A FRACTIONAL HADAMARD FORMULA AND APPLICATIONS

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Abstract. We consider the domain dependence of the best constant in the subcritical fractional Sobolev constant,

$$\lambda_{s,p}(\Omega) := \inf \left\{ [u]^2_{H^s(\mathbb{R}^N)},\ u \in C^\infty_c(\Omega),\ \|u\|_{L^p(\Omega)} = 1 \right\},$$

where $s \in (0,1)$, Ω is bounded of class $C^{1,1}$ and $p \in [1, \frac{2N}{N-2s})$ if $2s < N$, $p \in [1,\infty)$ if $2s \geq N = 1$. Explicitly, we derive formula for the one-sided shape derivative of the mapping $\Omega \mapsto \lambda_{s,p}(\Omega)$ under domain perturbations. In the case where $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer (e.g. $p = 1$ or $p = 2$), our result implies a nonlocal version of the classical variational Hadamard formula for the first eigenvalue of the Dirichlet Laplacian on Ω. Thanks to the formula for our one-sided shape derivative, we characterize smooth local minimizers of $\lambda_{s,p}(\Omega)$ under volume-preserving deformations, and we find that they are balls if $p \in \{1\} \cup [2,\infty)$. Finally, we consider the maximization problem for $\lambda_{s,p}(\Omega)$ among annular-shaped domains of fixed volume of the type $B \setminus \overline{B}'$, where B is a fixed ball and B' is ball whose position is varied within B. We prove that, for $p \in \{1,2\}$, the value $\lambda_{s,p}(B \setminus \overline{B}')$ is maximal when the two balls are concentric.

1. Introduction

Let $s \in (0,1)$ and Ω a bounded open subset of $\mathbb{R}^N$ of class $C^{1,1}$. The best constant in the subcritical fractional Sobolev inequality is given by

$$\lambda_{s,p}(\Omega) := \inf \left\{ [u]^2_{H^s(\mathbb{R}^N)}, \quad u \in \mathcal{H}^s_0(\Omega), \quad \|u\|_{L^p(\Omega)} = 1 \right\}, \tag{1.1}$$

where $p \in [1, \frac{2N}{N-2s})$ if $2s < N$ and $p \in [1,\infty)$ if $2s \geq N = 1$. Here and in the following, we consider the square of the fractional seminorm

$$[u]^2_{H^s(\mathbb{R}^N)} = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dxdy,$$

where the normalization constant is given by

$$b_{N,s} = s(1-s)\pi^{-N/2} 4^s \frac{\Gamma(\frac{N}{2} + s)}{\Gamma(2-s)}, \tag{1.2}$$

so that $[u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi$ with $\hat{u}$ the Fourier transform of u. Moreover, $\mathcal{H}^s_0(\Omega)$ is defined as the space of functions $w \in H^s(\mathbb{R}^N)$ with $w \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. We note that, since Ω has a continuous boundary by assumption, the space $\mathcal{H}^s_0(\Omega)$ is equivalently given as the closure of $C^\infty_c(\Omega)$ in $H^s(\mathbb{R}^N)$, see e.g. [15, Theorem 1.4.2.2].

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Thanks to the compact embedding $\mathcal{H}^s_0(\Omega) \hookrightarrow L^p(\Omega)$, a direct minimization argument shows that $\lambda_{s,p}(\Omega)$ admits a positive minimizer $u \in \mathcal{H}^s_0(\Omega)$ with $\|u\|_{L^p(\Omega)} = 1$. Moreover, every such minimizer solves, in the weak sense, the semilinear problem

$$(-\Delta)^s u = \lambda_{s,p}(\Omega) u^{p-1} \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{1.3}$$

where $(-\Delta)^s$ stands for the fractional Laplacian. Recall that for smooth functions $\varphi \in C^{1,1}_c(\mathbb{R}^N)$, the fractional Laplacian is given by

$$(-\Delta)^s\varphi(x) = b_{N,s} PV \int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\, dy = \frac{b_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2\varphi(x)-\varphi(x+y)-\varphi(x-y)}{|y|^{N+2s}}\, dy. \tag{1.4}$$

Of particular interest are the cases $p = 1$ and $p = 2$ which correspond to the fractional torsion rigidity problem and the first Dirichlet fractional eigenvalue problem, respectively. In these cases $\lambda_{s,p}(\Omega)$ possesses a unique positive minimizer.
Our goal in this paper is to derive a formula for a one-sided shape derivative of $\Omega \mapsto \lambda_{s,p}(\Omega)$. More precisely, we consider a family of deformation Φ_ε, $\varepsilon \in (-1, 1)$ with the following properties:

$$\begin{aligned} &\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N;\mathbb{R}^N) \text{ for } \varepsilon \in (-1,1),\ \Phi_0 = \mathrm{id}_{\mathbb{R}^N}, \text{ and} \\ &\text{the map } (-1,1) \to C^{0,1}(\mathbb{R}^N,\mathbb{R}^N),\ \varepsilon \to \Phi_\varepsilon \text{ is of class } C^2. \end{aligned} \tag{1.5}$$

We note that (1.5) implies that $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is a global diffeomorphism if $|\varepsilon|$ is small enough. From the variational characterization of $\lambda_{s,p}(\Omega)$ it is not difficult to see that the map $\varepsilon \mapsto \lambda_{s,p}(\Phi(\varepsilon,\Omega))$ is continuous. However, since $\lambda_{s,p}(\Omega)$ may not have a unique positive minimizer, one cannot expect this map to be differentiable. We therefore rely on determining the right derivative of $\varepsilon \mapsto \lambda_{s,p}(\Phi(\varepsilon,\Omega))$ from which we derive differentiability whenever $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer, thereby extending the classical Hadamard shape derivative formula for the first Dirichlet eigenvalue.

