ all\ x \in (a,b).$$

Proof. To check (i), it suffices to write $u = \chi_R u + (1-\chi_R)u$ with $\chi_R \in C_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $(-2R, 2R)$. Then one can easily see that $w \in H^1((-R,R)\times(0,R); t^{1-2s})$.
Now (ii) and (iii) are well known. □

We recall the classification of the functions of the form $\omega_\gamma := x^\gamma 1_{[0,\infty)}$ solving $(-\Delta)^s_{\mathbb{R}_+} x^\gamma = 0$ on $\mathbb{R}_+$, for $\gamma \geq 0$. Such functions are all known. Indeed for $\gamma \in (-1, 2s)$, we have, see e.g. [3],

$$(-\Delta)^s_{\mathbb{R}_+}\omega_\gamma = -\mu(\gamma,s)x^{-2s}\omega_\gamma \qquad \text{on } \mathbb{R}_+,$$

where

$$\mu(\gamma,s) = c_{1,s}\int_0^1 \frac{(t^\gamma-1)(1-t^{2s-1-\gamma})}{(1-t)^{1+2s}}\,dt.$$

We observe that the above integral is absolutely convergent and

$$\mu(\gamma,s) = 0 \qquad \text{if and only if} \qquad \gamma \in \{0; 2s-1\}. \tag{4.10}$$

We also notice, from (4.7), that

$$(-\Delta)^s\omega_\gamma = (a_s - \mu(\gamma,s))\,x^{-2s}\omega_\gamma \qquad \text{on } \mathbb{R}_+. \tag{4.11}$$

Moreover, by considering the extended function

$$w_\gamma(x,t) := b_s t^{2s}\int_{\mathbb{R}_+} \frac{\omega_\gamma(y)}{(|x-y|^2+t^2)^{(1+2s)/2}}\,dy, \tag{4.12}$$

we get, thanks to Lemma 4.4,

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla w_\gamma) = 0 & \text{in } \mathbb{R}^2_+ \\ -\lim_{t\to 0} t^{1-2s}\partial_t w_\gamma = \overline{\kappa}_s\,(a_s-\mu(\gamma,s))\,x^{-2s}w_\gamma & \text{on } \mathbb{R}_+ \\ w_\gamma = 0 & \text{on } \mathbb{R}_-. \end{cases} \tag{4.13}$$

We recall that $\omega_{\frac{2s-1}{2}}$ is the "virtual" positive ground state leading to the sharp fractional Hardy inequality on the half-line. Indeed, see e.g. [3], for every $u \in C_c^\infty(\mathbb{R}_+)$, we have

$$\frac{c_{1,s}}{2}\int_{\mathbb{R}_+\times\mathbb{R}_+}\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}\,dxdy \geq -\mu\left((2s-1)/2, s\right)\int_{\mathbb{R}_+} x^{-2s}u^2(x)\,dx.$$

The constant $-\mu(\frac{2s-1}{2}, s)$ is optimal and it is positive for $2s \neq 1$ and vanishes for $2s = 1$. From this and (4.8), we deduce the following sharp Hardy inequality that, for every $u \in C_c^\infty(\mathbb{R}_+)$,

$$\frac{c_{1,s}}{2}\int_{\mathbb{R}\times\mathbb{R}}\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}\,dxdy \geq \frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi}\int_{\mathbb{R}} x^{-2s}u^2(x)\,dx. \tag{4.14}$$

Here, we used that, see e.g. [3],

$$\frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi} = a_s - \mu\left((2s-1)/2, s\right). \tag{4.15}$$

By Lemma 4.4 and (4.14), we find that, for all $w \in C_c^1(\mathbb{R}^2)$, with $w = 0$ on $(-\infty, 0]\times\{0\}$,

$$\int_{\mathbb{R}^2_+}|\nabla w(x,t)|^2 t^{1-2s}\,dxdt \geq \overline{k}_s\frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi}\int_{\mathbb{R}} x^{-2s}w(x,0)^2\,dx. \tag{4.16}$$

Using polar coordinates, the above inequality yields a sharp inequality on S^1_+. Indeed, let $\psi \in C^1(0,\pi)$ such that $\psi(\pi)=0$ and $f \in C_c^\infty(\mathbb{R}_+)$. Letting $w(r\cos(\theta), r\sin(\theta)) = f(r)\psi(\theta)$ in (4.16), we get

$$\int_0^\infty (f'(r))^2 r^{2-2s}\,dr \int_0^\pi \sin(\theta)^{1-2s}\psi(\theta)^2 d\theta + \int_0^\infty (f(r))^2 r^{-2s}\,dr \int_0^\pi \sin(\theta)^{1-2s}(\psi'(\theta))^2 d\theta \geq \overline{k}_s \frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi} \int_0^\infty (f(r))^2 r^{-2s}\,dr \psi(0)^2.$$

This implies that

$$\frac{\int_0^\infty (f'(r))^2 r^{2-2s}\,dr}{\int_0^\infty (f(r))^2 r^{-2s}\,dr} \int_0^\pi \sin(\theta)^{1-2s}\psi(\theta)^2 d\theta + \int_0^\pi \sin(\theta)^{1-2s}(\psi'(\theta))^2 \geq \overline{k}_s \frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi}\psi(0)^2.$$

Using the classical optimal Hardy inequality, we obtain that

$$\inf_{f\in C_c^\infty(\mathbb{R}_+)} \frac{\int_0^\infty (f'(r))^2 r^{2-2s}\,dr}{\int_0^\infty (f(r))^2 r^{-2s}\,dr} = \frac{(2s-1)^2}{4}.$$

We then deduce immediately that

$$\frac{(2s-1)^2}{4}\int_0^\pi \sin(\theta)^{1-2s}\psi(\theta)^2 d\theta + \int_0^\pi \sin(\theta)^{1-2s}(\psi'(\theta))^2 d\theta \geq \overline{k}_s \frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi}\psi(0)^2. \tag{4.17}$$

We observe the following crucial fact that for $2s \neq 1$,

$$a_s < \frac{\Gamma\left(\frac{2s+1}{2}\right)^2}{\pi}, \tag{4.18}$$

which follows from (4.15) and the fact that $-\mu\left((2s-1)/2, s\right) > 0$ for $2s \neq 1$. We consider the eigenvalue problem on $H^1((0,\pi); \sin(\theta)^{1-2s})$ given by

$$\begin{cases} -(\sin(\theta)^{1-2s}\psi')' + \frac{(1-2s)^2}{4}\sin(\theta)^{1-2s}\psi = \lambda \sin(\theta)^{1-2s}\psi & \text{in } (0,\pi) \\ -\lim_{\theta\to 0}\sin(\theta)^{1-2s}\psi'(\theta) = \overline{\kappa}_s a_s \psi(0) \\ \psi(\pi) = 0. \end{cases} \tag{4.19}$$

By (4.18) and (4.17), for $2s \neq 1$, it possesses a sequence of increasing eigenvalue $0 < \lambda_1(s) < \lambda_2(s) \leq, \dots,$ with corresponding eigenfunctions ψ_k, normalized as

$$\int_0^\pi \sin(\theta)^{1-2s}\psi_k^2(\theta)\,d\theta = 1.$$

We have the following result.

Lemma 4.5. *Suppose that $s \in (0,1)$. Then the following statements hold.*

(*i*) *If $2s \neq 1$, then $\lambda_1(s) = \frac{(2s-1)^2}{4}$.*
(*ii*) *If $2s \neq 1$ then $\frac{2s-1}{2} + \sqrt{\lambda_2(s)} \geq 2s$.*
(*iii*) *If $2s = 1$, then (4.19) has a nontrivial solution for $\lambda = 0$ and $\psi_0(\theta) = \pi - \theta$.*

Proof. To prove (*i*), we consider w_γ defined by (4.12), with $\gamma = 0$. It is plain that $w_0(z) = w_0(z/|z|)$ and letting $\widehat{w_0}(\theta) = w_0(\cos(\theta), \sin(\theta))$, then $\widehat{w_0}$ solves (4.19), with $\lambda = \frac{(2s-1)^2}{4}$, pointwise, thanks to (4.13). Moreover, by a change of variable

$$\widehat{w_0}(\theta) = b_s \int_{-\cot(\theta)}^\infty \frac{1}{(1+t^2)^{\frac{2s+1}{2}}}\,dt.$$

From this, we find that $\widehat{w_0}'(\theta) = -b_s(\sin(\theta))^{2s-1}$. This implies that $w_0 \in H^1((0,\pi);\sin(\theta)^{1-2s})$, because $w_0 \in L^\infty(0,\pi)$. Since w_0 does not change sign we deduce that $\lambda_1(s) = \frac{(2s-1)^2}{4}$.

To prove (ii), we suppose on the contrary that $q_+ = \frac{2s-1}{2} + \sqrt{\lambda_2(s)} < 2s$. Then consider the function

$$U(r\cos(\theta), r\sin(\theta)) = r^{q_+}\psi_2(\theta).$$

Note that $q_+ > 0$, since $\frac{(2s-1)^2}{4} = \lambda_1(s) < \lambda_2(s)$. By direct computations, we have that U solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^2_+ \\ -t^{1-2s}\nabla U \cdot e_2 = \overline{\kappa}_s a_s x^{-2s} U & \text{on } \mathbb{R}_+ \\ U = 0 & \text{on } \mathbb{R}_-. \end{cases} \tag{4.20}$$

We observe that $U\big|_{\mathbb{R}} = \omega_{q_+}\psi_2(0)$. Recall that the extension of ω_{q_+} is given by w_{q_+} which solves (4.13). Since also $q_+ > \frac{2s-1}{2}$, we have that $U, w_{q_+} \in H^1_{loc}(\overline{\mathbb{R}^2_+}; t^{1-2s})$. It follows that $V := U - w_{q_+}\psi_2(0) \in C^0(\overline{\mathbb{R}^2_+})$ solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } \mathbb{R}^2_+ \\ V = 0 & \text{on } \mathbb{R}. \end{cases} \tag{4.21}$$

We then consider the odd reflection (in the t-direction) of V denoted by $\widetilde{V} \in H^1_{loc}(\mathbb{R}^2; |t|^{1-2s})$. It solves, weakly,

$$\operatorname{div}(|t|^{1-2s}\nabla\widetilde{V}) = 0 \qquad \text{in } \mathbb{R}^2. \tag{4.22}$$

Since by our assumption, the growth of V is of order $q_+ < 2$, it then follows from a well known Liouville theorem that $\widetilde{V}(x,t) = at|t|^{2s-1} + b + cx$, for some constants $a, b, c \in \mathbb{R}$, see e.g. [6]. Since $q_+ < 2s$, the continuity and oddness, with respect to t, of $\widetilde{V}$ imply that $\widetilde{V} = V \equiv 0$ on $\overline{\mathbb{R}^2_+}$. Hence $U = w_{q_+}\psi_2(0)$ on $\overline{\mathbb{R}^2_+}$. Now from (4.20) and the definition of w_{q_+}, we deduce that

$$\psi_2(0)(-\Delta)^s\omega_{q_+} = (-\Delta)^s(U\big|_{\mathbb{R}}) = a_s x^{-2s}U\big|_{\mathbb{R}} = (a_s - \mu(q_+,s))x^{-2s}\omega_{q_+}\psi_2(0) \quad \text{on } \mathbb{R}_+.$$

We then get $\mu(q_+,s) = 0$ and hence thanks to (4.10), either $q_+ = 0$ or $q_+ = 2s-1$. We now check that this leads to a contradiction. Indeed, each of these cases implies that $\sqrt{\lambda_2(s)} = \pm\frac{2s-1}{2}$ so that $\lambda_2(s) = \lambda_1(s)$ which is not possible.

Now (iii) follows by noticing that the explicit solutions to the first equation in (4.19), for $\lambda = 0$, are given by the affine functions. Note, in this case, that $\overline{\kappa}_{1/2} = 1$ and $a_{1/2} = \frac{1}{\pi}$. □

The following result shows that for $u \in H^s_{loc}(\overline{\mathbb{R}_+}) \cap L^1_s(\mathbb{R}_+)$ solving (4.6) then $u - u(0) \in L^2((0,R); x^{-2s})$ for all $R > 0$. With this property, we immediately deduce that $(u - u(0))1_{[0,\infty)} \in H^s_{loc}(\mathbb{R})$, thanks to (2.1). This will be useful in order to use Lemma 4.4.

Lemma 4.6. *Let $s \in (0,1)$ and $u \in H^s_{loc}(\overline{\mathbb{R}_+}) \cap L^1_s(\mathbb{R}_+)$ be a solution to*

$$(-\Delta)^s_{\mathbb{R}_+}u = 0 \qquad \text{on } \mathbb{R}_+.$$

Then $v := (u - u(0))1_{[0,\infty)} \in H^s_{loc}(\mathbb{R})$.

Proof. In view of (2.1), it suffices to prove that $\int_0^1 x^{-2s}v^2(x)\,dx < \infty$. In the case $2s \le 1$, this follows immediately from Theorem 3.3. Next, we consider the case $2s > 1$. Let $G \in C^1(\mathbb{R})$ be given by $G(t) = 0$ for $|t| < 1$ and $G(t) = t$ for $|t| \ge 2$. We define $v_k(x) = \frac{1}{k}G(kv(x))$. By Theorem 3.3, we have that $v \in C([0,1))$ and $v(0) = 0$. Therefore $\varphi v_k \in C_c(0,1) \cap H^s(0,1) \subset H^s_0(0,1)$ for all $\varphi \in C_c^\infty(0,1)$. By the Hardy inequality, see [19, Theorem 1.4.4.3] or [3], we have

$$\int_0^1 x^{-2s}(\varphi v_k)^2(x)\,dx \le C[(\varphi v_k)]^2_{H^s(0,1)} \le C[\varphi]^2_{H^s(0,1)}[v_k]^2_{H^s(0,1)} \le C[\varphi]^2_{H^s(0,1)}[v]^2_{H^s(0,1)}.$$

By Fatou's lemma, we deduce that $\int_0^1 x^{-2s}v^2(x)\,dx < \infty$ as desired. □

Proposition 4.7. *Let $u \in H^s_{loc}(\overline{\mathbb{R}_+})$ and $w \in H^s_{loc}(\mathbb{R}) \cap C^0(\mathbb{R})$, be such that*

$$(-\Delta)^s_{\overline{\mathbb{R}_+}} u = 0 \quad \text{in} \quad \mathbb{R}_+,$$

$$(-\Delta)^s_{\mathbb{R}_+} w = 0 \quad \text{in} \quad \mathbb{R}_+, \qquad w = 0 \qquad \text{on } (-\infty, 0]$$

and, for some constants $C > 0$ and $\varepsilon \in (0, 2s)$,

$$\|u\|^2_{L^2(-R,R)} \le CR^{1+2\varepsilon} \qquad \text{for all } R \ge 1 \tag{4.23}$$

and

$$\|w\|^2_{L^2(-R,R)} \le CR^{1+2\varepsilon} \qquad \text{for all } R \ge 1.$$

(i) *If $s \ne 1/2$ then u is a constant function on $\mathbb{R}_+$.*
(ii) *If $2s > 1$ then $w(x) = bx^{2s-1}$ for all $x \in \mathbb{R}_+$, for some constant $b \in \mathbb{R}$.*

Proof. By Lemma 4.6, the function $v = (u - u(0))1_{[0,\infty)} \in H^s_{loc}(\mathbb{R})$. We also have that (see (4.7))

$$(-\Delta)^s_{\mathbb{R}_+} v = (-\Delta)^s v - a_s x^{-2s} v = 0 \quad \text{in} \quad \mathbb{R}_+.$$

We then define $V(x,t) = (P(t,\cdot) \star v)(x,t)$. By Lemma 4.4, we have $V \in H^1_{loc}(\overline{\mathbb{R}^2_+}; t^{1-2s})$. Moreover,

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla V) = 0 & \text{in } \mathbb{R}^2_+ \\ -t^{1-2s}\nabla V \cdot e_2 = \overline{\kappa}_s a_s x^{-2s} V & \text{on } \mathbb{R}_+ \\ V = 0 & \text{on } \mathbb{R}_-. \end{cases} \tag{4.24}$$

For $s \ne 1/2$, we consider $(\psi_k)_k$, the complete sequence of eigenfunction corresponding to the eigenvalue problem (4.19). We then have that

$$V(r\cos(\theta), r\sin(\theta)) = \sum_{k=1}^\infty \alpha_k(r)\psi_k(\theta).$$

Then using (4.24), we find that the functions α_k solves the equation

$$\alpha_k'' + \frac{2-2s}{r}\alpha_k' + \left(\frac{(2s-1)^2}{4} - \lambda_k(s)\right)\alpha_k = 0 \qquad (0,\infty).$$

We then get

$$\alpha_k(r) = c_k^+ r^{\frac{2s-1}{2} + \sqrt{\lambda_k(s)}} + c_k^- r^{\frac{2s-1}{2} - \sqrt{\lambda_k(s)}}, \qquad \text{for some constants } c_k^\pm \in \mathbb{R}.$$

Noting that $V \in L^2((-1,1)\times(0,1);|z|^{-2}t^{1-2s})$ (by Hardy's inequality, see e.g. [17]) we thus get $c_k^- = 0$. It follows that

$$V(r\cos(\theta), r\sin(\theta)) = \sum_{k=1}^{\infty} c_k^+ r^{\frac{2s-1}{2}+\sqrt{\lambda_k(s)}}\psi_k(\theta).$$

From the growth assumption (4.23), the Parseval identity and Lemma 4.5 we have that $c_k^+ = 0$ for all $k \geq 2$. Now since $\lambda_1(s) = \frac{(2s-1)^2}{4}$, we obtain

$$V(r\cos(\theta), r\sin(\theta)) = c_1^+ r^{\frac{2s-1}{2}+\sqrt{\lambda_1(s)}}\psi_1(\theta) = c_1^+ r^{\frac{2s-1}{2}+\frac{2s-1}{2}\mathrm{sign}(2s-1)}\psi_1(\theta),$$

so that if $2s < 1$ then $V(z) = c_1^+\psi_1(0)$, while $V(r\cos(\theta), r\sin(\theta)) = c_1^+ r^{2s-1}\psi_1(\theta)$ in the case $2s > 1$. Recalling that $V(x,0) = v(x) = u(x) - u(0)$ for all $x \in \mathbb{R}_+$, we thus get $u(x) = u(0) + c_1^+\psi_1(0)x^{2s-1}$ for all $x \in \mathbb{R}_+$. Since[1] $(-\Delta)^s_{\mathbb{R}_+} x^{2s-1} \neq 0$ on $\mathbb{R}_+$, we deduce that u is a constant function on $\mathbb{R}_+$. The proof of (i) is thus complete.