In this paper, we consider a function δ, which coincides with the *signed* distance function $\mathrm{dist}(\cdot,\mathbb{R}^N \setminus \Omega) - \mathrm{dist}(\cdot,\Omega)$ in a neighborhood of $\partial\Omega$. Moreover, we suppose that δ is positive in Ω, negative in $\mathbb{R}^N \setminus \Omega$ and $\delta \in C^{1,1}(\mathbb{R}^N)$.

Our first main result is the following.

Theorem 1.1. *Let $\lambda_{s,p}(\Omega)$ be given by* (1.1) *and consider a family of deformations Φ_ε satisfying* (1.5)*. Then the map $\varepsilon \mapsto \theta(\varepsilon) := \lambda_{s,p}(\Phi_\varepsilon(\Omega))$ is right differentiable at $\varepsilon = 0$. Moreover*

$$\partial_+\theta(0) = \min\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 X\cdot\nu\, dx, \quad u \in \mathcal{H}\right\}, \tag{1.6}$$

where ν denotes the interior unit normal on $\partial\Omega$, $\mathcal{H}$ is the set of positive minimizers for $\lambda_{s,p}(\Omega)$, and $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$.

Here the function u/δ^s is defined as a limit. Namely for $x_0 \in \partial\Omega$,

$$\frac{u}{\delta^s}(x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega}} \frac{u}{\delta^s}(x).$$

We point out that, on $\partial\Omega$, the interior unit normal ν coincides with $\nabla\delta$. We also note that $u/\delta^s \in C^\alpha(\overline{\Omega})$, for some $\alpha > 0$, see e.g [22]. Moreover the expression u/δ^s, restricted on $\partial\Omega$, plays the role of a normal derivative, compared to the local case. In fact, we have that $\delta^{1-s}\nabla u\cdot\nabla\delta = su/\delta^s$ on $\partial\Omega$, see [9].
A one sided shape derivative in the case of degenerate eigenvalue was recently obtained in [10], where the authors considered the first nonzero Neuman eigenvalue which is not in general

simple. We observe that the constant $\Gamma(1+s)^2$ appears also in the fractional Pohozaev identity, see e.g. [23]. This is, to some extend, not surprising at least in the classical case since Pohozav's identity can be obtained using techniques of domain variation, see e.g. [26].
A natural consequence of Theorem 1.1 is that the map $\varepsilon \mapsto \theta(\varepsilon) := \lambda_{s,p}(\Phi(\varepsilon,\Omega))$ is differentiable whenever $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer. Indeed, applying Theorem 1.1 to the map $\varepsilon \mapsto \widetilde{\theta}(\varepsilon) := \lambda_{s,p}(\Phi(-\varepsilon,\Omega))$ yields

$$\partial_-\theta(0) = -\partial_+\widetilde{\theta}(0) = \max\left\{\Gamma(1+s)^2\int_{\partial\Omega}(u/\delta^s)^2 X\cdot\nu\,dx, \quad u\in\mathcal{H}\right\}.$$

As a consequence, we obtain the following result.

Corollary 1.2. *Let $\lambda_{s,p}(\Omega)$ be given by (1.1) and consider a family of deformations Φ_ε satisfying (1.5). Suppose that $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer $u \in H^s(\mathbb{R}^N)$. Then the map $\varepsilon \mapsto \theta(\varepsilon) := \lambda_{s,p}(\Phi(\varepsilon,\Omega))$ is differentiable at $\varepsilon = 0$. Moreover*

$$\theta'(0) = \Gamma(1+s)^2\int_{\partial\Omega}(u/\delta^s)^2 X\cdot\nu\,dx, \tag{1.7}$$

where $X := \partial_\varepsilon\Phi(0,\cdot)$.

As mentioned earlier, for $p = 1$ or $p = 2$, $\lambda_{s,p}(\Omega)$ admits a unique positive minimizer $u \in H^s(\mathbb{R}^N)$. Therefore Corollary 1.2 extends the classical Hadamard formula, for the first Dirichlet eigenvalue $\lambda_{1,2}(\Omega)$, in the fractional setting. We recall, see e.g. [17], that the Hadamard formula, is given by

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda_{1,2}(\Phi(\varepsilon,\Omega)) = \int_{\partial\Omega}|\nabla u|^2 X\cdot\nu\,dx. \tag{1.8}$$

We point out that, prior to this paper, a Hadamard formula in the fractional setting of the type (1.7) was obtained in [7] for the special case $p = 1$, $s = \frac{1}{2}$, $N = 2$ and Ω of class C^∞. We are not aware of any other previous work related to Theorem 1.1 or 1.2 in the fractional setting. An analogue of Corollary 1.2 for the case of the local p-Laplace operator was obtained in [2,14].