To obtain (ii) we simply carry out the above argument replacing $(u-u(0))1_{[0,\infty)}$ with w (recall that by assumption $\int_{-R}^{R} x^{-2s}w^2\,dx < \infty$ for all $R > 0$, by the Hardy inequality) and we deduce that $w(x) = c_1^+\psi_1(0)(x)_+^{2s-1}$. □

The following result contains a nonexistence result of nonegative supersolutions to linear equation with Hardy potential in 2 dimensions as considered in [18] under weaker form.

Proposition 4.8. *Let $u \in H^{1/2}_{loc}(\overline{\mathbb{R}_+}) \cap L^1_{1/2}(\mathbb{R}_+)$ be such that*

$$(-\Delta)^{1/2}_{\mathbb{R}_+} u = 0 \quad in \quad \mathbb{R}_+.$$

Then u is a constant function.

Proof. We first note that, letting $v = (u-u(0))1_{[0,\infty)}$, we have that $(-\Delta)^{1/2}_{\mathbb{R}_+} v = 0$ on $\mathbb{R}_+$. Moreover it satisfies

$$|v(x)| \leq C|x|^{\alpha_0}, \tag{4.25}$$

by Theorem 3.3. This implies that $v \in H^{1/2}_{loc}(\mathbb{R}) \cap C^{\alpha_0}_{loc}(\mathbb{R})$ because $x^{-1/2}v \in L^2(0,R)$. In addition by the interior regularity, we have that v is smooth in $\mathbb{R}_+$.

We wish to show that $v \equiv 0$ on $\mathbb{R}$. To this end, we write $v = v_+ - v_- = \max(v,0) - \max(-v,0)$. We will prove that $v_\pm = 0$.

Claim: $v_- = 0$ **on** $\mathbb{R}$. Suppose on the contrary that v_- is not identically zero. By the Kato-type inequality, we have

$$(-\Delta)^{1/2}v_- - \frac{1}{\pi}x^{-1}v_- = (-\Delta)^{1/2}_{\mathbb{R}_+}v_- \geq 0 \qquad \text{in } \mathbb{R}_+$$

in the weak sense. By the strong maximum principle, it is positive on $\mathbb{R}_+$. We let $V(t,x) = (P(t,\cdot)\star v)(x)$. Then by Lemma 4.4, we have that $V \in H^1_{loc}(\overline{\mathbb{R}^2_+})$ and

$$\begin{cases} \Delta V = 0 & \text{in } \mathbb{R}^2_+ \\ -\partial_t V \geq \frac{1}{\pi}x^{-1}V & \text{on } \mathbb{R}_+ \\ V = 0 & \text{on } \mathbb{R}_-. \end{cases} \tag{4.26}$$

[1]Communicated to the author by X. Ros-Oton and proved in [2,21].

Since $v_- \in C^{\alpha_0}_{loc}(\mathbb{R})$, we also have that $V \in C^{\alpha_0'}_{loc}(\overline{\mathbb{R}^2_+})$, for some $\alpha_0' > 0$. Therefore since $V(0) = 0$ by (4.25), we then deduce that

$$|V(z)| \leq C|z|^{\alpha_0'}. \tag{4.27}$$

We let $\tau(r) = \int_{S^1} V(r\cos(\theta), r\sin(\theta))\psi_0(\theta)\, d\theta$, where $\psi_0(\theta) = \pi - \theta > 0$ is given by Lemma 4.5-(iii). We have that $\tau \in H^1((0,R);r)$ for every $R > 0$ and thanks to (4.27) we have $r^{-1/2}\tau \in L^2((0,R);r)$. Then letting an arbitrary $f \in C^\infty_c(\mathbb{R})$, we can test (4.26) with $f\psi_0$ to deduce that

$$\int_{\mathbb{R}} \tau'(r) f'(r) r\, dr \geq 0,$$

so that

$$-(r\tau')' \geq 0 \qquad \text{on } (0,1). \tag{4.28}$$

Following [18], for $\delta \in (1/2, 1)$, we consider the function

$$\varphi_\delta(r) := (-\log(r))^{-\delta} \qquad \text{for } r \in (0, 1/e), \qquad \varphi_\delta(0) := 0.$$

Then $\varphi_\delta \in H^1((0,R);r)$ with

$$\int_0^{1/e} r^{-1}\varphi_\delta^2(r)\, dr = \frac{1}{2\delta - 1}. \tag{4.29}$$

A direct computation yields

$$-(r\varphi_\delta')' \leq 0 \qquad \text{on } (0,1). \tag{4.30}$$

We observe that by (4.27), $\tau(0) = 0$. Letting $K := \tau(1/e)$ then $(\tau - K\varphi_\delta)_- \in H^1_0((0,1/e);r) \cap C([0,1))$. Moreover, by combining (4.28) and (4.30) we see that $-(r(\tau - \varphi_\delta)')' \geq 0$ on $(0,1/e)$. Using $(\tau - K\varphi_\delta)_-$ as a test function in this differential inequality, we see that $[(\tau - K\varphi_\delta)_-]_{H^1_0((0,1/e);r)} = 0$. From this together with the fact that $(\tau - K\varphi_\delta)_-(0) = 0$, we deduce that $(\tau - K\varphi_\delta)_- = 0$ on $(0,1/e)$, so that $\tau \geq K\varphi_\delta$. But then (4.29) implies

$$\int_0^{1/e} \tau^2(r) r^{-1}\, dr \geq K \int_0^{1/e} \varphi_\delta^2(r) r^{-1}\, dr = \frac{K}{2\delta - 1}$$

and thus letting $\delta \searrow 1/2$, we reach a contradiction. Therefore $v_- \equiv 0$ on $\mathbb{R}_+$ as claimed.

As a consequence $v = v_+$. Now using precisely the same argument as above, we also deduce that $v_+ = 0$. □

4.2. **Proof of Theorem 4.1 (completed).**

Proof. If $\varepsilon < \min(2s, 1)$, then by Proposition 4.3, u is a function of x_N. Applying Poposition 4.7 and Proposition 4.8, we get the result.
We now consider the case $2s > \varepsilon \geq 1$. By Proposition 4.3 and a scaling, we have that

$$\begin{aligned} \|\nabla_{x'} u\|_{L^\infty(B_R^+)} &\leq C\left(\|u\|_{L^2(B_{2R}^+)} + R^{2s-1} \int_{|y|\geq R/2} |u(y)| 1_{\mathbb{R}^N_+}(y) |y|^{-N-2s}\, dy \right) \\ &\leq R^{\varepsilon - 1}, \end{aligned} \tag{4.31}$$

where we used that, since $\varepsilon < 2s$, then $\int_{|y|\geq R/2} |u(y)| 1_{\mathbb{R}^N_+}(y)|y|^{-N-2s}\, dy \leq CR^{-2s+\varepsilon}$. Note that $0 \leq \varepsilon - 1 < 2s - 1 < 1$. On the other hand, for $i = 1, \ldots, N-1$, we have that $\partial_{x_i} u \in H^s_{loc}(\overline{\mathbb{R}^N_+}) \cap L^1(\mathbb{R}^N_+; (1 + |x|^{N+2s})^{-1})$ by Proposition 4.2 and (4.31). Moreover it satisfies $(-\Delta)^s_{\mathbb{R}^N_+} \partial_{x_i} u = 0$ on $\mathbb{R}^N_+$. We then deduce, from Proposition 4.7-(i), Propositon 4.3 and (4.31),

that $\partial_{x_i} u$ is constant on $\mathbb{R}^N_+$ for all i. Hence $u(x', x_N) = a(x_N) + b(x_N) \cdot x'$. Now noting that $u(x+h) - u(x) = b(x_N)h$, for all $h \in \mathbb{R}^{N-1}$, we see that $(-\Delta)^s_{\mathbb{R}^N_+} b = (-\Delta)^s_{\mathbb{R}_+} b = 0$ and hence from the growth assumption of u and Proposition 4.7, we obtain $b(x_N) = b(0)$ for all $x_N > 0$. From this we deduce that $(-\Delta)^s_{\mathbb{R}^N_+} a(x_N) = 0$ and therefore $a(x_N) = a(0)$, thanks to Proposition 4.7-(i).

□

5. Regularity estimates on open sets

We define a class of kernels arising after performing a parameterization that locally flatten the underlying domain.

For $\psi \in C^{0,1}(\mathbb{R}^{N-1})$, we define the kernel $K^\psi_s : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty]$ by

$$K^\psi_s(x,y) := 1_{\mathbb{R}^N_+}(x) 1_{\mathbb{R}^N_+}(y) \frac{c_{N,s}}{|\Psi(x) - \Psi(y)|^{N+2s}}, \qquad \Psi(x) = \Psi(x', x_N) := (x', x_N + \psi(x')). \tag{5.1}$$

We have

$$|\Psi(x) - \Psi(y)|^2 = |x-y|^2 + \mu(x,y), \qquad \mu(x,y) := 2(x_N - y_N)(\psi(x') - \psi(y')) + (\psi(x') - \psi(y'))^2. \tag{5.2}$$

By the fundamental theorem of calculus,

$$\frac{|\mu(x,y)|}{|x-y|^2} \le \left(2 + \int_0^1 |\nabla\psi(x' + t(y'-x'))|\, dt\right) \int_0^1 |\nabla\psi(x' + t(y'-x'))|\, dt. \tag{5.3}$$

Recalling (3.2) for the definition of K^+_s. Then for $x, y \in \mathbb{R}^N_+$, we have

$$\begin{aligned} &|x-y|^{N+2s}\left(K^\psi_s(x,y) - K^+_s(x,y)\right) \\ &= -(2s+N)|x-y|^{N+2s}\mu(x,y) \int_0^1 \left(|x-y|^2 + t\mu(x,y)\right)^{-\frac{N+2s+2}{2}}\, dt \\ &= -(2s+N)c_{N,s}\frac{\mu(x,y)}{|x-y|^2} \int_0^1 \left(1 + t\frac{\mu(x,y)}{|x-y|^2}\right)^{-\frac{N+2s+2}{2}}\, dt. \end{aligned} \tag{5.4}$$

We then get from (5.3), for all $x, y \in \mathbb{R}^N_+$

$$|x-y|^{N+2s}|K^\psi_s(x,y) - K^+_s(x,y)| \le \frac{c_{N,s}(N+2s)(2 + \|\nabla\psi\|_{L^\infty(\mathbb{R}^{N-1})})\|\nabla\psi\|_{L^\infty(\mathbb{R}^{N-1})}}{\left(1 - (2 + \|\nabla\psi\|_{L^\infty(\mathbb{R}^{N-1})})\|\nabla\psi\|_{L^\infty(\mathbb{R}^{N-1})}\right)^{\frac{N+2s+2}{2}}}. \tag{5.5}$$

In the following, for a kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to [-\infty, \infty]$, we define the bilinear form

$$\mathcal{D}_K(u,v) = \frac{1}{2}\int_{\mathbb{R}^{2N}} (u(x) - u(y))(v(x) - v(y))K(x,y)\, dxdy. \tag{5.6}$$

5.1. **A priori estimates.** We recall the following Caccioppoli type inequality, see e.g. [16, Lemma 9.1].

Lemma 5.1. *Let Ω be an open set, $h \in L^1_{loc}(\Omega)$ and $\mathcal{K}$ a nonnegative and symmetric function defined on $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^N$ such that*

$$\int_{\mathbb{R}^{2N}} (w(x) - w(y))(\varphi(x) - \varphi(y))\mathcal{K}(x,y)\, dxdy = \int_{\mathbb{R}^N} h(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^\infty_c(\Omega).$$

Then for all $\varepsilon > 0$ and all $\varphi \in C_c^\infty(\Omega)$, we have

$$(1-\varepsilon)\int_{\mathbb{R}^{2N}} (w(x)-w(y))^2\varphi^2(y)\mathcal{K}(x,y)\,dydx \le \int_{\mathbb{R}^N} |h(x)||v(x)|\varphi^2(x)\,dx + \varepsilon^{-1}\int_{\mathbb{R}^{2N}} w^2(x)(\varphi(x)-\varphi(y))^2\mathcal{K}(x,y)\,dydx.$$

Next, we prove the following result toward the regularity up to the boundary in C^1 domains.

Proposition 5.2. *Let $s_0, \overline{\alpha} \in (0,1)$ and $p > \frac{N}{2s_0}$. Then there exist $\varepsilon_0, C > 0$ such that for every $\psi \in C^1(\mathbb{R}^{N-1})$, $s \in [s_0, 1)$ and for every $g \in L^p(\mathbb{R}^N)$, $v \in H^s(\mathbb{R}^N_+)$ satisfying*

$$\|\nabla\psi\|_{L^\infty(\mathbb{R}^{N-1})} < \varepsilon_0$$

and

$$\mathcal{D}_{K_s^\psi}(v,\varphi) = \int_{B_2^+} g(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2),$$

we have

$$\sup_{r>0} r^{-2\alpha-N} \sup_{z\in B_1'} \|v - v_{B_r^+(z)}\|^2_{L^2(B_r^+(z))} \le C(\|v\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)})^2, \tag{5.7}$$

where $u_A = \frac{1}{|A|}\int_A f(x)\,dx$ and $\alpha := \min(2s - N/p, \overline{\alpha})$.

Proof. Assume that the assertion in the proposition does not hold, then for every $n \in \mathbb{N}$, there exist $\psi_n \in C^1(\mathbb{R}^{N-1})$, $s_n \in [s_0, 1)$, $g_n \in L^p(\mathbb{R}^N)$, $v_n \in H^s(\mathbb{R}^N_+)$, with

$$\|g_n\|_{L^p(\mathbb{R}^N)} + \|v_n\|_{L^2(\mathbb{R}^N_+)} \le 1$$

satisfying

$$\|\nabla\psi_n\|_{L^\infty(\mathbb{R}^{N-1})} < \frac{1}{n} \tag{5.8}$$

and

$$\mathcal{D}_{K_{s_n}^{\psi_n}}(v_n,\varphi) = \int_{B_2^+} g_n(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2), \tag{5.9}$$

while

$$\sup_{r>0} r^{-N-2\alpha_n} \sup_{z\in B_1^+} \|v_n - (v_n)_{B_r(z)}\|^2_{L^2(B_r(z))} > n.$$

where $\alpha_n := \min(\overline{\alpha}, 2s_n - N/p))$. Consequently, there exists $\overline{r}_n > 0$ and $z_n \in B_1'$ such that

$$\overline{r}_n^{-N-2\alpha_n}\|v_n - (v_n)_{B^+_{\overline{r}_n}(z_n)}\|^2_{L^2(B^+_{\overline{r}_n}(z_n))} > n/2. \tag{5.10}$$

We consider the (well defined, because $\|v_n\|_{L^2(\mathbb{R}^N_+)} \le 1$) nonincreasing function $\Theta_n : (0,\infty) \to [0,\infty)$ given by

$$\Theta_n(\overline{r}) = \sup_{r\in[\overline{r},\infty)} r^{-N-2\alpha_n}\|v_n - (v_n)_{B_r^+(z_n)}\|^2_{L^2(B_r^+(z_n))}.$$

Obviously, for $n \ge 2$, by (5.10),

$$\Theta_n(\overline{r}_n) > n/2 \ge 1. \tag{5.11}$$

Hence, provided $n \ge 2$, there exists $r_n \in [\overline{r}_n, \infty)$ such that

$$\Theta_n(r_n) \ge r_n^{-N-2\alpha_n}\|v_n - (v_n)_{B^+_{r_n}(z_n)}\|^2_{L^2(B^+_{r_n}(z_n))} \ge \Theta_n(\overline{r}_n) - 1/2 \ge (1-1/2)\Theta_n(\overline{r}_n) \ge \frac{1}{2}\Theta_n(r_n),$$

where we used the monotonicity of Θ_n for the last inequality, while the first inequality comes from the definition of Θ_n. In particular, thanks to (5.11), $\Theta_n(r_n) \ge n/4$. Now since

$\|\overline{v}_n\|_{L^2(\mathbb{R}^N)} \leq 1$, we have that $r_n^{-N-2\alpha} \geq n/8$, so that $r_n \to 0$ as $n \to \infty$. We now define the sequence of functions

$$w_n(x) = \Theta_n(r_n)^{-1/2} r_n^{-\alpha_n} \left\{ v_n(r_n x + z_n) - \frac{1}{|B_1^+|} \int_{B_1^+} v_n(r_n x + z_n)\, dx \right\},$$

which, satisfies

$$\|w_n\|^2_{L^2(B_1^+)} \geq \frac{1}{2} \qquad \text{and} \qquad \int_{B_1^+} w_n(x)\, dx = 0 \qquad \text{for every } n \geq 2. \tag{5.12}$$

Using that, for every $r > 0$ and $z \in B_1'$, $\|v_n - (v_n)_{B_r^+(z)}\|^2_{L^2(B_r^+(z))} \leq r^{N+2\alpha_n}\Theta_n(r)$ and the monotonicity of Θ_n, by [16, Lemma 3.1], we find that

$$\|w_n\|^2_{L^2(B_R^+)} \leq C R^{N+2\alpha_n} \qquad \text{for every } R \geq 1 \text{ and } n \geq 2, \tag{5.13}$$

for some constant $C = C(N, s_0, p) > 0$.
We define

$$K_n(x,y) = K_{s_n}^{\widetilde{\psi}_n}(x,y), \qquad \text{where } \widetilde{\psi}_n(x) = \frac{1}{r_n}\psi(r_n x + z_n).$$

Since $|z_n| \leq 1$, we have that $B_{\frac{1}{2r_n}}(0) \subset B_{\frac{2}{r_n}}(\frac{-z_n}{r_n})$. Letting

$$\overline{g}_n(x) = \frac{r_n^{2s-\alpha_n}}{\Theta_n(r_n)^{\frac{1}{2}}} g_n(r_n x + z_n),$$

then by a change of variable, we obtain that, for all $\varphi \in C_c^\infty(B_{\frac{1}{2r_n}}(0))$,

$$\frac{1}{2}\int_{\mathbb{R}^{2N}} (w_n(x) - w_n(y))(\varphi(x) - \varphi(y)) K_n(x,y)\, dxdy = \int_{\mathbb{R}^N_+} \overline{g}_n(x)\varphi(x)\, dx. \tag{5.14}$$

We fix $M > 1$ and let $n \geq 2$ large, so that $1 < M < \frac{1}{2r_n}$. Let $\chi_M \in C_c^\infty(B_{4M})$ be such that $\chi_M = 1$ on B_{2M} and define $W_n := \chi_M w_n$. Then by letting $\varphi = \chi_{M/4} \in C_c^\infty(B_M)$, we find that

$$\frac{1}{2}\int_{\mathbb{R}^{2N}} (W_n(x) - W_n(y))(\varphi(x) - \varphi(y)) K_n(x,y)\, dxdy = \int_{\mathbb{R}^N_+} \widetilde{g}_n(x)\varphi(x)\, dx, \tag{5.15}$$

with

$$\|\widetilde{g}_n\|_{L^p(\mathbb{R}^N)} \leq \|\overline{g}_n\|_{L^p(\mathbb{R}^N)} + C(M).$$

By Lemma 5.1, we have

$$\begin{aligned}
&(1-\varepsilon)\int_{\mathbb{R}^{2N}} (W_n(x) - W_n(y))^2 \varphi^2(y) K_n(x,y)\, dydx \leq 2\int_{B_M^+} |\widetilde{g}_n(x)||W_n(x)|\varphi^2(x)\, dx \\
&+ 2\varepsilon^{-1}\int_{\mathbb{R}^{2N}} W_n^2(x)(\varphi(x) - \varphi(y))^2 K_n(x,y)\, dydx \qquad (5.16)\\
&\leq 2\int_{B_M^+} |\widetilde{g}_n(x)||W_n(x)|\varphi^2(x)\, dx + \varepsilon^{-1}\frac{c_{N,s_n}}{1-s_n} C(M)\|w_n\|_{L^2(B_{2M})}.
\end{aligned}$$

By Young's inequality and Sobolev inequality, we have that

$$\begin{aligned}\int_{B_M^+} |\widetilde{g}_n(x)|W_n(x)|\varphi^2(x)\,dx &\le \varepsilon \int_{B_M^+} |\widetilde{g}_n(x)|W_n^2(x)\varphi^2(x)\,dx + C_\varepsilon \int_{B_M^+} |\widetilde{g}_n(x)|\varphi^2(x)\,dx \\ &\le \varepsilon C(1-s_n)\|\varphi\widetilde{g}_n\|_{L^p(\mathbb{R}^N)}\|\varphi W_n\|^2_{H^s(\mathbb{R}^N_+)} + C(M)\|\widetilde{g}_n\|_{L^p(\mathbb{R}^N)} \\ &\le C\varepsilon\|\widetilde{g}_n\|_{L^p(\mathbb{R}^N)} \int_{\mathbb{R}^{2N}} (W_n(x)-W_n(y))^2\varphi^2(y)K_n(x,y)\,dydx \\ &+ C\|w_n\|_{L^2(B_{2M})} + C(M)\|\widetilde{g}_n\|_{L^p(\mathbb{R}^N)} \\ &\le C(M)\left(\varepsilon \int_{\mathbb{R}^{2N}} (W_n(x)-W_n(y))^2\varphi^2(y)K_n(x,y)\,dydx + 1\right),\end{aligned}$$

where we used (5.13) in the last inequality. Recall that $W_n = w_n$ on $B_{M/2}$. Using the above inequality in (5.16), provided ε is small, we obtain

$$(1-s_n)[w_n]^2_{H^{s_n}\left(B^+_{M/2}\right)} \le \int_{\mathbb{R}^{2N}} (W_n(x)-W_n(y))^2\varphi^2(y)K_n(x,y)\,dydx \le C(M). \tag{5.17}$$

Let $\overline{s} = \lim_{n\to\infty} s_n$. Thanks to (5.17), we have that w_n is bounded in $H^{s_n}_{loc}(\overline{\mathbb{R}^N_+})$. Hence by a diagonal argument, up to a subsequence, there exists $w \in H^{\overline{s}}_{loc}(\overline{\mathbb{R}^N_+})$ such that

$$w_n \to w \qquad \text{in } L^2_{loc}(\mathbb{R}^N) \tag{5.18}$$

and, for all $\sigma < \overline{s}$,

$$w_n \to w \qquad \text{in } H^{\sigma}_{loc}(\mathbb{R}^N). \tag{5.19}$$

Therefore passing to the limit in (5.12) and (5.13), we obtain

$$\|w\|^2_{L^2(B_1^+)} \ge \frac{1}{2} \qquad \text{and} \qquad \int_{B_1^+} w(x)\,dx = 0 \tag{5.20}$$

and

$$\|w\|^2_{L^2(B_R^+)} \le CR^{N+2\beta} \qquad \text{for every } R \ge 1, \tag{5.21}$$

where $\beta = \min(2\overline{s} - N/p, \overline{\alpha})$. We observe that (5.21) and (5.18) imply

$$\int_{\mathbb{R}^N_+} \frac{|w_n(x) - w(x)|}{1+|x|^{N+2s_n}}\,dx \to 0 \qquad \text{as } n \to \infty. \tag{5.22}$$

Next by (5.14) and the choice of α_n, we have for all $\phi \in C_c^\infty(B_M)$, with M as above,

$$\begin{aligned}&\frac{1}{2}\left|\int_{\mathbb{R}^{2N}} (w_n(x)-w_n(y))(\phi(x)-\phi(y))K_n(x,y)\,dxdy\right| \le \|\overline{g}_n\|_{L^p(\mathbb{R}^N)}\|\phi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)} \\ &\le \Theta_n(r_n)^{-1/2} r_n^{2s-\alpha_n-N/p}\|g_n\|_{L^p(\mathbb{R}^N)}\|\phi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)} \\ &\le \Theta_n(r_n)^{-1/2}\|\phi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^N)}.\end{aligned} \tag{5.23}$$

If $\overline{s} < 1$, then from (5.8) and (5.4), we get

$$|K_n(x,y) - K^+_{s_n}(x,y)| \le \frac{C}{n}|x-y|^{-N-2s_n} \qquad \text{as } n \to \infty. \tag{5.24}$$

Hence by (5.24), (5.19), (5.22) and using that $\Theta_n(r_n) \to \infty$ as $n \to \infty$, we deduce from (5.23) that

$$\int_{\mathbb{R}^N_+\times\mathbb{R}^N_+} (w(x)-w(y))(\phi(x)-\phi(y))|x-y|^{-N-2\overline{s}}\,dxdy = 0, \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^N).$$

By Theorem 4.1, we deduce that $w \equiv Const.$ on $\mathbb{R}^N_+$. This leads to a contradiction in (5.20).

If now $\overline{s} = 1$, then applying Lemma 8.1 and using (5.23), we obtain

$$\int_{\mathbb{R}^N_+} \nabla w(x) \cdot \nabla \phi(x)\, dx = 0 \qquad \text{for all } \phi \in C^\infty_c(\mathbb{R}^N).$$

From this and (5.21), it follows from the well known Liouville theorem for the Laplacian that $w \equiv Const.$ on $\mathbb{R}^N_+$. This leads to a contradiction in (5.20).

□

We state the following result from [26] that we will use frequently in the following.

Lemma 5.3. *Let $u \in L^\infty(B_1^+)$ and suppose that for every $x = (x', x_N) \in B_1^+$,*

$$|u(x)| + [u]_{C^\gamma(B_{\frac{x_N}{2}}(x))} \le C, \qquad \text{for some constant } C > 0 \text{ and } \gamma \in (0,1).$$

Then there exists $\overline{C} > 0$ and $r > 0$ depending only on N, C, γ such that

$$\|u\|_{C^\gamma(B_r)} \le \overline{C}.$$

As a consequence of Proposition 5.2, we get the

Corollary 5.4. *Let $s_0, \overline{\alpha} \in (0,1)$. Let $\psi \in C^1(\mathbb{R}^{N-1})$, $s \in [s_0, 1)$ and $p > \frac{N}{2s_0}$. Consider $g \in L^p(\mathbb{R}^N)$, $v \in H^s(\mathbb{R}^N_+)$ satisfying*

$$\mathcal{D}_{K_s^\psi}(v, \varphi) = \int_{B_2^+} g(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^\infty_c(B_2), \tag{5.25}$$

Then there exist $\varepsilon_0, C > 0$ depending only on s_0, N, p and $\overline{\alpha}$ such that if

$$[\psi]_{C^1(\mathbb{R}^{N-1})} < \varepsilon_0,$$

we have

$$\|v\|_{C^\alpha(\overline{B_1^+})} \le C \left(\|v\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)} \right), \tag{5.26}$$

with $\alpha := \min(\overline{\alpha}, 2s - N/p)$.

Proof. In view of (5.7), by a classical iteration argument (see e.g. [9]), we can find a constant $C_0 = C_0(N, s_0, p, \beta, \overline{c}, \varepsilon_0) > 0$ and a function $m \in L^\infty(B_1')$, with $\|m\|_{L^\infty(B_1')} \le C_0$ such that

$$\sup_{r>0} r^{-2\alpha - N} \sup_{z \in B_1'} \|\overline{v} - m(z)\|^2_{L^2(B_r^+(z))} \le C_0 (\|v\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)})^2, \tag{5.27}$$

Without loss of generality, we assume that $\|v\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)} \le 1$. Let $\xi := (\xi', \xi_N) \in B_1^+$ and $\rho := \xi_N/2$. We define

$$u_\xi(y) = v(\xi + \rho y) - m(\xi') = v(\xi' + 2\rho e_N + \rho y) - m(\xi')$$

and $\widetilde{g}(y) = g(\xi + \rho y)$. We have, for all $\varphi \in C^\infty_c(B_2)$

$$\mathcal{D}_{K_{\rho,s}}(u_\xi, \varphi) = \rho^{2s} \int_{B_2} g(\xi + \rho y)\varphi(y),$$

where $K_{\rho,s}(x,y) = 1_{\{x_N>-2\}}(x)1_{\{y_N>-2\}}(y)\frac{c_{N,s}}{|\Psi_\rho(x)-\Psi_\rho(y)|^{N+2s}}$ and $\Psi_\rho(x) = \frac{1}{\rho}\Psi(\rho x+\xi)$. By interior regularity, see e.g. [16, Theorem 4.1], we have

$$\begin{aligned}\|u_\xi\|_{C^\alpha(B_{1/2})} &\le C\left(\|u_\xi\|_{L^\infty(B_1)} + \int_{\{|y|\ge 1/2\}\{y_N>-2\}} \frac{|u_\xi(y)|}{|y|^{N+2s}}\,dy + \rho^{2s}\|\widetilde{g}\|_{L^p(\mathbb{R}^N)}\right)\\ &\le C\left(\|u_\xi\|_{L^\infty(B_1)} + \int_{\{|y|\ge 1/2\}\{y_N>-2\}} \frac{|u_\xi(y)|}{|y|^{N+2s}}\,dy + \rho^{2s-N/p}\right). \qquad (5.28)\end{aligned}$$

By (5.27), we have

$$\|u_\xi\|_{L^\infty(B_1)} \le C\rho^\alpha. \qquad (5.29)$$

On the other hand, letting

$$\begin{aligned}w_\xi(y) &:= \big(u(\xi+y)-m(\xi')\big)\,1_{\{x_N>-2\}}(y)\\ &= \big(u(\xi'+2\rho e_N+y)-m(\xi')\big)\,1_{\{x_N>-2\}}(y),\end{aligned}$$

and using Corollary 5.27, we get

$$\begin{aligned}&\int_{|y|\ge 1/2\cap\mathbb{R}^N_+} \frac{|u_\xi(y)|}{|y|^{N+2s}}\,dy = \rho^{2s}\int_{|y|\ge\rho/2}|y|^{-N-2s}|w_\xi(y)|\,dy\\ &\le \rho^{2s}\sum_{k=0}^\infty \int_{\rho 2^k\ge|y|\ge\rho 2^{k-1}}|y|^{-N-2s}|w_\xi(y)|\,dy \le \rho^{2s}\sum_{k=0}^\infty (2^k\rho)^{-N-2s}\int_{\rho 2^{k+1}\ge|y|}|w_\xi(y)|\,dy\\ &\le \rho^{2s}\sum_{k=0}^\infty (2^k\rho)^{-N-2s}(\rho 2^{k+1})^N\int_{B^+_{2^{k+2}\rho}}|u(\xi'+\zeta)-m(\xi')|\,d\zeta\\ &\le C\rho^{2s}\sum_{k=0}^\infty (2^k\rho)^{-N-2s}(\rho 2^k)^{N+\alpha} \le C\rho^\alpha\sum_{k=0}^\infty 2^{-k(2s-\alpha)} \le C\rho^\alpha.\end{aligned}$$

Hence, combining this with (5.29) and (5.28), we then get

$$\|u_\xi\|_{C^{1,\delta}(B_{1/2})} \le C\rho^\alpha. \qquad (5.30)$$

Consequently,

$$|v(\xi)| \le |m(\xi')| + C\rho^\alpha \le C$$

and

$$[v-m(\xi')]_{C^\alpha(B_{\rho/2}(\xi))} = [v]_{C^\alpha(B_{\rho/2}(\xi))} \le C.$$

We can apply Lemma 5.3 to deduce that

$$\|v\|_{C^\alpha(B^+_{1/2})} \le C$$

and the proof is complete. □

5.2. **Higher order regularity estimates.** The following result provides the first steps toward the higher order boundary regularity. Next, we define, for $i = 1,\dots,N$, $\zeta\in\mathbb{R}^N$, $r>0$ and $w\in L^2_{loc}(\mathbb{R}^N)$,

$$P_{w,r}(x) = w_{B^+_r} + \sum_{i=1}^{N-1}\frac{x_i}{\|y_i\|^2_{L^2(B^+_r)}}\int_{B^+_r} w(y)y_i\,dy =: w_{B^+_r} + \sum_{i=1}^{N-1} x_i p^i_w(r), \qquad (5.31)$$

where for a measurable subset $A \in \mathbb{R}^N$ and a measurable function u, we write $u_A = \frac{1}{|A|}\int_A u(y)\,dy$. We note that $P_{w,r}$ is the $L^2(B_r)$-projection of $x \mapsto w(x) - w_{B_r^+}$ on the finite dimension subspace of $L^2(B_r)$, $\text{span}\{x_1, \dots, x_{N-1}\}$. We next consider the regularity of the tangential gradient of our solutions.

Proposition 5.5. *Let $s \in (1/2, 1)$, $\beta \in (0, 2s-1)$, $2s-N/p > 1$ and $\varepsilon < \min(2s-N/p-1, \beta)$. Then there exist $\varepsilon_0, C > 0$ such that for every $\psi \in C^{1,\beta}(\mathbb{R}^{N-1})$ and for every $g \in L^p(\mathbb{R}^N)$, $v \in H^s(\mathbb{R}^N_+)$ satisfying*

$$\mathcal{D}_{K_s^\psi}(v, \varphi) = \int_{B_2^+} g(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2), \tag{5.32}$$

$$\|g\|_{L^p(\mathbb{R}^N)} + \|v\|_{L^2(\mathbb{R}^N_+)} \le 1$$

and

$$[\nabla\psi]_{C^{0,\beta}(\mathbb{R}^{N-1})} < \frac{1}{4}, \qquad \psi(0) = |\nabla\psi(0)| = 0$$

then we have

$$\sup_{r>0} r^{-\frac{N}{2}-\gamma_1}\|v - P_{v,r}\|_{L^2(B_r)} \le C, \tag{5.33}$$

where $\gamma_1 := \min(2s - N/p, 1 + \beta) - \varepsilon > 1$.