Our next result provides a characterization of local minima of $\Omega \mapsto \lambda_{s,p}(\Omega)$.

Corollary 1.3. *Let $p \in \{1\}\cup[2,\infty)$. Suppose that Ω, an open set of class C^3, is a volume constrained local minimum for $\Omega \mapsto \lambda_{s,p}(\Omega)$. Then Ω is a ball.*
Here we call Ω a constrained local minimum for $\Omega \mapsto \lambda_{s,p}(\Omega)$ if for all families of deformations Φ_ε satisfying (1.5) and the volume invariance condition $|\Phi_\varepsilon(\Omega)| = |\Omega|$ for $\varepsilon \in (-1,1)$, there exists $\varepsilon_0 \in (0,1)$ such that $\lambda_{s,p}(\Phi(\varepsilon,\Omega)) \geq \lambda_{s,p}(\Omega)$ for $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$.

Corollary 1.3 is a consequence of Theorem 1.1, from which we derive that if Ω is a constraint local minimum then any element $u \in \mathcal{H}$ satisfies the overdetermined condition $u/\delta^s \equiv Const$ on $\partial\Omega$. Therefore by the rigidity result in [13] we find that Ω must be a ball.
We recall that the authors in [7] considered also shape minimization problem of $\lambda_{s,p}$ for $p = 1$, $s = \frac{1}{2}$, $N = 2$ and among domains Ω of class C^∞. They showed in [7] that such minimizers are discs.

Next we consider the optimization problem of $\Omega \mapsto \lambda_{s,p}(\Omega)$ for $p \in \{1,2\}$ and Ω a punctured ball, with the hole having the shape of ball. We show that, as the hole moves in Ω then $\lambda_{s,p}(\Omega)$ is maximal when the two balls are concentric. This problem was first solved by Hersch [18] in the case $s = 1$ and $N = 2$; for subsequent generalizations in the local case $s = 1$; see [5,16,20].

Theorem 1.4. *Let $p \in \{1,2\}$, $B_1(0)$ be the unit centered ball and $\tau \in (0,1)$. Define*

$$\mathcal{A} := \{a \in B_1(0) \, : \, B_\tau(a) \subset B_1(0)\}.$$

Then the map $\mathcal{A} \to \mathbb{R}$, $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its maximum at $a = 0$.

The proof of Theorem 1.4 is inspired by the argument given in [16, 20] for the local case $s = 1$. It uses the fractional Hadamard formula in Corollary 1.2 and maximum principles for anti-symmetric functions. Our proof also shows that the map $a \mapsto \lambda_{s,p}(B_1(0) \setminus \overline{B_\tau(a)})$ takes its minimum when the boundary of the ball $B_\tau(a)$ touches the one of $B_1(0)$, see Section 5 below.

The proof of Theorem 1.1 is inspired by [10]. It is mainly based on the use of test functions in the variational characterization of $\lambda_{s,p}(\Omega)$ and $\lambda_{s,p}(\Phi(\varepsilon,\Omega))$. In the case of $\lambda_{s,p}(\Phi(\varepsilon,\Omega))$, it is important to make a change of variable so that $\lambda_{s,p}(\Phi(\varepsilon,\Omega))$ is determined by minimizing an ε-dependent family of seminorms among functions $u \in H^s(\mathbb{R}^N)$ vanishing outside the fixed set Ω, see Section 2 below. An obvious choice of test functions are minimizers u and v_ε for $\lambda_{s,p}(\Omega)$ and $\lambda_{s,p}(\Phi(\varepsilon,\Omega))$, respectively. However, due to the fact that u is only of class C^s up to the boundary, we cannot obtain a boundary integral term directly from the divergence theorem. In particular, the integration by parts formula given in [23, Theorem 1.9] does not apply to general vector fields X which appear in (1.6). Hence, we need to replace u with $\tilde{\zeta}_k u$, where $\tilde{\zeta}_k$ is a cut-off function vanishing in a $\frac{1}{k}$-neighborhood of $\partial\Omega$. This leads to upper and lower estimates of $\lambda_{s,p}(\Phi(\varepsilon,\Omega))$ up to order $o(\varepsilon)$, where the first order term is given by an integral involving $(-\Delta)^s(\zeta_k u)$ and $\nabla(\zeta_k u)$. We refer the reader to Section 4 below for more precise information. A highly nontrivial task is now to pass to the limit as $k \to \infty$ in order to get boundary integrals involving $\psi := u/\delta^s$. This is most difficult part of the paper. We refer to Proposition 2.4 and Section 6 below for more details.

The paper is organized as follows. In Section 2, we provide preliminary results on convergence properties of integral functional, inner approximations of functions in $\mathcal{H}^s_0(\Omega)$ and on properties of minimizers of (1.1). In Section 3, we introduce notation related to domain deformations and related quantities.