Proof. Assume that the assertion does not hold, then for every integer $n \ge 2$, there exist $\psi_n \in C^{1,\beta}(\mathbb{R}^{N-1})$, $g_n \in L^p(\mathbb{R}^N)$, $v_n \in H^s(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$, with

$$\|g_n\|_{L^p(\mathbb{R}^N)} + \|v_n\|_{L^2(\mathbb{R}^N_+)} \le 1$$

satisfying

$$\mathcal{D}_{K_s^{\psi_n}}(v_n, \varphi) = \int_{B_2^+} g_n(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2), \tag{5.34}$$

while

$$\sup_{r>0} r^{-N-\gamma_1}\|v_n - P_{v_n,r}\|_{L^2(B_r)} > n$$

and

$$[\nabla\psi_n]_{C^{0,\beta}(\mathbb{R}^{N-1})} < \frac{1}{4}, \qquad \psi_n(0) = |\nabla\psi_n(0)| = 0. \tag{5.35}$$

Define

$$\Theta_n(\overline{r}) = \sup_{r\in[\overline{r},\infty)} r^{-\frac{N}{2}-\gamma_1}\|v_n - P_{v_n,r_n}\|_{L^2(B_r)}.$$

Proceeding as in the proof of Proposition 5.2, we can find a sequence r_n, tending to zero, such that, for all $n \ge 2$,

$$r_n^{-\frac{N}{2}-\gamma_1}\|v_n - P_{v_n,r_n}\|_{L^2(B_{r_n})} \ge \frac{1}{2}\Theta_n(r_n) \ge \frac{n}{4}.$$

We now define the sequence of functions

$$w_n(x) = \frac{r_n^{-\gamma_1}}{\Theta_n(r_n)}\left\{v_n(r_n x) - P_{v_n,r_n}(r_n x)\right\}. \tag{5.36}$$

It satisfies

$$\|w_n\|_{L^2(B_1)} \ge \frac{1}{2} \tag{5.37}$$

and for every $n \ge 2$, $i = 1 \dots, N-1$,

$$\int_{B_1} w_n(x)\,dx = 0 \qquad \text{and} \qquad \int_{B_1} w_n(x)x_i\,dx = 0. \tag{5.38}$$

By the definition and monotonicity of Θ_n, we can use similar arguments as in [16, 28], to obtain

$$\|w_n\|_{L^2(B_R)} \le CR^{N/2+\gamma_1} \qquad \text{for every } R \ge 1 \text{ and } n \ge 2, \tag{5.39}$$

for some constant $C = C(N, \gamma_1) > 0$.

For the following, we put $\hat{g}_n(x) = \frac{r_n^{2s-\gamma_1}}{\Theta_n(r_n)^{\frac{1}{2}}} g_n(r_n x)$, $\widetilde{\psi}_n(x') = \frac{1}{r_n}\psi(r_n x')$ and

$$K_n(x, y) = K_s^{\widetilde{\psi}_n}(x, y),$$

Then letting

$$P_n(x) := \frac{r_n^{\beta-\gamma_1}}{\Theta_n(r_n)^{\frac{1}{2}}} P_{v_n,r_n}(r_n x) = \frac{r_n^{1+\beta-\gamma_1}}{\Theta_n(r_n)^{\frac{1}{2}}} \sum_{i=1}^{N-1} p^i_{v_n}(r_n) x_i,$$

by a change of variable, we thus get

$$\mathcal{D}_{K_n}(w_n, \varphi) - r_n^{-\beta}\mathcal{D}_{K_n}(P_n, \varphi) = \int_{\mathbb{R}^N_+} \hat{g}_n(x)\varphi(x)\, dx.$$

Since $(-\Delta)^s_{\mathbb{R}^N_+} x_i = 0$ on $\mathbb{R}^N_+$, letting $K'_n(x, y) = r_n^{-\beta}(K_n(x, y) - K_s^+(x, y))$, we have

$$\mathcal{D}_{K_n}(w_n, \varphi) - \mathcal{D}_{K'_n}(P_n, \varphi) = \int_{\mathbb{R}^N_+} \hat{g}_n(x)\varphi(x)\, dx. \tag{5.40}$$

We fix $M > 1$ and let $n \ge 2$ large so that $1 < M < \frac{1}{2r_n}$. Let $\chi_M \in C_c^\infty(B_{4M})$ be such that $\chi_M = 1$ on B_{2M} and define $W_n := \chi_M w_n$. Then by letting $\varphi = \chi_{M/4} \in C_c^\infty(B_M)$, we find that

$$\mathcal{D}_{K_s^{\psi_n}}(W_n, \varphi) - \mathcal{D}_{K'_n}(P_n, \varphi) = \int_{\mathbb{R}^N_+} \widetilde{g}_n(x)\varphi(x)\, dx \tag{5.41}$$

with

$$\|\widetilde{g}_n\|_{L^p(\mathbb{R}^N)} \le \|\hat{g}_n\|_{L^p(\mathbb{R}^N)} + C(M).$$

From (5.3), (5.4) and (5.35), we then obtain, for all $x, y \in \mathbb{R}^N_+$,

$$|K'_n(x, y)| = r_n^{-\beta}\left|K_n(x, y) - K_n^+(x, y)\right| \le c_{N,s}C(N)\frac{r_n^{-\beta}\min(|r_n x'|^\beta + |r_n y'|^\beta, 1)}{|x - y|^{N+2s}}. \tag{5.42}$$

By Corollary 5.4, we have that $\|v_n\|_{C^{1-\varepsilon}(B_1^+)} \le C_\varepsilon$, for every $\varepsilon \in (0, 1)$. Hence Recalling (5.31), we thus get

$$|p^i_{v_n}(r_n)| \le C_\varepsilon r_n^{-\varepsilon}. \tag{5.43}$$

From this, we get

$$\|P_n\|_{H^s(B_R)} \le \frac{C(R)}{\Theta_n(r_n)} \qquad \text{for all } R > 0 \tag{5.44}$$

and, since $\beta < 2s - 1$,

$$\|P_n\|_{L^1(\mathbb{R}^N_+;(1+|x|)^{-N-2s+\beta})} \le \frac{C}{\Theta_n(r_n)}. \tag{5.45}$$

Using the argument in the proof of Proposition 5.2 based on Lemma 5.1 and the fractional Sobolev inequality, we obtain a constant positive constant $C(M)$ only depending on M and

N such that

$$C(M)\int_{\mathbb{R}^{2N}}(W_n(x)-W_n(y))^2\varphi^2(y)K_n(x,y)\,dydx\le 1 \\ +\left|\int_{\mathbb{R}^{2N}}(P_n(x)-P_n(y))(W_n(x)\varphi(x)-W_n(y)\varphi(y))K_n(x,y)\,dydx\right|. \tag{5.46}$$

We now use (5.42), (5.43), (5.44) and (5.45) to estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{2N}}(P_n(x)-P_n(y))(W_n(x)\varphi(x)-W_n(y)\varphi(y))K_n'(x,y)\,dydx\right| \\
&\le\left|\int_{B_{2M}\times B_{2M}}(P_n(x)-P_n(y))(W_n(x)\varphi(x)-W_n(y)\varphi(y))K_n'(x,y)\,dydx\right| \\
&+2\int_{B_M}|W_n(x)\varphi(x)|\int_{\mathbb{R}^N\setminus B_{2M}}|P_n(x)-P_n(y)||K_n'(x,y)|\,dydx \\
&\le C(M)\left((1-s)\varepsilon[W_n\varphi]^2_{H^s(B^+_{2M})}+[P_n]^2_{H^s(B_{2M})}+\int_{\mathbb{R}^N\setminus B_{2M}}\frac{|P_n(y)|}{|y|^{N+2s-\beta}}\,dy+1\right) \\
&\le C(M)\left((1-s)\varepsilon[W_n\varphi]^2_{H^s(B^+_{2M})}+1\right).
\end{aligned}$$

Combining this with (5.46), we then deduce that

$$(1-s)[w_n]^2_{H^s(B^+_{M/4})}\le C(M).$$

Thanks to (5.17), we have that w_n is bounded in $H^s_{loc}(\overline{\mathbb{R}^N_+})$. Hence by a diagonal argument, up to a subsequence, there exists $w\in H^s_{loc}(\overline{\mathbb{R}^N_+})$ such that

$$w_n\to w \qquad \text{in } L^2_{loc}(\mathbb{R}^N)$$

and, for all $\sigma<s$,

$$w_n\to w \qquad \text{in } H^\sigma_{loc}(\mathbb{R}^N).$$

Moreover $w_n\to w$ in $L^1(\mathbb{R}^N_+;(1+|x|^{N+2s})^{-1})$, thanks to (5.39). Passing to the limit in (5.39), we obtain $\|w\|_{L^2(B^+_R)}\le CR^{\frac{N}{2}+\gamma_1}$ for all $R\ge 1$. Next as above for all $\phi\in C^\infty_c(B_M)$, by (5.42), (5.44) and (5.45), we can estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{2N}}(P_n(x)-P_n(y))(\phi(x)-\phi(y))K_n'(x,y)\,dydx\right| \\
&\le\frac{C(N)}{\Theta_n(r_n)}\left([P_n]_{H^s(B_{2M})}[\phi]_{H^s(B_{2M})}+\|\phi\|_{L^\infty(B_M)}\left(1+\|P_n\|_{L^1(\mathbb{R}^N_+;(1+|x|)^{-N-2s+\beta})}\right)\right) \\
&\le\frac{C(N)}{\Theta_n(r_n)}\|\phi\|_{C^{0,1}(\mathbb{R}^N)}.
\end{aligned}$$

It is also easy to see that $\|\hat{g}_n\|_{L^p(\mathbb{R}^N)}\le\frac{1}{\Theta_n(r_n)}$. Therefore, by (5.42), passing to the limit in (5.40), we find that $(-\Delta)^s_{\mathbb{R}^N_+}w=0$ on $\mathbb{R}^N_+$. We then deduce from Theorem 4.1 that $w(x',x_N)=a+c\cdot x'$, so that we reach a contradiction by passing to the limit in (5.37) and (5.38). □

Corollary 5.6. *Let $s \in (1/2, 1)$, $\beta < 2s-1$ and $2s-N/p > 1$. Put $\gamma = \min(2s-N/p, 1+\beta)$. Consider $\psi \in C^{1,\beta}(\mathbb{R}^{N-1})$, with $\|\psi\|_{C^{1,\beta}(\mathbb{R}^{N-1})} \le \frac{1}{4}$ and $\psi(0) = |\nabla\psi(0)| = 0$. Let $g \in L^p(\mathbb{R}^N)$ and $v \in H^s(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$ satisfying*

$$\mathcal{D}_{K_s^\psi}(v, \varphi) = \int_{B_2^+} g(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2),$$

$$\|g\|_{L^p(\mathbb{R}^N)} + \|v\|_{L^2(\mathbb{R}^N_+)} \le 1.$$

Then there exist $\overline{C} > 0$ depending only on N, s, p and β and a constant $D \in \mathbb{R}^{N-1}$ satisfying

$$|D| \le \overline{C}, \tag{5.47}$$

and

$$\|v - v(0) - D \cdot x'\|_{L^2(B_r^+)} \le \overline{C} r^{\frac{N}{2}+\gamma}.$$

Proof. In view of (5.33), using e.g. the argument in [9, 16, 28], we can find a constant C depending only on N, s_0, β, p, N and ε and some constant $D \in \mathbb{R}^{N-1}$, $d \in \mathbb{R}$ such that

$$|D| + |d| \le C,$$

and for every $r > 0$

$$\|v - d - D \cdot x'\|_{L^2(B_r^+)} \le C r^{\min(2s-N/p, 1+\beta)-\varepsilon}.$$

By continuity of v on $\overline{\mathbb{R}^N_+}$, thanks to Corollary 5.4, we deduce that $v(0) = d$. From the above result we also deduce that $\frac{1}{|B_r^+|}\|v - v(0)\|_{L^2(B_r^+)} \le Cr$. Therefore, we repeat the argument in Proposition 5.5 with $\gamma_1 = \min(2s - N/p, 1 + \beta)$ to deduce that

$$\|v - v(0) - D \cdot x'\|_{L^2(B_r^+)} \le \overline{C} r^{N/2+\min(2s-N/p,1+\beta)}.$$

Note that the reason for considering $\gamma_1 := \min(2s - N/p, 1 + \beta) - \varepsilon$ was just because we applied Corollary 5.4 which implied (5.43). □

Next, we prove the following crucial result.

Corollary 5.7. *Under the hypothesis of Corollary 5.6, we have*

$$\sup_{\xi \in B_1^+ \cap \mathbb{R}e_N} |\nabla v(\xi)| + \sup_{\xi \in B_1^+ \cap \mathbb{R}e_N} [\nabla v]_{C^{\gamma-1}(B_{\xi_N/2}(\xi))} \le C. \tag{5.48}$$

Moreover $x_N \mapsto \partial_{x_N} v(0, x_N)$ is continuous at $x_N = 0$ and

$$\partial_{x_N} v(0) = 0, \tag{5.49}$$

with the constant C depending only on N, s, p and β.

Proof. Let $\xi := (0, 2\rho) \in B_1^+ \cap \mathbb{R}e_N$. We define

$$u_\xi(y) = v(\xi + \rho y) - v(0) - D \cdot (\rho y') = v(2\rho e_N + \rho y) - v(0) - D \cdot (2\rho e_N + \rho y)'$$

and $\widetilde{g}(y) = g(\xi + \rho y)$ and $U(y) := \rho^{1+\beta} D \cdot y'$. We have, for all $\varphi \in C_c^\infty(B_1)$,

$$\mathcal{D}_{K_{\rho,s}}(u_\xi, \varphi) + \rho^{-\beta}\mathcal{D}_{K_{\rho,s}}(U, \varphi) = \rho^{2s}\int_\Sigma \widetilde{g}(y)\varphi(y)\,dy,$$

where $K_{\rho,s}(x, y) = 1_\Sigma(x)1_\Sigma(y)\frac{c_{N,s}}{|\Psi_\rho(x)-\Psi_\rho(y)|^{N+2s}}$, $\Sigma := \{x_N > -2\}$ and $\Psi_\rho(x) = \frac{1}{\rho}\Psi(\rho x + \xi)$. Since, by a change of variable,

$$p.v. \int_{\mathbb{R}^N} \frac{U(x) - U(y)}{|\Psi_\rho(x) - \Psi_\rho(y)|^{N+2s}}\,dy = 0 \qquad \text{for all } x \in \mathbb{R}^N,$$

letting $F_\xi(x) := c_{N,s}\rho^{-\beta}\int_{\mathbb{R}^N\setminus\Sigma}\frac{U(x)-U(y)}{|\Psi_\rho(x)-\Psi_\rho(y)|^{N+2s}}\,dx$, we then get

$$\mathcal{D}_{K_{\rho,s}}(u_\xi,\varphi) = \int_{B_1}\left(\rho^{2s}\widetilde{g}(y) - F_\xi(y)\right)\varphi(y)\,dy, \tag{5.50}$$

To estimate F_ξ we observe that $(-\Delta)^s_{\mathbb{R}^N\setminus\Sigma}x_i = 0$ on $\mathbb{R}^N\setminus\Sigma$, for $i = 1,\dots,N-1$. Consequently, by (5.4), for $x \in B_1$, we have

$$\begin{aligned}
|F_\xi(x)| &= c_{N,s}\left|\int_{\mathbb{R}^N\setminus\Sigma}(U(x)-U(y))\rho^{-\beta}\left(\frac{1}{|\Psi_\rho(x)-\Psi_\rho(y)|^{N+2s}} - \frac{1}{|x-y|^{N+2s}}\right)dy\right| \\
&\le C(N)c_{N,s}\int_{\mathbb{R}^N\setminus\Sigma}|U(x)-U(y)|\frac{\rho^{-\beta}\int_0^1|\nabla\psi(\rho x' + t(\rho y' - \rho x'))|\,dt}{|x-y|^{N+2s}}\,dy \\
&\le C(N)c_{N,s}[U]_{C^{0,1}(\mathbb{R}^N)}\int_{\mathbb{R}^N\setminus\Sigma}\frac{|x|^\beta + |y|^\beta}{|x-y|^{N+2s-1}}\,dy \le C(N)\frac{c_{N,s}}{2s-1+\beta}\rho^{1+\beta}|D|.
\end{aligned}$$

By [15, Theorem 1.3], applied to (5.50), and using (5.47), we have

$$\begin{aligned}
\|\nabla u_\xi\|_{C^{\gamma-1}(B_{1/2})} &\le C\left(\|u_\xi\|_{L^2(B_1)} + \int_{|y|\ge 1/2}\frac{|u_\xi(y)|}{|y|^{N+2s}}\,dy + \rho^{2s}\|\widetilde{g}\|_{L^p(\mathbb{R}^N)} + \rho^{1+\beta}\right) \\
&\le C\left(\|u_\xi\|_{L^2(B_1)} + \int_{|y|\ge 1/2\cap\mathbb{R}^N_+}\frac{|u_\xi(y)|}{|y|^{N+2s}}\,dy + \rho^\gamma\right).
\end{aligned} \tag{5.51}$$

By Corollary 5.6, we have

$$\|u_\xi\|_{L^2(B_1)} \le C\rho^\gamma, \tag{5.52}$$

where $\gamma := \min(2s - N/p, 1+\beta)$. We let

$$\begin{aligned}
w_\xi(y) := u_\xi(y/\rho) &= \left(v(\xi+y) - v(0) - D\cdot y'\right)1_\Sigma(y) \\
&= (v(2\rho e_N + y) - v(0) - D\cdot(2\rho e_N + y))\,1_\Sigma(y).
\end{aligned}$$

Then, using Corollary 5.6, we get

$$\begin{aligned}
&\int_{|y|\ge 1/2\cap\Sigma}\frac{|u_\xi(y)|}{|y|^{N+2s}}\,dy = \rho^{2s}\int_{|y|\ge\rho/2}|y|^{-N-2s}|w_\xi(y)|\,dy \\
&\le \rho^{2s}\sum_{k=0}^\infty\int_{\rho 2^k\ge|y|\ge\rho 2^{k-1}}|y|^{-N-2s}|w_\xi(y)|\,dy \le \rho^{2s}\sum_{k=0}^\infty(2^k\rho)^{-N-2s}\int_{\rho 2^{k+1}\ge|y|}|w_\xi(y)|\,dy \\
&\le \rho^{2s}\sum_{k=0}^\infty(2^k\rho)^{-N-2s}\|v - v(0) - D\cdot\zeta\|_{L^1(B^+_{\rho 2^{k+2}})} \\
&\le C\rho^{2s}\sum_{k=0}^\infty(2^k\rho)^{-N-2s}(\rho 2^k)^{N+\gamma} \le C\rho^\gamma\sum_{k=0}^\infty 2^{-k(2s-\gamma)} \le C\rho^\gamma.
\end{aligned}$$

Hence, combining this with (5.52) and (5.53), we then get

$$\|\nabla u_\xi\|_{C^{\gamma-1}(B_{1/2})} \le C\rho^\gamma. \tag{5.53}$$

Scaling back, we get, for all $\xi \in B_1^+ \cap \mathbb{R}e_N$,

$$|\nabla v(\xi)| + [\nabla v]_{C^{\gamma-1}(B_\rho(\xi))} \le C.$$

This proves (5.48). Note that the above estimate also implies that $\|\nabla v\|_{C^{\gamma-1}(B^+_{1/2}\cap\mathbb{R}e_N)} \le C$, thanks to Lemma 5.3 and that

$$|\partial_{x_N} v(\xi)| \le C\rho^{\gamma-1}.$$

Hence, letting $\rho = \xi_N/2 \to 0$, we obtain $\partial_{x_N} v(0) = 0$, which is (5.49). □

6. The Dirichlet problem

As mentioned earlier, thanks to the fractional Hardy inequality with boundary singularity $H^s_0(\Omega) \subset L^2(\Omega;\delta^{-2s}(x))$, for $s \neq 1/2$, for every Lipschitz open set Ω, see e.g. [19]. As a consequence, we may identify the space $H^s_0(\Omega)$ with the Sobolev space

$$\mathcal{H}^s_s(\Omega) := \{u \in H^s(\mathbb{R}^N) \,:\, u = 0 \qquad \text{in } \mathbb{R}^N \setminus \Omega\}.$$

See e.g. [4, 5, 10] we have the following result.

Lemma 6.1. *Let $s \in (0, 1/2)$ and $u \in \mathcal{H}^s_0(\mathbb{R}^N_+)$ and $g \in L^\infty(\mathbb{R}^N)$ such that*

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N_+} u = g & \text{in } B^+_2, \\ u = 0 & \text{in } B^-_2. \end{cases} \tag{6.1}$$

Then

$$\|u\|_{C^{2s-1}(B^+_1)} \le C\left(\|u\|_{L^2(B_2)} + \int_{|y|\ge\frac14} \frac{|u(y)|}{|y|^{N+2s}}\,dy + \|g\|_{L^\infty(\mathbb{R}^N)}\right).$$

As a consequence of this result we prove the following

Lemma 6.2. *Let $u \in H^s_{loc}(\mathbb{R}^N)$ be such that*

$$\begin{cases} (-\Delta)^s_{\mathbb{R}^N_+} u = 0 & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \mathbb{R}^N_+ \end{cases} \tag{6.2}$$

and, for some constants $C > 0$ and $\varepsilon \in (0, 2s)$,

$$\|u\|_{L^2(B_R)} \le CR^{\frac{N}{2}+\varepsilon} \qquad \text{for all } R \ge 1. \tag{6.3}$$

Then $u(x', x_N) = a(x_N)^{2s-1}_+$ for some constant $a \in \mathbb{R}$.

Proof. Using the same argument as in the proof of Proposition 4.2, we find that

$$\|\nabla^2_{x'} u\|_{L^\infty(B^+_1)} \le C\left(\|u\|_{L^2(B^+_2)} + \int_{|y|\ge1/2} |u(y)||y|^{-N-2s}\,dy\right).$$

Hence letting $u_R(x) = u(Rx)$ which is a solution to (6.2), we then get

$$\begin{aligned}\|\nabla^2_{x'} u_R\|_{L^\infty(B^+_1)} &\le C\left(\|u_R\|_{L^2(B^+_2)} + \int_{|y|\ge1/2} |u_R(y)||y|^{-N-2s}\,dy\right) \\ &\le R^{\varepsilon-2} + R^{2s}\int_{|y|\ge1/2R} |u(y)||y|^{-N-2s}.\end{aligned}$$

From this we deduce that

$$\|\nabla^2_{x'} u\|_{L^\infty(B^+_R)} \le CR^{\varepsilon-2}.$$

Hence, letting $R \to \infty$, we deduce that $u(x', x_N) = a(x_N) + b(x_N)\cdot x'$. Since $x \mapsto u(x+h) - u(x)$ solves (6.2) for all $h \in \partial\mathbb{R}^N_+$, we deduce that, for $i = 1\ldots, N$, the function b_i solves (6.2). But then Proposition 4.7-(*ii*) implies that $b_i(x_N) = c_i(x_N)^{2s-1}_+$, for some constant c_i, so that

(6.3) implies that $c_i = 0$. It then follows that a solves (6.2) and thus Proposition 4.7-(ii) yields the desired result. □

Next, we state the following result for which its proof is based on a blow up argument as above and the classification result given by Lemma 6.2. Since its proof is very similar (even simpler) to the one of Proposition 5.2, to avoid redundancy, we omit it.

Proposition 6.3. *Let $s_0 \in (1/2, 1)$, $\varepsilon \in (0, 2s_0 - 1)$, $p > \frac{N}{2s_0}$. Put $\gamma := \min(2s - N/p, 2s - 1 - \varepsilon)$. Then there exist $\varepsilon_0, C > 0$ such that for every $s \in [s_0, 1)$, $\psi \in C^1(\mathbb{R}^{N-1})$ and for every $g \in L^p(\mathbb{R}^N)$, $w \in \mathcal{H}^s_0(\mathbb{R}^N_+)$ satisfying*

$$[\psi]_{C^1(\mathbb{R}^{N-1})} < \varepsilon_0$$

and

$$\mathcal{D}_{K^\psi_s}(w, \varphi) = \int_{B_2^+} g(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^\infty_c(B_2^+),$$

we have

$$\sup_{r>0} r^{-2\gamma-N} \sup_{z\in B_1'} \|w\|^2_{L^2(B_r(z))} \le C(\|w\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^p(\mathbb{R}^N)})^2. \tag{6.4}$$

As a consequence we have the following result.

Corollary 6.4. *Let $s_0 \in (1/2, 1)$, $\varepsilon \in (0, 2s_0 - 1)$, $p > \frac{N}{2s_0}$ and $s \in [s_0, 1)$. We put $\gamma := \min(2s - N/p, 2s - 1 - \varepsilon)$. Let $\psi \in C^1(\mathbb{R}^{N-1})$ $g \in L^p(\mathbb{R}^N)$, $w \in \mathcal{H}^s_0(\mathbb{R}^N_+)$ satisfying and*

$$\mathcal{D}_{K^\psi_s}(w, \varphi) = \int_{B_2^+} g(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^\infty_c(B_2^+). \tag{6.5}$$

Then there exists $\overline{C}, \varepsilon_0$ depending only on N, s_0, p and ε such that, provided $[\psi]_{C^1(\mathbb{R}^{N-1})} < \varepsilon_0$,

$$\|w\|_{C^\gamma(B_{1/2})} \le C(\|w\|_{L^2(B_2)} + \|w\|_{L^1(\mathbb{R}^N; \frac{1}{1+|x|^{N+2s}})} + \|g\|_{L^p(\mathbb{R}^N)}).$$

Proof. Let $\chi \in C^\infty_c(B_4)$ with $\chi \equiv 1$ on $B_{1/2}$. Then we have that $u = \chi_{B_4} w$ solves

$$\mathcal{D}_{K^\psi_s}(u, \varphi) = \int_{B_2^+} f(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^1_c(B_{1/2}),$$

for some function $f \in L^p(\mathbb{R}^N)$ satisfying $\|f\|_{L^p(\mathbb{R}^N)} \le C(\|w\|_{\mathcal{L}^1_s} + \|g\|_{L^p(\mathbb{R}^N)})$. Let $x_0 = (x_0', x_0 \cdot e_N) \in B_1^+$ and define $\rho = \frac{x_0 \cdot e_N}{2}$, so that $B_\rho(x_0) \subset B_{3\rho}(x_0') \cap \mathbb{R}^N_+$. Next, we define $v_\rho(x) := u(\rho x + x_0)$ and $f_\rho(x) := \rho^{2s} f(\rho x + x_0)$. It is plain that $v \in H^s_{loc}(B_2) \cap L^1(\mathbb{R}^N; \frac{1}{1+|x|^{N+2s}})$ and

$$\mathcal{D}_{K_{\rho,s}}(v_\rho, \varphi) = \int_{B_2^+} f_\rho(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^1_c(B_1),$$

where $K_{\rho,s}(x, y) = 1_{\{x_N > -2\}}(x) 1_{\{y_N > -2\}}(y) \frac{c_{N,s}}{|\Psi_\rho(x) - \Psi_\rho(y)|^{N+2s}}$ and $\Psi_\rho(x) = \frac{1}{\rho}\Psi(\rho x + x_0)$. Note that $\|f_\rho\|_{L^p(\mathbb{R}^N)} \le \rho^{2s-N/p} \|f\|_{L^p(\mathbb{R}^N)}$. By the interior regularity, see e.g. [16, Theorem 4.1].

$$\|v_\rho\|_{C^{\min(1-\varepsilon, 2s-N/p)}(B_{1/2})} \le C \left(\|v_\rho\|_{L^2(B_1)} + \|v_\rho\|_{L^1(\mathbb{R}^N; \frac{1}{1+|x|^{N+2s}})} + \|f_\rho\|_{L^p(\mathbb{R}^N)} \right). \tag{6.6}$$

By (6.4) and Hölder's inequality, we have

$$\|v_\rho\|_{L^2(B_1)} \le \rho^{-N/2} \|u\|_{L^2(B_{3\rho}(z))} \le C\rho^\gamma \quad \text{and} \quad \|u\|_{L^1(B_r(z))} \le C r^{N+\gamma} \quad \text{for } r > 0. \tag{6.7}$$

Using the second estimate in (6.7), we get

$$\begin{aligned}
&\int_{|x|\geq 1} |x|^{-N-2s}|v_\rho(x)|\,dx = \rho^{2s}\int_{|y-x_0|\geq\rho} |y-x_0|^{-N-2s}|u(y)|\,dy\\
&\leq \rho^{2s}\sum_{k=0}^{\infty}\int_{\rho 2^{k+1}\geq|y-x_0|\geq\rho 2^k} |y-x_0|^{-N-2s}|u(y)|\,dy\\
&\leq \rho^{2s}\sum_{k=0}^{\infty}(2^k\rho)^{-N-2s}\int_{\rho 2^{k+1}\geq|y-x_0|}|u(y)|\,dy \leq \rho^{2s}\sum_{k=0}^{\infty}(2^k\rho)^{-N-2s}\|u\|_{L^1(B_{\rho 2^{k+3}}(z))}\\
&\leq C\rho^{2s}\sum_{k=0}^{\infty}(2^k\rho)^{-N-2s}(\rho 2^k)^{N+\gamma} \leq C\rho^\gamma\sum_{k=0}^{\infty}2^{-k(2s-\gamma)} \leq C\rho^\gamma.
\end{aligned}$$

We then conclude that $\|v_\rho\|_{L^1(\mathbb{R}^N;\frac{1}{1+|x|^{N+2s}})} \leq C\rho^\gamma$. It follows from (6.7) and (6.6), that

$$\|v_\rho\|_{C^\gamma(B_{1/2})} \leq C(\rho^\gamma + \rho^{2s-N/p}).$$

Scaling back, we get

$$|u(x_0)| + \|u\|_{C^\gamma(B_{\rho/2}(x_0))} \leq C,$$

which implies the claimed estimate, thanks to Lemma 5.3. □

The next result provided the first steps toward the higher order boundary regularity for the regional fractional Dirichlet problem. We first define for $w \in L^2_{loc}(\mathbb{R}^N)$ and $r > 0$,

$$Q_{w,r}(x) = \frac{(x_N)_+^{2s-1}}{\int_{B_r}(y_N)_+^{2(2s-1)}\,dy}\int_{B_r} w(y)(y_N)_+^{2s-1}\,dy =: (x_N)_+^{2s-1}q_w(r). \tag{6.8}$$

We have the following result which is crucial to deduce higher order regularity of v/δ^{2s-1} up to the boundary.

Proposition 6.5. *Let $s \in (1/2, 1)$, $p > N$ and $\beta \in (0, 1)$. Then there exist $\varepsilon_0, C > 0$ such that for every $\psi \in C^{1,\beta}(\mathbb{R}^{N-1})$ and for every $g \in L^p(\mathbb{R}^N)$, $v \in \mathcal{H}^s_0(\mathbb{R}^N_+)$ satisfying*

$$\mathcal{D}_{K_s^\psi}(v,\varphi) = \int_{B_2^+} g(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2^+),$$

$$\|g\|_{L^p(\mathbb{R}^N)} + \|v\|_{L^2(\mathbb{R}^N)} \leq 1$$

and

$$\|\nabla\psi\|_{C^{0,\beta}(\mathbb{R}^{N-1})} < \frac{1}{4}, \qquad \psi(0) = |\psi(0)| = 0,$$

then we have

$$\sup_{r>0} r^{-\frac{N}{2}-\gamma}\|v - Q_{v,r}\|_{L^2(B_r)} \leq C, \tag{6.9}$$

where $\gamma := \min(2s - N/p, \beta + 2s - 1) - \varepsilon > 2s - 1$.

Proof. Assume that the assertion does not hold, then for every $n \in \mathbb{N}$, $\psi_n \in C^{1,\beta}(\mathbb{R}^{N-1})$, $g_n \in L^p(\mathbb{R}^N)$, $v_n \in \mathcal{H}^s_0(\mathbb{R}^N_+)$, with

$$\|g_n\|_{L^p(\mathbb{R}^N)} + \|v_n\|_{L^2(\mathbb{R}^N)} \leq 1$$

satisfying

$$\mathcal{D}_{K_s^{\psi_n}}(v_n,\varphi) = \int_{B_2^+} g_n(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_2^+), \tag{6.10}$$

while

$$\sup_{r>0} r^{-\frac{N}{2}-\gamma}\|v_n - Q_{v_n,r}\|_{L^2(B_r)} > n$$

and

$$\|\nabla\psi_n\|_{C^{0,\beta}(\mathbb{R}^{N-1})} < \frac{1}{4}, \qquad \psi_n(0) = |\psi_n(0)| = 0, \tag{6.11}$$

Consequently, there exists $\overline{r}_n > 0$ such that

$$\overline{r}_n^{-\frac{N}{2}-\gamma}\|v_n - Q_{v_n,\overline{r}_n}\|_{L^2(B_{\overline{r}_n})} > n/2.$$

We consider the (well defined, because $\|\overline{v}_n\|_{L^2(\mathbb{R}^N)} \le 1$) nonincreasing function $\Theta_n : (0,\infty) \to [0,\infty)$ given by

$$\Theta_n(\overline{r}) = \sup_{r\in[\overline{r},\infty)} r^{-\frac{N}{2}-\gamma}\|v_n - Q_{v_n,r}\|_{L^2(B_r)}.$$

Proceeding as in the proof of Proposition 5.2, we can find a sequence r_n tending to zero such that for $n \ge 2$,

$$\Theta_n(r_n) > n/4. \tag{6.12}$$

Hence, provided $n \ge 2$, there exists $r_n \in [\overline{r}_n, \infty)$ such that

$$r_n^{-\frac{N}{2}-\gamma}\|v_n - Q_{v_n,r_n}\|_{L^2(B_{r_n})} \ge \frac{1}{2}\Theta_n(r_n) \ge \frac{n}{8}.$$

We now define the sequence of functions

$$w_n(x) = \frac{r_n^{-\frac{N}{2}-\gamma}}{\Theta_n(r_n)}\left\{v_n(r_n x) - Q_{v_n,r_n}(r_n x)\right\}.$$

It satisfies

$$\|w\|_{L^2(B_1)} \ge \frac{1}{2}, \qquad \text{and} \qquad \int_{B_1} w_n(x)(x_N)_+^{2s-1}\,dx = 0 \qquad \text{for every } n \ge 2. \tag{6.13}$$

Using similar arguments as in [16, 24], we find that

$$\|w_n\|_{L^2(B_R)} \le CR^{\frac{N}{2}+\gamma} \qquad \text{for every } R \ge 1 \text{ and } n \ge 2, \tag{6.14}$$

for some constant $C = C(N,\gamma) > 0$.