In Section 4 we establish a preliminary variant of Theorem 1.1, which is given in Proposition 4.1. In this variant, the constant $\Gamma(1+s)^2$ in (1.6) is replaced by an implicitly given value which still depends on cut-off data. The proofs of the main results, as stated in this introduction, are then completed in Section 5. Finally, Section 6 is devoted to the proof of the main technical ingredient of the paper, which is given by Proposition 2.4.

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2. Notations and preliminary results

We start with an elementary but useful observation.

Lemma 2.1. *Let $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, and let $(v_k)_k$ be a sequence in $H^s(\mathbb{R}^N)$ with $v_k \to v$ in $H^s(\mathbb{R}^N)$. Then we have*

$$\lim_{k\to\infty} \int_{\mathbb{R}^{2N}} \frac{(v_k(x) - v_k(y))^2 \mu(x,y)}{|x-y|^{N+2s}} dxdy = \int_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))^2 \mu(x,y)}{|x-y|^{N+2s}} dxdy.$$

Proof. We have

$$\Big|\int_{\mathbb{R}^{2N}} \frac{(v_k(x)-v_k(y))^2-(v(x)-v(y))^2\mu(x,y)}{|x-y|^{N+2s}}dxdy\Big| \leq \|\mu\|_{L^\infty}\int_{\mathbb{R}^{2N}} \frac{|(v_k(x)-v_k(y))^2-(v(x)-v(y))^2|}{|x-y|^{N+2s}}dxdy,$$

where

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|(v_k(x)-v_k(y))^2-(v(x)-v(y))^2|}{|x-y|^{N+2s}}dxdy\\ &=\int_{\mathbb{R}^{2N}} \frac{|[(v_k(x)-v(x))-(v_k(y)-v(y))][(v_k(x)+v(x))-(v_k(y)+v(y))]|}{|x-y|^{N+2s}}dxdy\\ &\leq \frac{2}{b_{N,s}}[v_k-v]_{H^s(\mathbb{R}^N)}[v_k+v]_{H^s(\mathbb{R}^N)} \to 0 \qquad \text{as } k\to\infty.\end{aligned}$$

□

Throughout the remainder of this paper, we fix $\rho \in C_c^\infty(-2,2)$, with $0\leq\rho\leq 1$, $\rho\equiv 1$ on $(-1,1)$, and we define

$$\zeta(t)=1-\rho(t),\quad \rho_k(t)=\rho(kt) \quad\text{and}\quad \zeta_k(t)=1-\rho_k(t) \qquad \text{for } t\in\mathbb{R},\ k\in\mathbb{N}. \tag{2.1}$$

Lemma 2.2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain and let $u\in\mathcal{H}_0^s(\Omega)$. Moreover, for $k\in\mathbb{N}$, let $u_k\in\mathcal{H}_0^s(\Omega)$ denote inner approximations of u defined by $u_k(x)=u(x)\zeta_k(\delta(x))$ for $x\in\mathbb{R}^N$. Then we have*

$$u_k\to u \qquad \text{in } H^s(\mathbb{R}^N).$$

Proof. In the following, we let $\beta_k:=\rho_k\circ\delta:\mathbb{R}^N\to\mathbb{R}$ for $k\in\mathbb{N}$. Moreover, the letter $C>0$ stands for various constants independent of k. Clearly, it suffices to show that

$$u\beta_k\in\mathcal{H}_0^s(\Omega) \text{ for } k \text{ sufficiently large} \qquad\text{and}\qquad [u\beta_k]_{H^s(\mathbb{R}^N)}\to 0 \quad \text{as } k\to\infty. \tag{2.2}$$

For $\varepsilon>0$, we put $A_\varepsilon=\{x\in\Omega\,:\,\delta(x)<\varepsilon\}$. Since $u\beta_k$ vanishes in $\mathbb{R}^N\setminus A_{\frac{2}{k}}$, $0\leq\beta_k\leq 1$ on $\mathbb{R}^N$ and $|\beta_k(x)-\beta_k(y)|\leq C\min\{k|x-y|,1\}$ for $x,y\in\mathbb{R}^N$, we observe that

$$\begin{aligned}&\frac{1}{b_{N,s}}[\beta_k u]^2_{H^s(\mathbb{R}^N)}=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(x)\beta_k(x)-u(y)\beta_k(y)]^2}{|x-y|^{N+2s}}dydx\\ &=\frac{1}{2}\int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}}\frac{[u(x)\beta_k(x)-u(y)\beta_k(y)]^2}{|x-y|^{N+2s}}\,dydx+\int_{A_{\frac{2}{k}}}u(x)^2\beta_k(x)^2\int_{\mathbb{R}^N\setminus A_{\frac{4}{k}}}|x-y|^{-N-2s}\,dydx\\ &\leq\frac{1}{2}\int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}}\frac{\big[u(x)\big(\beta_k(x)-\beta_k(y)\big)+\beta_k(y)\big(u(x)-u(y)\big)\big]^2}{|x-y|^{N+2s}}\,dydx\\ &\quad+C\int_{A_{\frac{2}{k}}}u(x)^2\operatorname{dist}(x,\mathbb{R}^N\setminus A_{\frac{4}{k}})^{-2s}dx \qquad (2.3)\\ &\leq\int_{A_{\frac{4}{k}}}u^2(x)\int_{A_{\frac{4}{k}}}\frac{(\beta_k(x)-\beta_k(y))^2}{|x-y|^{N+2s}}\,dydx+\int_{A_{\frac{4}{k}}}\int_{A_{\frac{4}{k}}}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}dydx\\ &\quad+C\int_{A_{\frac{2}{k}}}u(x)^2\delta^{-2s}(x)dx\end{aligned}$$