We define

$$K_n(x,y) = \frac{r_n^{N+2s}}{2}K_s^{\psi_n}(r_n x, r_n y) = \frac{1}{2}K_s^{\widetilde{\psi}_n}(x,y),$$

where $\widetilde{\psi}_n(x) := \frac{1}{r_n}\psi_n(r_n x)$. Let $\hat{g}_n(x) = \frac{r_n^{2s-\gamma}}{\Theta_n(r_n)}g_n(r_n x)$ and

$$Q_n(x) = \frac{r_n^{\beta-\gamma}}{\Theta_n(r_n)}Q_{v_n,r_n}(r_n x) = \frac{r_n^{\beta+2s-1-\gamma}q_{v_n}(r_n)}{\Theta_n(r_n)}(x_N)_+^{2s-1}.$$

By a change of variable, we get

$$\|\hat{g}_n\|_{L^p(\mathbb{R}^N)} \le \frac{1}{\Theta_n(r_n)}. \tag{6.15}$$

Moreover, for all $\varphi \in C_c^\infty(B^+_{\frac{1}{2r_n}}(0))$,

$$\mathcal{D}_{K_n}(w_n,\varphi) + r_n^{-\beta}\mathcal{D}_{K_n}(Q_n,\varphi) = \int_{\mathbb{R}^N_+} \overline{g}_n(x)\varphi(x)\,dx.$$

We define $K_n'(x,y) := r_n^{-\beta}\left(K_n(x,y) - K_n^+(x,y)\right)$, so that

$$|K_n'(x,y)| = r_n^{-\beta}\left|K_n(x,y) - K_n^+(x,y)\right| \le c_{N,s}C(N)\frac{r_n^{-\beta}\min(|r_n x'|^\beta + |r_n y'|^\beta, 1)}{|x-y|^{N+2s}}. \tag{6.16}$$

Using that $(-\Delta)^s_{\mathbb{R}^N_+} x_N^{2s-1} = 0$ on $\mathbb{R}^N_+$, we thus obtain

$$\mathcal{D}_{K_n}(w_n,\varphi) + \mathcal{D}_{K_n'}(Q_n,\varphi) = \int_{\mathbb{R}^N_+} \overline{g}_n(x)\varphi(x)\,dx. \tag{6.17}$$

By Corollary 6.4, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|q_{v_n}(r_n)| \le C_\varepsilon r_n^{-\varepsilon}. \tag{6.18}$$

From this, we get

$$\|Q_n\|_{H^s(B_R)} \le \frac{C(R)}{\Theta_n(r_n)} \qquad \text{for all } R > 0 \tag{6.19}$$

and, since $\beta < 1$,

$$\|Q_n\|_{L^1(\mathbb{R}^N_+;(1+|x|)^{-N-2s+\beta})} \le \frac{C}{\Theta_n(r_n)} \tag{6.20}$$

Moreover, as in the proof of Proposition 5.5 based on Lemma 5.1, we can show that, for fixed $M > 1$ and $n \ge 2$ large, so that $1 < M < \frac{1}{2r_n}$,

$$(1-s)[w_n]^2_{H^s(B_M)} \le C(M).$$

Consequently, up to a subsequence, there exists $w \in H^s_{loc}(\mathbb{R}^N)$ such that

$$w_n \to w \qquad \text{weakly in } H^s_{loc}(\mathbb{R}^N) \tag{6.21}$$

and, by (6.14),

$$\int_{\mathbb{R}^N} \frac{|w_n(x) - w(x)|}{1+|x|^{N+2s}}\,dx \to 0 \qquad \text{as } n \to \infty. \tag{6.22}$$

Passing to the limit in (6.13), we obtain

$$\|w\|_{L^2(B_1)} \ge \frac{1}{2} \qquad \text{and} \qquad \int_{B_1} w(x)(x_N)_+^{2s-1}\,dx = 0. \tag{6.23}$$

Next by (6.17), (6.19), (6.20) and (6.15), we have for all $\phi \in C_c^\infty(B_M^+)$, with M as above,

$$\frac{1}{2}\left|\int_{\mathbb{R}^{2N}} (w_n(x) - w_n(y))(\phi(x) - \phi(y))K_n(x,y)\,dxdy\right| \le \frac{C}{\Theta_n(r_n)}\|\phi\|_{C^{0,1}(\mathbb{R}^N)}. \tag{6.24}$$

Now in view of (6.16), (6.21), (6.22) and (6.24), we can let $n \to \infty$ in (6.24), to obtain $(-\Delta)^s_{\mathbb{R}^N_+} w = 0$ on $\mathbb{R}^N_+$ and $w = 0$ on $\mathbb{R}^N \backslash \mathbb{R}^N_+$. Passing to the limit in (6.14) yields $\|w\|_{L^2(B_R)} \le CR^{N/2+\gamma}$ for all $R \ge 1$. By Lemma 6.2, and since $\gamma < 2s$, we find that $w(x) = b(x_N)_+^{2s-1}$, for some $b \in \mathbb{R}$. This is in contradiction with (6.23).

□

Corollary 6.6. *Let $s \in (1/2, 1)$, $2s - N/p > 2s - 1$ and $\beta \in (0,1)$. Let $\psi \in C^{1,\beta}(\mathbb{R}^{N-1})$ be such that $\|\nabla\psi\|_{C^{0,\beta}(\mathbb{R}^{N-1})} < \frac{1}{4}$ and $\psi(0) = |\psi(0)| = 0$. Let $g \in L^p(\mathbb{R}^N)$, $v \in \mathcal{H}^s_0(\mathbb{R}^N_+)$ satisfying*

$$\mathcal{D}_{K_s^\psi}(v,\varphi) = \int_{B_2^+} g(x)\varphi(x)\,dx \qquad \textit{for all } \varphi \in C_c^\infty(B_2^+),$$

with

$$\|g\|_{L^p(\mathbb{R}^N)} + \|v\|_{L^2(\mathbb{R}^N)} \le 1$$

Then there exist positive constant C_0 depending only on N, s, p and β and a constant $Q \in \mathbb{R}$ satisfying

$$|Q| \leq C_0 \tag{6.25}$$

and

$$\sup_{r>0} r^{-\min(2s-N/p,\beta+2s-1)-N/2} \|v - Q(x_N)_+^{2s-1}\|_{L^2(B_r)} \leq C_0.$$

Proof. Since $\min(2s - N/p, \beta + 2s - 1) - \varepsilon > 2s - 1$, in view of (6.9), we can use similar arguments as in [16, 24], to obtain a constant $Q \in \mathbb{R}$ satisfying

$$|Q| + \sup_{r>0} r^{-\min(2s-N/p,\beta+2s-1)-\varepsilon-N/2} \|v - Q(x_N)_+^{2s-1}\|_{L^2(B_r)} \leq C_0.$$

This implies, recalling (6.8), that $q_v(r) \leq C$. We can thus repeat the proof of Proposition 6.5 using $\min(2s - N/p, \beta + 2s - 1)$ in the place of $\gamma = \min(2s - N/p, \beta + 2s - 1) - \varepsilon$ to get the desired estimate. We recall that the correction ε was considered in the proof of Proposition 6.5 in order to have a control on the estimate (6.18). □

Corollary 6.7. *Under the hypothesis of Corollary 6.6, there exists a constant C depending only on N, s, p and β with the following properties.*

(i) For all $\xi \in B_1^+ \cap \mathbb{R}e_N$,

$$|v(\xi)/\xi_N^{2s-1}| + [v]_{C^{2s-1}(B_{\xi_N/2}(\xi))} + [v/x_N^{2s-1}]_{C^{\min(1-N/p,\beta)}(B_{\xi_N/2}(\xi))} \leq C.$$

(ii) If $2s - \frac{N}{p} > 1$ and $\beta > 2s - 1$ then for all $\xi \in B_{1/2}^+ \cap \mathbb{R}e_N$,

$$|\xi_N^{2-2s} \nabla v(\xi)| + [x_N^{2-2s} \nabla v]_{C^{2s-\frac{N}{p}-1}(B_{\xi_N/2}(\xi))} \leq C. \tag{6.26}$$

Moreover $[x_N^{2-2s} \nabla v](0, \cdot)$ and $[v/x_N^{2s-1}](0, \cdot)$ are continuous at $x_N = 0$ and satisfy

$$[x_N^{2-2s} \nabla v](0) = (2s - 1)[v/x_N^{2s-1}](0)e_N. \tag{6.27}$$

Proof. Let $\xi := (0, 2\rho) \in B_1^+$ with $\rho < 1/2$. We define $u_\xi(y) = v(\xi+\rho y) - Q((\xi+\rho y)\cdot e_N)_+^{2s-1} = v(\xi + \rho y) - Q(2\rho + \rho y_N)_+^{2s-1}$ and $g_\xi(y) = \rho^{2s} g(\xi + \rho y)$. We have

$$\mathcal{D}_{K_{\xi,s}}(u_\xi, \varphi) + \rho^{-\beta} \mathcal{D}_{K_{\xi,s}}(U_\xi, \varphi) = \int_{\mathbb{R}^N} g_\xi(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C_c^\infty(B_1), \tag{6.28}$$

where $K_{\xi,s}(x, y) = 1_\Sigma(x)1_\Sigma(y)\frac{c_{N,s}}{|\Psi_\xi(x)-\Psi_\xi(y)|^{N+2s}}$, $\Sigma := \{(y', y_N) : y_N > -2\}$, $\Psi_\xi(y) = \frac{1}{\rho}\Psi(\rho y + \xi)$ and

$$U_\xi(y) := \rho^{2s-1+\beta}(y_N + 2)_+^{2s-1}.$$

By a change of variable, we get $(-\Delta)_\Sigma^s U_\xi = 0$ on Σ. We then define

$$K'_{\xi,s}(x, y) = \rho^{-\beta} \left(K_{\xi,s}(x, y) - 1_\Sigma(x)1_\Sigma(y)\frac{c_{N,s}}{|x - y|^{N+2s}} \right).$$

We thus get from (6.28),

$$\mathcal{D}_{K_{\xi,s}}(u_x, \varphi) + \mathcal{D}_{K'_{\xi,s}}(U_\xi, \varphi) = \int_{\mathbb{R}^N} g_\xi(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C_c^\infty(B_1). \tag{6.29}$$

Clearly by (5.4), $K'_{\xi,s}$ satisfies

$$||x - y|^{N+2s} K'_{\xi,s}(x, y)| \leq C(|x|^\beta + |y|^\beta).$$

We let $\chi \in C_c^\infty(B_{1/2})$ with $\chi = 1$ on $B_{1/4}$ and put $\widetilde{U}_\xi = \chi U_\xi \in C_c^2(\mathbb{R}^N)$. Then thanks (6.29), we have the following equation

$$\mathcal{D}_{K_{\xi,s}}(u_x, \varphi) + \mathcal{D}_{K'_{\xi,s}}(\widetilde{U}_\xi, \varphi) = \int_{\mathbb{R}^N} G_\xi(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_{1/8}), \tag{6.30}$$

where, for all $x \in B_{1/8}$,

$$G_\xi(x) = g_\xi(x) + \int_{|y|>1/2} (1-\chi(y))U_\xi(y)K'_{\xi,s}(x,y)\,dy.$$

By (6.30) and interior regularity (see e.g. [15, Corollary 3.5]) and using (6.25), we have that

$$\begin{aligned}\|u_\xi\|_{C^{2s-1}(B_{1/9})} &\le C\left(\|u_\xi\|_{L^2(B_1)} + \int_{|y|\ge 1/4} |u_\xi(y)||y|^{-N-2s}\,dy + \|G_\xi\|_{L^p(\mathbb{R}^N)}\right)\\ &\le C\left(\|u_\xi\|_{L^2(B_1)} + \int_{|y|\ge 1/4} |u_\xi(y)||y|^{-N-2s}\,dy + \rho^{2s-N/p} + \rho^{\beta+2s-1}\right).\end{aligned}$$

We put $\gamma = \min(2s - N/p, \beta + 2s - 1)$. In view of Corollary 6.6 we have

$$\|u_\xi\|_{L^2(B_1)} \le C\rho^{N/2+\gamma}. \tag{6.31}$$

Moreover, using similar arguments as in the proof of Corollary 6.4, we can estimates the terms

$$\int_{|y|\ge 1/4} |u_\xi(y)||y|^{-N-2s}\,dy \le C\rho^\gamma. \tag{6.32}$$

We then obtain

$$\|u_\xi\|_{C^{2s-1}(B_{1/9})} \le C\left(\rho^\gamma + \rho^{2s-N/p} + \rho^{2s+\beta-1}\right) \le C\rho^\gamma. \tag{6.33}$$

In particular, provided $\beta \le 2s-1$,

$$\|u_\xi\|_{C^{\gamma-(2s-1)}(B_{1/9})} \le C\rho^\gamma.$$

Scaling back, we get

$$\|v - Qy_N^{2s-1}\|_{L^\infty(B_{\rho/9}(\xi))} \le C\rho^\gamma, \qquad [v - Qy_N^{2s-1}]_{C^{\gamma-(2s-1)}(B_{\rho/9}(\xi))} \le C\rho^{2s-1}. \tag{6.34}$$

The above three estimates and (6.33) imply that

$$[v]_{C^{2s-1}(B_{\rho/9}(\xi))} + |v(\xi)/\xi_N^{2s-1}| + [v/y_N^{2s-1}]_{C^{\gamma-(2s-1)}(B_{\rho/9}(\xi))} \le C, \tag{6.35}$$

thanks to (6.25). We thus get (i) for $\beta \le 2s-1$.
We now prove (ii) and (i) in the case $\beta > 2s-1$. We define the new kernel

$$\mathcal{K}_\xi(x,y) = c_{N,s}\rho^{-\beta}\left(\frac{1}{|\Psi_\xi(x) - \Psi_\xi(y)|^{N+2s}} - \frac{1}{|x-y|^{N+2s}}\right)$$

and rewrite (6.30) as

$$\mathcal{D}_{K_{\xi,s}}(u_\xi, \varphi) = \int_{\mathbb{R}^N} \left(g_\xi(x) + F_\xi^1(x) - F_\xi^2(x)\right)\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(B_{1/8}), \tag{6.36}$$

where

$$F_\xi^1(x) = \int_{\mathbb{R}^N\setminus\Sigma} \left(\widetilde{U}_\xi(x) - \widetilde{U}_\xi(y)\right)\mathcal{K}_\xi(x,y)dy, \quad F_\xi^2(x) = p.v.\int_{\mathbb{R}^N} \left(\widetilde{U}_\xi(x) - \widetilde{U}_\xi(y)\right)\mathcal{K}_\xi(x,y)dy.$$

Since $|x-y|\geq 1/2$ for $x\in B_{1/8}$ and $y\in\mathbb{R}^N\setminus\Sigma$, using (5.4), we easily see that

$$|F^1_\xi(x)|\leq C(N)\|\widetilde{U}_\xi\|_{L^\infty(\mathbb{R}^N)}\leq C(N)\rho^{\beta+2s-1}. \tag{6.37}$$

On the other hand, using polar coordinates, w can write

$$\begin{aligned}F^2_\xi(x)=&\frac{1}{2}\int_{S^{N-1}}\int_0^\infty\frac{2\widetilde{U}_\xi(x)-\widetilde{U}_\xi(x+r\theta)-\widetilde{U}_\xi(x-r\theta)}{r^{1+2s}}(\mathcal{A}_\xi(x,r,\theta)+\mathcal{A}_\xi(x,-r,\theta))\,drd\theta\\&+\frac{1}{2}\int_{S^{N-1}}\int_0^\infty\frac{\widetilde{U}_\xi(x+r\theta)-\widetilde{U}_\xi(x-r\theta)}{r^{1+2s}}(\mathcal{A}_\xi(x,r,\theta)-\mathcal{A}_\xi(x,-r,\theta))\,drd\theta,\end{aligned}$$

where

$$\mathcal{A}_\xi(x,r,\theta)=r^{N+2s}\mathcal{K}_\xi(x,x+r\theta)=c_{N,s}\rho^{-\beta}\left(\frac{1}{|\int_0^1 D\Psi(\rho x+tr\rho\theta)\theta|^{N+2s}}-1\right).$$

Recall that $\Psi(y+\xi)=y+(0,y_N+2\rho+\psi(y'))$. As a consequence, for $x\in B_{1/8}$, $r>0$ and $\theta\in\partial B_1$,

$$|\mathcal{A}_\xi(x,r,\theta)|\leq C(|x|^\beta+r^\beta),\qquad |\mathcal{A}_\xi(x,r,\theta)-\mathcal{A}_\xi(x,r,-\theta)|\leq Cr^\beta.$$