$$\leq Ck^2 \int_{A_{\frac{4}{k}}} u^2(x) \int_{B_{\frac{1}{k}}(x)} |x-y|^{2-2s-N} dydx + C \int_{A_{\frac{4}{k}}} u^2(x) \int_{\mathbb{R}^N \setminus B_{\frac{1}{k}}(x)} |x-y|^{-N-2s} dydx$$

$$+ \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx + C \int_{A_{\frac{2}{k}}} u(x)^2 \delta^{-2s}(x) dx \tag{2.4}$$

$$\leq Ck^{2s} \int_{A_{\frac{4}{k}}} u^2(x) dx + \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx + C \int_{A_{\frac{2}{k}}} u(x)^2 \delta^{-2s}(x) dx$$

$$\leq C \int_{A_{\frac{4}{k}}} u^2(x) \delta^{-2s}(x) dx + \int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx. \tag{2.5}$$

Now, since Ω has a Lipschitz boundary, using $\int_{\mathbb{R}^N \setminus \Omega} |x-y|^{-N-2s}\, dy \sim \delta^{-2s}(x)$ see e.g [3], we get

$$\int_\Omega u^2(x) \delta^{-2s}(x) dx \leq C \int_\Omega u^2(x) \int_{\mathbb{R}^N \setminus \Omega} |x-y|^{-N-2s}\, dydx \leq C[u]^2_{H^s(\mathbb{R}^N)},$$

and therefore

$$\int_{A_{\frac{4}{k}}} u^2(x) \delta^{-2s}(x) dx \to 0 \qquad \text{as } k \to \infty. \tag{2.6}$$

Moreover, since also

$$\int_\Omega \int_\Omega \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx \leq \frac{2}{b_{N,s}} [u]^2_{H^s(\mathbb{R}^N)},$$

we have

$$\int_{A_{\frac{4}{k}}} \int_{A_{\frac{4}{k}}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dydx \to 0 \qquad \text{as } k \to \infty. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we obtain (2.2), as required. □

From now on, we fix a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^N$, and we let

$$C^s_0(\overline{\Omega}) = \left\{ w \in C^s(\overline{\Omega}) : w = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \right\}.$$

We recall the following regularity properties for (positive) minimizers for $\lambda_{s,p}(\Omega)$.

Lemma 2.3. *Let $u \in \mathcal{H}^s_0(\Omega)$ be a positive minimizer for $\lambda_{s,p}(\Omega)$. Then $u \in C^\infty(\Omega) \cap C^s_0(\overline{\Omega})$. Moreover, $\psi := \frac{u}{\delta^s} \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, and there exists a constant $c = c(N,s,\Omega,\alpha,p) > 0$ such that*

$$\|\psi\|_{C^\alpha(\overline{\Omega})} \leq c \tag{2.8}$$

and

$$|\nabla \psi(x)| \leq c\delta^{\alpha-1}(x) \qquad \text{for all } x \in \Omega. \tag{2.9}$$

Proof. By standard arguments in the calculus of variations, u is a weak solution of (1.3). By [24, Proposition 1.3] we have that $u \in L^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$ follows by interior regularity theory (see e.g. [21]) and the fact that the function $t \mapsto t^{p-1}$ is of class C^∞ on $(0,\infty)$. Moreover, the regularity up to the boundary $u \in C^s_0(\overline{\Omega})$ is proved in [22], where also the C^α-bound for the function $\psi := \frac{u}{\delta^s}$ is established for some $\alpha > 0$. Finally, (2.9) is proved in [9]. □

The computation of one-sided shape derivatives as given in Theorem 1.1 will be carried out in Section 4, and it requires the following key technical proposition. Since its proof is long and quite involved, we postpone the proof to Section 6 below.

Proposition 2.4. *Let $X \in C^0(\overline{\Omega}, \mathbb{R}^N)$, and let $u \in C^s_0(\overline{\Omega}) \cap C^1(\Omega)$ be a function such that $\psi := \frac{u}{\delta^s} : \Omega \to \mathbb{R}$ satisfies* (2.8) *and* (2.9). *Moreover, put $U_k := u[\zeta_k \circ \delta] \in C^{1,1}_c(\Omega)$. Then*

$$\lim_{k\to\infty} \int_\Omega \nabla U_k \cdot X\Big(u(-\Delta)^s[\zeta_k \circ \delta] - I(u, \zeta_k \circ \delta)\Big)\, dx = -\kappa_s \int_{\partial\Omega} \psi^2 X \cdot \nu \, dx,$$

where

$$\kappa_s := -\int_{\mathbb{R}} h'(r)(-\Delta)^s h(r)\, dr \qquad \text{with} \quad h(r) := r_+^s \zeta(r) = \max(r,0)^s \zeta(r), \tag{2.10}$$

and where we use the notation

$$I(u,v)(x) := b_{N,s} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dy \tag{2.11}$$

for $u \in C^s_c(\mathbb{R}^N)$, $v \in C^{0,1}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

Remark 2.5. *The minus sign in the definition of the constant κ_s in* (2.10) *might appear a bit strange at first glance. We shall see later that, defined in this way, κ_s has a positive value. A priori it is not clear that the value of κ_s does not depend on the particular choice of the function ζ. This follows a posteriori once we have established in Proposition 4.1 below that this constant appears in Theorem 1.1. This will then allow us to show that $\kappa_s = \frac{\Gamma(1+s)^2}{2}$ by applying the resulting shape derivative formula to a one-parameter family of concentric balls, see Section 5 below. A more direct, but quite lengthy computation of κ_s is possible via the logarithmic Laplacian, which has been introduced in [4].*

3. Domain perturbation and the associated variational problem

We fix a map $\Phi : (-1,1) \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.5). Moreover, for $\varepsilon \in (-1,1)$, we write $\Phi_\varepsilon(x) = \Phi(\varepsilon, x)$. As in the introduction, we then define $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$. In order to study the dependence of $\lambda_{s,p}(\Omega_\varepsilon)$ on ε, it is convenient to pull back the problem on the fixed domain Ω via a change of variable. For this we let $\mathrm{Jac}_{\Phi_\varepsilon}$ denote the Jacobian determinant of the map $\Phi_\varepsilon \in C^{1,1}(\mathbb{R}^N)$, and we define the kernels

$$K_\varepsilon(x,y) := b_{N,s} \frac{\mathrm{Jac}_{\Phi_\varepsilon}(x)\mathrm{Jac}_{\Phi_\varepsilon}(y)}{|\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|^{N+2s}} \qquad \text{and} \qquad K_0(x,y) = b_{N,s}\frac{1}{|x-y|^{N+2s}}. \tag{3.1}$$

Then (1.5) gives rise to the well known expansions

$$\mathrm{Jac}_{\Phi_\varepsilon}(x) = 1 + \varepsilon \mathrm{div} X(x) + O(\varepsilon^2), \qquad \partial_\varepsilon \mathrm{Jac}_{\Phi_\varepsilon}(x) = \mathrm{div} X(x) + O(\varepsilon) \tag{3.2}$$

uniformly in $x \in \mathbb{R}^N$, where $X := \partial_\varepsilon\big|_{\varepsilon=0} \Phi_\varepsilon \in C^{0,1}(\mathbb{R}^N; \mathbb{R}^N)$ and therefore $\mathrm{div} X$ is a.e. defined on $\mathbb{R}^N$. From (1.5), we also get

$$|\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|^{-N-2s} = |x-y|^{-N-2s}\left(1 + 2\varepsilon \frac{x-y}{|x-y|} \cdot P_X(x,y) + O(\varepsilon^2)\right)^{-\frac{N+2s}{2}},$$

and

$$\partial_\varepsilon |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)|^{-N-2s} = |x-y|^{-N-2s}\left((N+2s)\frac{x-y}{|x-y|} \cdot P_X(x,y) + O(\varepsilon)\right),$$

uniformly in $x, y \in \mathbb{R}^N$, $x \neq y$ with

$$P_X \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N), \qquad P_X(x,y) = \frac{X(x)-X(y)}{|x-y|}.$$

Moreover by (3.2) and the fact that $\partial_\varepsilon \Phi_\varepsilon,\ X \in C^{0,1}(\mathbb{R}^N)$, we have that

$$K_\varepsilon(x,y) = K_0(x,y) + \varepsilon \partial_\varepsilon\Big|_{\varepsilon=0} K_\varepsilon(x,y) + O(\varepsilon^2) K_0(x,y), \tag{3.3}$$

and

$$\partial_\varepsilon K_\varepsilon(x,y) = \partial_\varepsilon\Big|_{\varepsilon=0} K_\varepsilon(x,y) + O(\varepsilon) K_0(x,y), \tag{3.4}$$

uniformly in $x, y \in \mathbb{R}^N$, $x \neq y$, where

$$\partial_\varepsilon\Big|_{\varepsilon=0} K_\varepsilon(x,y) = \Big[(N+2s)\frac{x-y}{|x-y|} \cdot P_X(x,y) - (\mathrm{div} X(x) + \mathrm{div} X(y))\Big] K_0(x,y). \tag{3.5}$$

In particular, it follows from (3.3) and (3.5) that there exist $\varepsilon_0, C > 0$ such that

$$\frac{1}{C} K_0(x,y) \le K_\varepsilon(x,y) \le C K_0(x,y) \qquad \text{for all } x, y \in \mathbb{R}^N,\ x \neq y \text{ and } \varepsilon \in (-\varepsilon_0, \varepsilon_0). \tag{3.6}$$

For $v \in H^s(\mathbb{R}^N)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we now define

$$\mathcal{V}_v(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 K_\varepsilon(x,y) dx dy. \tag{3.7}$$