Since $\widetilde{U}_\xi\in C^2(\mathbb{R}^N)$ and $1>\beta>2s-1>0$, we obtain

$$\|F^2_\xi\|_{L^\infty(B_{1/8})}\leq C(N)\|\widetilde{U}_\xi\|_{C^2(\mathbb{R}^N)}\leq C(N)\rho^{\beta+2s-1}.$$

From this, (6.37), (6.31), (6.32) and the fact that $\|g_\xi\|_{L^p(\mathbb{R}^N)}\leq\rho^{2s-N/p}$, we can apply [16, Theorem 4.1] to (6.36) and get

$$\|u_\xi\|_{C^\alpha(B_{1/9})}\leq C\rho^\gamma,$$

for all $\alpha\in(0,\min(2s-N/p,1))$. In particular,

$$\|u_\xi\|_{C^{\gamma-(2s-1)}(B_{1/9})}+\|u_\xi\|_{C^{2s-1}(B_{1/9})}\leq C\rho^\gamma, \tag{6.38}$$

Then as above, we obtain

$$[v]_{C^{2s-1}(B_{\rho/9}(\xi))}+|v(\xi)/\xi_N^{2s-1}|+[v/y_N^{2s-1}]_{C^{\gamma-(2s-1)}(B_{\rho/9}(\xi))}\leq C. \tag{6.39}$$

We thus get (i) for $\beta>2s-1$.
We now finalize the proof of (ii). Here, we recall that $2s-N/p>1$ and $\beta>2s-1$. Then [15, Theorem 1.3] applied to the equation (6.36) yields

$$\|u_\xi\|_{C^{1,\sigma}(B_{1/9})}\leq C\rho^\gamma, \tag{6.40}$$

where $\sigma=2s-\frac{N}{p}-1$. Hence by scaling back, we obtain

$$\|\nabla\left(v-Qy_N^{2s-1}\right)\|_{L^\infty(B_{\rho/9}(\xi))}\leq C\rho^{\gamma-1},\qquad [\nabla\left(v-Qy_N^{2s-1}\right)]_{C^\sigma(B_{\rho/9}(\xi))}\leq C\rho^{\gamma-1-\sigma}. \tag{6.41}$$

Let $w(y)=y_N^{2-2s}\nabla\left(v-Qy_N^{2s-1}\right)$. Then, from the above estimates, for $y_1,y_2\in B_{\rho/9}(\xi)$, we have

$$\begin{aligned}&|w(y_1)-w(y_2)|\leq C|y_1-y_2|\rho^{1-2s}\|\nabla\left(v-Qy_N^{2s-1}\right)\|_{L^\infty(B_{\rho/9}(\xi))}\\&+C\rho^{2-2s}|y_1-y_2|^\sigma[\nabla\left(v-Qy_N^{2s-1}\right)]_{C^\sigma(B_{\rho/9}(\xi))}\\&\leq C|y_1-y_2|\rho^{1-2s+\gamma-1}+C\rho^{2-2s+\gamma-1-\sigma}|y_1-y_2|^\sigma\\&=C|y_1-y_2|^{\sigma+1-\sigma}\rho^{1-2s+\gamma-1}+C\rho^{2-2s+\gamma-1-\sigma}|y_1-y_2|^\sigma\leq C\rho^{\min(2-2s,N/p)}|y_1-y_2|^\sigma.\end{aligned}$$

We then get, recalling that $\sigma = 2s - \frac{N}{p} - 1$,

$$[w]_{C^{2s-\frac{N}{p}-1}(B_{\rho/9}(\xi))} \leq C\rho^{\min(2-2s,N/p)} \leq C.$$

Moreover by (6.41), we have

$$|w(\xi)| \leq C\xi_N^{\gamma+1-2s}. \tag{6.42}$$

Note also that $y_N^{2-2s}\nabla v(y) = w(y) + (2s-1)Qe_N$, we thus deduce (6.26).
Next, from the two estimates above, we can apply Lemma 5.3 to deduce that

$$\|w\|_{C^{2s-\frac{N}{p}-1}(B^+_{1/9}\cap\mathbb{R}e_N)} \leq C.$$

Therefore passing to the limit in (6.42), we see that $w(0) = 0$, so that $\lim_{\xi\to 0}\xi_N^{2-2s}\nabla v(\xi) = (2s-1)Qe_N$. In addition (6.35), (6.39) and Lemma 5.3 imply that $v/y_N^{2s-1}(0,\cdot) \in C^{\min(1-N/p,\beta)}(0,1)$. Now, letting $\xi \to 0$ in (6.34), we find that $[v/y_N^{2s-1}](0) = Q$ and we get (6.27). □

7. Proof of the main results

We start with the following result which allows us to study an equivalent problem in the half-space. Let Ω be an open subset of $\mathbb{R}^N$ of class $C^{1,\beta}$, with $0 \in \partial\Omega$ and $\beta \geq 0$. Up to scaling and translations, we may assume that $B_1 \cap \partial\Omega$ is contained in the graph of a $C^{1,\beta}$ function. For $q \in \partial\Omega$, let $\nu(q)$ denote the unit interior normal vector of Ω at q and denote $T_q\partial\Omega = \nu(q)^\perp$ the tangent plane of $\partial\Omega$ at q. By the local inversion theorem, there exist $r_0 > 0$ depending only on N, β and Ω such that for any $q \in B_{r_0} \cap \partial\Omega$, there exist an orthonormal basis $(E_i(q)\dots, E_{N-1}(q))$ of $T_q\partial\Omega$ and $\widetilde{\phi} \in C^{1,\beta}(B'_{r_0})$ such that

$$x' \mapsto q + \sum_{i=1}^{N-1} x_iE_i(q) + \widetilde{\phi}(x')\nu(q) : B'_{r_0} \to \partial\Omega$$

with $\widetilde{\phi}(0) = |\nabla\widetilde{\phi}(0)| = 0$. We let $\eta \in C_c^\infty(B'_{r_0})$ with $\eta \equiv 1$ on $B'_{r_0/2}$ and put $\phi = \eta\widetilde{\phi}$. For all $q \in B_{r_0} \cap \partial\Omega$, we define

$$\Upsilon : \mathbb{R}^N \to \mathbb{R}^N, \qquad \Upsilon(x', x_N) = q + \sum_{i=1}^{N-1} x_iE_i(q) + \big(x_N + \phi(x')\big)\,\nu(q).$$

Then, see e.g. [16], Υ is a global $C^{1,\beta}$ diffeomorphism with $\mathrm{Det}D\Upsilon(x) = 1$ for all $x \in \mathbb{R}^N$. We define $Q_\delta := B'_\delta \times (-\delta,\delta)$ and $\mathcal{Q}_\delta := \Upsilon(Q_\delta)$. We have that Υ parameterizes $\Omega \cap \mathcal{Q}_{r_0/2}$ over $Q_{r_0/2} \cap \mathbb{R}^N_+$. It is easy to see that

$$\|D\Upsilon - Id\|_{L^\infty(\mathbb{R}^N)} = \|\nabla\phi\|_{L^\infty(\mathbb{R}^{N-1})} = \|\nabla\phi\|_{L^\infty(B'_{r_0})} \to 0 \qquad \text{as } r_0 \to 0.$$

Since, recalling (5.2), for all $y' \in B'_{r_0/2}$ and $x \in Q_{r_0/2}$,

$$|\Upsilon(x', x_N) - \Upsilon(y', 0)|^2 = |(x', x_N) - (y', 0)|^2 + \big(O(x_N|x' - y'|) + O(|x' - y'|^2)\big)\,\|\nabla\phi\|_{L^\infty(B'_{r_0})},$$

decreasing r_0 if necessary, by Young's inequality, we have that

$$\frac{x_N}{2} \leq \mathrm{dist}(\Upsilon(x', x_N), \partial\Omega) \leq 2x_N \quad \text{ for all } x \in Q_{r_0/2} \cap \mathbb{R}^N_+.$$

We state the following result.

Lemma 7.1. *Let Ω, r_0, ϕ and Υ be as above. For $s \in [s_0, 1)$, with $s_0 \in (0,1)$, let $u \in H^s_{loc}(\overline{\Omega}) \cap L^2(\Omega)$ be a solution to (1.3) and for $s \in [s_0, 1)$, with $s_0 \in (1/2, 1)$, we let $v \in H^s_0(\Omega)$ be a solution to (1.4), with $f \in L^p(\Omega)$ and $p > \frac{N}{2s_0}$. Then there exist $\hat{u} \in H^s(\mathbb{R}^N_+)$, $\hat{v} \in H^s_0(\mathbb{R}^N_+)$, $\hat{f} \in L^p(\mathbb{R}^N)$ and $\psi \in C^{1,\beta}(\mathbb{R}^{N-1})$ such that*

$$\mathcal{D}_{K_s^\psi}(\hat{u}, \varphi) = \int_{B_2} \hat{f}(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^1_c(B_2)$$

and

$$\mathcal{D}_{K_s^\psi}(\hat{v}, \varphi) = \int_{B_2} \hat{f}(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^1_c(B_2^+),$$

where $\psi(x) = \frac{1}{r}\phi(rx)$ and $r = \frac{r_0}{40}$. Moreover, $\hat{u} = u \circ \Upsilon(rx)$, $\hat{v} = v \circ \Upsilon(rx)$,

$$\|\hat{u}\|_{L^2(\mathbb{R}^N_+)} \le C\|u\|_{L^2(\Omega)}, \qquad \|\hat{v}\|_{L^2(\mathbb{R}^N)} \le C\|v\|_{L^2(\Omega)}$$

and $\|\hat{f}\|_{L^p(\mathbb{R}^N)} \le C\left(\|u\|_{L^2(\Omega)} + \|\hat{f}\|_{L^p(\Omega)}\right)$. Here, $C = C(N, \Omega, r_0, s_0)$.

Proof. We will prove the result in the Neumann case. The proof of the Dirichlet case is similar. Let $\nu(x,y) = \frac{c_{N,s}}{2|x-y|^{N+2s}}$. Put $r = \frac{r_0}{40}$ and let $\widetilde{u} = u\chi_r$, with $\chi_r \in C^\infty_c(\mathcal{Q}_{10r})$ such that $\chi_r \equiv 1$ on $\mathcal{Q}_{9r}$. We then get

$$(-\Delta)^s_{\overline{\Omega}}\widetilde{u} = \widetilde{f},$$

for some function $\widetilde{f} \in L^p(\mathbb{R}^N)$ and satisfying

$$\|\widetilde{f}\|_{L^p(\mathbb{R}^N)} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\mathbb{R}^N)}\right). \tag{7.1}$$

Next, for $\varphi \in C^\infty_c(\mathcal{Q}_{7r})$,

$$\begin{aligned}
&\int_\Omega \widetilde{f}(x)\varphi(x)\, dx = \int_{\Omega\times\Omega} (\widetilde{u}(x) - \widetilde{u}(y))(\varphi(x) - \varphi(y))\nu(x,y)\, dxdy \\
&= \int_{\Omega\cap\mathcal{Q}_{8r}\times\Omega\cap\mathcal{Q}_{8r}} (\widetilde{u}(x) - \widetilde{u}(y))(\varphi(x) - \varphi(y))\nu(x,y)\, dxdy \\
&+ 2\int_{\Omega\cap\mathcal{Q}_{8r}}\int_{\Omega\setminus\mathcal{Q}_{8r}} (\widetilde{u}(x) - \widetilde{u}(y))(\varphi(x) - \varphi(y))\nu(x,y)\, dxdy \\
&= \int_{\Omega\cap\mathcal{Q}_{8r}\times\Omega\cap\mathcal{Q}_{8r}} (\widetilde{u}(x) - \widetilde{u}(y))(\varphi(x) - \varphi(y))\nu(x,y)\, dxdy \\
&+ 2\int_{\Omega\cap\mathcal{Q}_{7r}} \widetilde{u}(x)\varphi(x)\int_{\Omega\setminus\mathcal{Q}_{8r}} \nu(x,y)\, dxdy + 2\int_{\Omega\cap\mathcal{Q}_{7r}} \varphi(x)\int_{\Omega\setminus\mathcal{Q}_{8r}} \widetilde{u}(y)\nu(x,y)\, dxdy.
\end{aligned}$$

Letting

$$\widetilde{V}(x) = 21_{\mathcal{Q}_{7r}}(x)\int_{\Omega\setminus\mathcal{Q}_{8r}} \nu(x,y)\, dy, \quad F(x) = \widetilde{f}(x) - 21_{Q_{7r}}(x)\int_{\Omega\setminus\mathcal{Q}_{8r}} \widetilde{u}(y)\nu(x,y)\, dy,$$

we deduce that

$$\int_{\Omega\cap\mathcal{Q}_{8r}\times\Omega\cap\mathcal{Q}_{8r}} (\widetilde{u}(x) - \widetilde{u}(y))(\varphi(x) - \varphi(y))\nu(x,y)\, dxdy + \int_\Omega \widetilde{V}(x)\widetilde{u}(x)\varphi(x)\, dx = \int_\Omega F(x)\varphi(x) \tag{7.2}$$

and we note that

$$\min_{\mathcal{B}_{7r}} \widetilde{V} > 0, \qquad \|\widetilde{V}\|_{L^\infty(\mathbb{R}^N)} \le C, \qquad \|F\|_{L^p(\mathbb{R}^N)} \le C(N,\Omega)(\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\mathbb{R}^N)}). \tag{7.3}$$

Let $w = \widetilde{u} \circ \Upsilon \in H^s(\mathbb{R}^N_+)$, $\overline{V} = \widetilde{V} \circ \Upsilon$ and $\overline{F} = F \circ \Upsilon \in L^p(\mathbb{R}^N)$. Then by a change of variable and using (7.2), we get, for $\varphi \in C_c^\infty(B_{4r})$,

$$\begin{aligned}
&\mathcal{D}_{K_s^\phi}(w,\varphi) := \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} (w(x)-w(y))(\varphi(x)-\varphi(y))K_s^\phi(x,y)\,dxdy \\
&= \frac{1}{2}\int_{B_{6r}^+\times B_{6r}^+} (w(x)-w(y))(\varphi(x)-\varphi(y))K_s^\phi(x,y)\,dxdy \\
&+ \int_{B_{6r}^+}\int_{\mathbb{R}^N\setminus B_{6r}^+} (w(x)-w(y))(\varphi(x)-\varphi(y))K_s^\phi(x,y)\,dxdy \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left(-\overline{V}(x)w(x)+\overline{F}(x)\right)\varphi(x)\,dx + \int_{B_{4r}^+} w(x)\varphi(x)\int_{\mathbb{R}^N\setminus B_{6r}^+} K_s^\phi(x,y)\,dxdy \\
&+ \int_{B_{4r}^+} \varphi(x)\int_{\mathbb{R}^N\setminus B_{6r}^+} w(y)K_s^\phi(x,y)\,dxdy.
\end{aligned}$$

We then get, for all $\varphi \in C_c^\infty(B_{4r})$,

$$\mathcal{D}_{K_s^\phi}(w,\varphi) + \int_{\mathbb{R}^N} H(x)u(x)\varphi(x)\,dx = \int_{\mathbb{R}^N} G(x)\varphi(x)\,dx,$$

where

$$H(x) := \overline{V}(x) + 1_{B_{5r}}(x)\int_{\mathbb{R}^N\setminus B_{6r}^+} K_s^\phi(x,y)\,dy$$

and

$$G(x) := 1_{B_{5r}}(x)\int_{\mathbb{R}^N\setminus B_{6r}^+} w(y)K_s^\phi(x,y)\,dy + \overline{F}(x).$$

We can now scale by putting $\hat{u}(x) = w(rx)$, $g(x) = 1_{B_4}(x)G(rx)$, $V(x) = 1_{B_4}(x)H(rx)$ and $\psi(x) = \frac{1}{r}\phi(rx)$. We then have that

$$\hat{u} \in H^s(\mathbb{R}^N_+), \qquad \|\hat{u}\|_{L^2(\mathbb{R}^N_+)} \le C\|u\|_{L^2(\Omega)}, \tag{7.4}$$

$$\|g\|_{L^p(\mathbb{R}^N)} \le C\left(\|f\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^2(\Omega)}\right), \tag{7.5}$$

$$\min_{B_2} V > 0, \qquad \|V\|_{L^\infty(B_2)} \le C \tag{7.6}$$

and for every $\varphi \in C_c^\infty(B_2)$,

$$\mathcal{D}_{K_s^\psi}(\hat{u},\varphi) + \int_{\mathbb{R}^N} V(x)\hat{u}(x)\varphi(x)\,dx = \int_{\mathbb{R}^N} g(x)\varphi(x)\,dx. \tag{7.7}$$

Now by the Kato inequality and (7.6), for all $\varphi \in C_c^\infty(B_2)$ with $\varphi \ge 0$, we have that

$$\mathcal{D}_{K_s^\psi}(|\hat{u}|,\varphi) \le \mathcal{D}_{K_s^\psi}(|\hat{u}|,\varphi) + \int_{\mathbb{R}^N} V(x)|\hat{u}(x)|\varphi(x)\,dx \le \int_{\mathbb{R}^N} |g(x)|\varphi(x)\,dx.$$

By a direct minimization argument, there exists a unique function $\eta \in H^s(\mathbb{R}^N_+)$ such that $\eta = |\hat{u}|$ on $\mathbb{R}^N_+ \setminus B_2$ and for all $\varphi \in C_c^\infty(B_2)$,

$$\mathcal{D}_{K_s^\psi}(\eta,\varphi) = \int_{\mathbb{R}^N} |g(x)|\varphi(x)\,dx.$$

By construction, we have that

$$|\hat{u}| \le \eta \qquad \text{in } B_2. \tag{7.8}$$

Applying Corollary 5.4, we deduce that

$$\|\eta\|_{L^\infty(B_1^+)} \leq C(N,s_0,p)\left(\|\eta\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)}\right) \leq C(N,s_0,p)\left(\|\hat{u}\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)}\right). \tag{7.9}$$

From this and (7.8), we finally get

$$\|\hat{u}\|_{L^\infty(B_1^+)} \leq C\left(\|\hat{u}\|_{L^2(\mathbb{R}^N_+)} + \|g\|_{L^p(\mathbb{R}^N)}\right) \leq C\left(\|u\|_{L^2(\Omega)} + \|g\|_{L^p(\mathbb{R}^N)}\right). \tag{7.10}$$

We can now rewrite (7.7) in the form,

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (\hat{u}(x)-\hat{u}(y))(\varphi(x)-\varphi(y))K_s^\psi(x,y)\,dxdy = \int_{\mathbb{R}^N} \hat{f}(x)\varphi(x)\,dx,$$

with $\hat{f} = 1_{\mathbb{R}^N_+}(g - 1_{B_1}V\hat{u})$. We note that by (7.10), (7.5) and (7.4),

$$\|\hat{f}\|_{L^p(\mathbb{R}^N)} \leq C\left(\|f\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^2(\Omega)}\right).$$

The proof is thus complete in the Neumann case. Note that for the Dirichlet case, the only change in the above argument is to apply Corollary 6.4 in the place of Corollary 5.4 in order to get (7.9). □

7.1. **Proof of the main results (completed).** We are now in position to complete the proofs of the main results in the first section. For the following, we assume for simplicity that $\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)} \leq 1$, $\|f\|_{L^p(\Omega)} + \|v\|_{L^2(\Omega)} \leq 1$ and Ω is as in the beginning of Section 7. For $x \in \mathbb{R}^N$, we let $\delta(x)$ to be $\mathrm{dist}(x,\partial\Omega)$ for x in a tubular neighbourhood of $\partial\Omega$ and Lipschitz continuous and positive elsewhere.