Then, by (1.1), (1.5) and a change of variable, we have the following variational characterization for $\lambda_{s,p}(\Omega_\varepsilon)$:

$$\begin{aligned} \lambda_{s,p}^\varepsilon := \lambda_{s,p}(\Omega_\varepsilon) &= \inf\left\{ [u]^2_{H^s(\mathbb{R}^N)} \ :\ u \in \mathcal{H}^s_0(\Omega_\varepsilon), \quad \int_{\Omega_\varepsilon} |u|^p \, dx = 1 \right\} \\ &= \inf\left\{ \mathcal{V}_v(\varepsilon) \ :\ v \in \mathcal{H}^s_0(\Omega), \quad \int_\Omega |v|^p \mathrm{Jac}_{\Phi_\varepsilon}(x) \, dx = 1 \right\} \quad \text{for } \varepsilon \in (-\varepsilon_0, \varepsilon_0). \end{aligned} \tag{3.8}$$

As mentioned earlier, we prefer to use (3.8) from now on where the underlying domain is fixed and the integral terms depend on ε instead. It follows from (3.3) and (3.4) that, for given $v \in H^s(\mathbb{R}^N)$, the function $\mathcal{V}_v : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is of class C^1 with

$$\mathcal{V}_v'(0) = \frac{1}{2} \int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 \partial_\varepsilon\big|_{\varepsilon=0} K_\varepsilon(x,y) dx dy, \tag{3.9}$$

where $\partial_\varepsilon\big|_{\varepsilon=0} K_\varepsilon(x,y)$ is given in (3.5),

$$|\mathcal{V}_v'(0)| \le C [v]^2_{H^s(\mathbb{R}^N)} \qquad \text{with a constant } C > 0 \tag{3.10}$$

and we have the expansions

$$\mathcal{V}_v(\varepsilon) = \mathcal{V}_v(0) + \varepsilon \mathcal{V}_v'(0) + O(\varepsilon^2)[v]^2_{H^s(\mathbb{R}^N)}, \qquad \mathcal{V}_v'(\varepsilon) = \mathcal{V}_v'(0) + O(\varepsilon)[v]^2_{H^s(\mathbb{R}^N)} \tag{3.11}$$

with $O(\varepsilon)$, $O(\varepsilon^2)$ independent of v. From (3.2), (3.6) and the variational characterization (3.8), it is easy to see that

$$\frac{1}{C} \le \lambda_{s,p}^\varepsilon \le C \qquad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0) \text{ with some constant } C > 0.$$

Using this and (3.2), (3.6) once more, we can show that

$$\frac{1}{C} \le \|v_\varepsilon\|_{L^p(\Omega)} \le C \qquad \text{and} \qquad \frac{1}{C} \le \|v_\varepsilon\|_{H^s(\mathbb{R}^N)} \le C. \tag{3.12}$$

for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and every minimizer $v_\varepsilon \in \mathcal{H}^s_0(\Omega)$ for (3.8) with a constant $C > 0$.

The following lemma is essentially a corollary of Lemma 2.1.

Lemma 3.1. *Let $(v_k)_k$ be a sequence in $H^s(\mathbb{R}^N)$ with $v_k \to v$ in $H^s(\mathbb{R}^N)$. Then we have*

$$\lim_{k\to\infty} \mathcal{V}_{v_k}(0) = \mathcal{V}_v(0) \qquad \textit{and} \qquad \lim_{k\to\infty} \mathcal{V}'_{v_k}(0) = \mathcal{V}'_v(0).$$

Proof. The first limit is trivial since $\mathcal{V}_v(0) = [v]^2_{H^s(\mathbb{R}^N)}$ for $v \in H^s(\mathbb{R}^N)$. The second limit follows from Lemma 2.1, (3.5) and (3.9) by noting that $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ for the function

$$\mu(x,y) = (N+2s)\frac{x-y}{|x-y|} \cdot P_X(x,y) - (\mathrm{div}X(x) + \mathrm{div}X(y)).$$

□

4. One-sided Shape derivative computations

We keep using the notation of the previous sections, and we recall in particular the variational characterization of $\lambda^\varepsilon_{s,p}$ given in (3.8). The aim of this section is to prove the following result.

Proposition 4.1. *We have*

$$\partial^+_\varepsilon\Big|_{\varepsilon=0}\lambda^\varepsilon_{s,p} = \inf\left\{2\kappa_s\int_{\partial\Omega}(u/\delta^s)^2 X\cdot\nu\,dx,\, u\in\mathcal{H}\right\},$$

where $\mathcal{H}$ is the set of positive minimizers for $\lambda^0_{s,p} := \lambda_{s,p}(\Omega)$, $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$ and κ_s is given by (2.10).

The proof of Proposition 4.1 requires several preliminary results. We start with a formula for the derivative of the function given by (3.7).