Proof of Theorem 1.1 (completed). The statement in the theorem follows from Lemma 7.1 and Corollary 5.4. □

Proof of Theorem 1.2 (completed). We put $\gamma := \min(2s-N/p, 1+\beta)$. Then in view of Lemma 7.1 and Corollary 5.7, we can find two constants $\overline{r}, C > 0$, depending only on N, s, p, Ω and β such that for any $x \in B_{\overline{r}}\cap\Omega$, we have that $|\nabla u(x)| + [\nabla u]_{C^{\gamma-1}(B_\rho(x))} \leq C$ and, $\nabla u(q)\cdot\nu(q) = 0$, with $\rho = \delta(x)/4$ and $|x-q| = \delta(x)$. Applying Lemma 5.3, after flattening the boundary, we get the stated results. □

Proof of Theorem 1.3 (completed). The proof follows from Lemma 7.1 and Corollary 6.4. □

Proof of Theorem 1.4 (completed). In view of Lemma 7.1 and Corollary 6.7-(i), we obtain two constants $\overline{r} = \overline{r}(N,\Omega,\beta) > 0$ and $C = C(N,s,p,\Omega,\beta) > 0$ such that for any $x \in B_{\overline{r}}\cap\Omega$, we have that

$$|v(x)/\delta^{2s-1}(x)| + [v]_{C^{2s-1}(B_\rho(x))} + [v/\delta^{2s-1}]_{C^{\min(1-N/p,\beta)}(B_\rho(x))} \leq C$$

with $\rho = \delta(x)/4$. Applying Lemma 5.3, after flattening the boundary, we get (i). Next, we let $2s - N/p > 1$ and $\beta > 2s-1$. Then in this case we use Corollary 6.7-(ii) to get

$$|\delta^{2s-2}(x)\nabla v(x)| + [\delta^{2s-2}\nabla v]_{C^{2s-\frac{N}{p}-1}B_\rho(x)} \leq C$$

and $\lim_{\delta(x)\to 0}\left(\delta(x)^{2-2s}\nabla v(x)\right) = (2s-1)[v/\delta^{2s-1}](q)\nu(q)$, where $|x-q| = \delta(x)$. We then apply Lemma 5.3, after flattening the boundary, to get (ii). □

Remark 7.2. *We point out that provided $p > \frac{N}{2s_0}$ and $\beta < 2s_0 - 1$, the constant C in Theorem 1.2 and Theorem 1.4-(i) can be chosen to remain bounded as $s \to 1$, while for Theorem 1.4-(ii), we need to assume that $\beta = 1$. Then the constant C appearing in Theorem 1.2 and Theorem 1.4 remains bounded as $s \to 1$. This follows from Lemma 8.1 and the blow up argument that is used in the proof of Proposition 5.5 and Proposition 6.5.*

8. Appendix

This section is devoted to prove the convergence of the nonlocal problem to the local problem as $s \to 1$.

Lemma 8.1. *(i) Let Ω be an open subset of $\mathbb{R}^N$. Then for all $u, v \in C_c^1(\mathbb{R}^N)$, we have*

$$\lim_{s\to 1} c_{N,s} \int_\Omega \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy = \gamma_N \int_\Omega \nabla u(x)\cdot \nabla v(x)\, dx.$$

(ii) Let $\psi_n \in C^1(\mathbb{R}^{N-1})$ be such that $[\nabla \psi_n]_{C^1(\mathbb{R}^{N-1})} \le \frac{1}{n}$ and $s_n \in (0,1)$ with $s_n \to 1$. Let $w_n \in H^{s_n}_{loc}(\overline{\mathbb{R}^N_+})$ with $w_n \to w \in L^2_{loc}(\overline{\mathbb{R}^N_+})$ and suppose that there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$(1-s_n)[w_n]_{H^{s_n}_{loc}(\mathbb{R}^N_+)} + \int_{\mathbb{R}^N_+} \frac{|w_n(x)|}{1+|x|^{N+2s_n}}\, dx \le C.$$

Then, for all $\phi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \mathcal{D}_{K^{\psi_n}_{s_n}}(w_n, \phi) = \frac{\gamma_N}{2} \int_{\mathbb{R}^N_+} \nabla w(x)\cdot \nabla \phi(x) dx.$$

Here, $\gamma_N = \lim_{s\to 1} \frac{c_{N,s}|B_1|}{2(1-s)}$.

Proof. Let $u, v \in C_c^\infty(\mathbb{R}^N)$ and $R > 0$ be such that Supp$u$, Supp$v \subset B_R$. We estimate

$$\begin{aligned}
&c_{N,s} \int_\Omega \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \\
&= c_{N,s} \int_{\Omega\cap B_{2R}} \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \\
&+ c_{N,s} \int_{\Omega\setminus B_{2R}} \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \\
&= c_{N,s} \int_{\Omega\cap B_{2R}} \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy \\
&+ c_{N,s} \int_{\Omega\cap B_R} u(y)v(y)dy \int_{\Omega\setminus B_{2R}} \frac{1}{|x-y|^{N+2s}}\, dx \\
&= c_{N,s} \int_{\Omega\cap B_{2R}} \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy + O(c_{N,s})[u]_{C^1(\mathbb{R}^N)}[v]_{C^1(\mathbb{R}^N)}. \qquad (8.1)
\end{aligned}$$

By Taylor expansion, for all $\varepsilon < 0$, there exits $r_\varepsilon > 0$ such that for $y \in B(x, r_\varepsilon)$

$$u(x) - u(y) = \nabla u(x)\cdot(x-y) + O(\varepsilon|x-y|)[u]_{C^1(\mathbb{R}^N)}.$$

To alleviate the notations, we assume that $[u]_{C^1(\mathbb{R}^N)}[v]_{C^1(\mathbb{R}^N)} \leq 1$. We then have

$$\begin{aligned}
&c_{N,s}\int_{\Omega\cap B_{2R}} dx \int_{\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dy\\
&= c_{N,s}\int_{\Omega\cap B_{2R}} dx \int_{|x-y|\leq r_\varepsilon} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dy\\
&+ c_{N,s}\int_{\Omega\cap B_{2R}} dx \int_{|x-y|\geq r_\varepsilon} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dy\\
&= c_{N,s}\int_{\Omega\cap B_{2R}} dx \int_{|x-y|\leq r_\varepsilon} \frac{\nabla u(x)\cdot(x-y)\nabla v(x)\cdot(x-y)}{|x-y|^{N+2s}}\,dy\\
&+ c_{N,s}\int_{\Omega\cap B_{2R}} dx \int_{|x-y|\leq r_\varepsilon} \frac{O(\varepsilon|x-y|^2)}{|x-y|^{N+2s}}\,dy + c_{N,s}\int_{\Omega\cap B_{2R}} dx \int_{|x-y|\geq r_\varepsilon} \frac{O(1)}{|x-y|^{N+2s}}\,dy.
\end{aligned}$$

We then obtain

$$\begin{aligned}
&c_{N,s}\int_{\Omega\cap B_{2R}}\int_{\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy\\
&= c_{N,s}\int_{\Omega}\int_{S^{N-1}} \nabla u(x)\cdot\theta\nabla v(x)\cdot\theta\,d\theta\int_0^{r_\varepsilon} r^{1-2s}dr + O(\varepsilon\frac{c_{N,s}}{1-s}) + c_{N,s}O(r_\varepsilon^{-2s})\\
&= \frac{c_{N,s}|B_1|r_\varepsilon^{2-2s}}{2(1-s)}\int_{\Omega}\nabla u(x)\cdot\nabla v(x)dx + \left(O(\varepsilon\frac{c_{N,s}}{1-s}) + c_{N,s}O(r_\varepsilon^{-2s})\right).
\end{aligned}$$

Using this in (8.1), we conclude that

$$\begin{aligned}
c_{N,s}\int_{\Omega}\int_{\Omega}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy &= \frac{c_{N,s}|B_1|r_\varepsilon^{2-2s}}{2(1-s)}\int_{\Omega}\nabla u(x)\cdot\nabla v(x)dx\\
&+ \left(O(\varepsilon\frac{c_{N,s}}{1-s}) + c_{N,s}O(r_\varepsilon^{-2s}\right)[u]_{C^1(\mathbb{R}^N)}[v]_{C^1(\mathbb{R}^N)}. \qquad (8.2)
\end{aligned}$$

Since $c_{N,s} = O(1-s)$, then letting $s\to 1$ and then $\varepsilon\to 0$, we get (i).
We now prove (ii). Recall that $\Psi_n(x',x_N) = (x', x_N+\psi_n(x))$ (see Section 5). We define $\Omega_n := \Psi_n(\mathbb{R}^N_+)$. Let $\phi,\varphi\in C_c^\infty(\mathbb{R}^N)$ and define $u_n := \phi\circ\Psi_n$ and $v_n := \varphi\circ\Psi_n$. By a change of variable, we get

$$\mathcal{D}_{K_{s_n}^{\psi_n}}(\phi,\varphi) = \frac{c_{N,s_n}}{2}\int_{\Omega_n}\int_{\Omega_n}\frac{(u_n(x)-u_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+2s_n}}\,dxdy$$

By (8.2), we have

$$\begin{aligned}
c_{N,s_n}\int_{\Omega_n}\int_{\Omega_n}\frac{(u_n(x)-u_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+2s_n}}\,dxdy &= \frac{c_{N,s_n}|B_1|r_\varepsilon^{2-2s_n}}{2(1-s_n)}\int_{\Omega_n}\nabla u_n(x)\cdot\nabla v_n(x)dx\\
&+ \left(O(\varepsilon\frac{c_{N,s_n}}{1-s_n}) + c_{N,s_n}O(r_\varepsilon^{-2s_n}\right)[u_n]_{C^1(\mathbb{R}^N)}[v_n]_{C^1(\mathbb{R}^N)}.
\end{aligned}$$

Using now that $[\nabla\psi_n]_{C^1(\mathbb{R}^{N-1})} \leq \frac{1}{n}$, the fact that $c_{N,s_n} = O(1-s_n)$ and the dominated convergence theorem, we deduce that

$$2\lim_{n\to\infty}\mathcal{D}_{K_{s_n}^{\psi_n}}(\phi,\varphi) = \gamma_N\int_{\mathbb{R}^N_+}\nabla\phi(x)\cdot\nabla\varphi(x)dx + O(\varepsilon)[\phi]_{C^1(\mathbb{R}^N)}[\varphi]_{C^1(\mathbb{R}^N)}.$$

Now letting $\varepsilon \to 0$, we get, for all $\phi, \varphi \in C_c^\infty(\mathbb{R}^N)$,

$$2 \lim_{n\to\infty} \mathcal{D}_{K_{s_n}^{\psi_n}}(\phi, \varphi) = \gamma_N \int_{\mathbb{R}^N_+} \nabla\phi(x) \cdot \nabla\varphi(x) dx. \tag{8.3}$$

In addition, for $\phi, \varphi \in H^1(\mathbb{R}^N_+)$, we have that

$$|\mathcal{D}_{K_{s_n}^{\psi_n}}(\phi, \varphi)| \le C(N)\|\phi\|_{H^1(\mathbb{R}^N_+)}\|\varphi\|_{H^1(\mathbb{R}^N_+)}.$$

Therefore the symmetric bilinear form $\mathcal{D}_{K_{s_n}^{\psi_n}}$ form-converges in $H^1(\mathbb{R}^N_+) \times H^1(\mathbb{R}^N_+)$ to a symmetric bilinear form $\mathcal{D}_\infty : H^1(\mathbb{R}^N_+) \times H^1(\mathbb{R}^N_+) \to \mathbb{R}$. Namely, for all $\phi, \varphi \in H^1(\mathbb{R}^N_+)$,

$$\lim_{n\to\infty} \mathcal{D}_{K_{s_n}^{\psi_n}}(\phi, \varphi) = \mathcal{D}_\infty(\phi, \varphi).$$

By density and (8.3), we have that for all $\phi, \varphi \in H^1(\mathbb{R}^N_+)$,

$$2 \lim_{n\to\infty} \mathcal{D}_{K_{s_n}^{\psi_n}}(\phi, \varphi) = 2\mathcal{D}_\infty(\phi, \varphi) = \gamma_N \int_{\mathbb{R}^N_+} \nabla\phi(x) \cdot \nabla\varphi(x) dx. \tag{8.4}$$

Fix $M \ge 1$ and consider $\chi_M \in C_c^\infty(B_{2M})$ such that $\chi_M = 1$ on B_M. Then for $\phi \in C_c^\infty(B_M/2)$, we get

$$\mathcal{D}_{K_{s_n}^{\psi_n}}(\chi_M w_n, \phi) = \mathcal{D}_{K_{s_n}^{\psi_n}}(\chi_M w, \phi) + \mathcal{D}_{K_{s_n}^{\psi_n}}(\chi_M (w_n - w), \phi). \tag{8.5}$$

Letting $v_n = \chi_M(w_n - w)$, we have

$$2\left|\mathcal{D}_{K_{s_n}^{\psi_n}}(v_n, \phi)\right| = c_{N,s_n} \left| \int_{\mathbb{R}^N_+ \times \mathbb{R}^N_+} \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s_N}} dxdy \right|$$
$$+ \int_{\mathbb{R}^N \times \mathbb{R}^N} |(v_n(x) - v_n(y))(\phi(x) - \phi(y))| \left| K_{s_n}^{\psi_n}(x,y) - K_{s_n}^+(x,y) \right| dxdy.$$

Using this, the fact that $w_n \to w$ in $H^\sigma_{loc}(\overline{\mathbb{R}^N_+})$ for every $\sigma \in (0,1)$ and (5.4), we obtain

$$\left|\mathcal{D}_{K_{s_n}^{\psi_n}}(\chi_M(w_n - w), \phi)\right| = o(1) \qquad \text{as } n \to \infty. \tag{8.6}$$

Now, by a direct computation, for $\phi \in C_c^\infty(B_{M/2})$, we also have that

$$\mathcal{D}_{K_{s_n}^{\psi_n}}(\chi_M w_n, \phi) = \mathcal{D}_{K_{s_n}^{\psi_n}}(w_n, \phi) + c_{N,s_n} \int_{\mathbb{R}^N} G_n(x)\phi(x)\, dx,$$

where $G_n(x) = 1_{B_{M/2}} \int_{\{|y|\ge M\}\cap\mathbb{R}^N_+} |x-y|^{-N-2s_n} w_n(y)\, dy$. Since $c_{N,s_n} = O(1-s_n) \to 0$ as $n \to \infty$, by the above identity, (8.5), (8.6) and (8.4), we deduce that

$$\lim_{n\to\infty} \mathcal{D}_{K_{s_n}^{\psi_n}}(w_n, \phi) = \frac{\gamma_N}{2} \int_{\mathbb{R}^N_+} \nabla w(x) \cdot \nabla\phi(x) dx.$$

□

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M.M.F.: African Institute for Mathematical Sciences in Senegal, KM 2, Route de Joal, B.P. 14 18. Mbour, Sénégal

E-mail address: mouhamed.m.fall@aims-senegal.org, mouhamed.m.fall@gmail.com