Lemma 4.2. *Let $U \in C^{1,1}_c(\Omega)$. Then*

$$\mathcal{V}'_U(0) = -2\int_{\mathbb{R}^N}\nabla U\cdot X(-\Delta)^s U dx. \tag{4.1}$$

Proof. By (3.5) and (3.11),

$$\begin{aligned}\mathcal{V}'_U(0) =& \frac{-(N+2s)b_{N,s}}{2}\int_{\mathbb{R}^{2N}}(U(x)-U(y))^2\frac{(x-y)\cdot(X(x)-X(y))}{|x-y|^{N+2s+2}}dxdy\\ &+\frac{1}{2}\int_{\mathbb{R}^{2N}}(U(x)-U(y))^2K_0(x,y)(\mathrm{div}X(x)+\mathrm{div}X(y))dxdy.\end{aligned}$$

Using that $\nabla|z|^{-N-2s} = -(N+2s)z|z|^{-N-2s-2}$ and the divergence theorem, we obtain

$$\begin{aligned}\mathcal{V}'_U(0) =& \frac{-(N+2s)b_{N,s}}{2}\lim_{\mu\to 0}\int_{|x-y|>\mu}(U(x)-U(y))^2\frac{(x-y)\cdot(X(x)-X(y))}{|x-y|^{N+2s+2}}dxdy\\ &+\frac{1}{2}\int_{\mathbb{R}^{2N}}(U(x)-U(y))^2K_0(x,y)(\mathrm{div}X(x)+\mathrm{div}X(y))dxdy\\ =& -\frac{1}{2}\lim_{\mu\to 0}\int_{|x-y|>\mu}(U(x)-U(y))^2K_0(x,y)(\mathrm{div}X(x)+\mathrm{div}X(y))dxdy\\ &-\lim_{\mu\to 0}\int_{|x-y|>\mu}(U(x)-U(y))(\nabla U(x)\cdot X(x)+\nabla U(y)\cdot X(y))K_0(x,y)dxdy\\ &+\lim_{\mu\to 0}\int_{\mathbb{R}^N}\int_{|x-y|=\mu}(U(x)-U(y))^2(X(x)-X(y))\cdot\frac{y-x}{|x-y|}K_0(x,y)\,d\sigma(y)\,dx\\ &+\frac{1}{2}\int_{\mathbb{R}^{2N}}(U(x)-U(y))^2K_0(x,y)(\mathrm{div}X(x)+\mathrm{div}X(y))dxdy.\end{aligned}$$

We thus get

$$
\begin{aligned}
&\mathcal{V}_U'(0) = -\lim_{\mu\to 0}\int_{|x-y|>\mu}(U(x)-U(y))(\nabla U(x)\cdot X(x)+\nabla U(y)\cdot X(y))K_0(x,y)dxdy\\
&+\lim_{\mu\to 0}\int_{\mathbb{R}^N}\int_{|x-y|=\mu}(U(x)-U(y))^2(X(x)-X(y))\cdot\frac{y-x}{|x-y|}K_0(x,y)\,d\sigma_\mu(y)dx\\
&=-2\lim_{\mu\to 0}\int_{\mathbb{R}^N}(-\Delta)^s_\mu U(x)\nabla U(x)\cdot X(x)\,dx\\
&+b_{N,s}\lim_{\mu\to 0}\mu^{-N-1-2s}\int_{\mathbb{R}^N}\int_{|x-y|=\mu}(U(x)-U(y))^2(X(x)-X(y))\cdot(y-x)\,d\sigma(y)\,dx \qquad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
(-\Delta)^s_\mu U_k(x) &= b_{N,s}\int_{|x-y|\geq\mu}\frac{U_k(x)-U_k(y)}{|x-y|^{N+2s}}\,dy\\
&=\frac{b_{N,s}}{2}\int_{|y|>\mu}\frac{2U_k(x)-U_k(x-y)-U_k(x+y)}{|y|^{N+2s}}\,dy. \qquad (4.3)
\end{aligned}
$$

Using that $U,X\in C^{0,1}(\mathbb{R}^N)$, we find that

$$
\Big|\int_{|x-y|=\mu}(U(x)-U(y))^2(X(x)-X(y))\cdot(y-x)\,d\sigma(y)\Big|\leq C\mu^{N+3}
$$

with a constant $C>0$ independent of x. Moreover, since U is compactly supported in Ω, setting $N_\mu(\Omega):=\{x\in\mathbb{R}^N\,:\,\delta(x)>-\mu\}$ for $\mu>0$, we conclude that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{|x-y|=\mu}(U(x)-U(y))^2(X(x)-X(y))\cdot(y-x)\,d\sigma(y)\,dx\\
&\quad=\int_{N_{2\mu}(\Omega)}\int_{|x-y|=\mu}(U(x)-U(y))^2(X(x)-X(y))\cdot(y-x)\,d\sigma(y)=O(\mu^{N+3})\quad\text{as }\mu\to 0.
\end{aligned}
$$

Going back to (4.2), we thus see that

$$
\mathcal{V}_U'(0)=-2\lim_{\mu\to 0}\int_{\mathbb{R}^N}(-\Delta)^s_\mu U(x)\nabla U(x)\cdot X(x). \qquad (4.4)
$$

On the other hand, since $U\in C^{1,1}_c(\Omega)$, we have that

$$
\sup_{\mu\in(0,1)}\|(-\Delta)^s_\mu U\|_{L^\infty(\mathbb{R}^N)}<\infty\quad\text{and}\quad\nabla U\cdot X\in C^{0,1}_c(\Omega),
$$

so we can apply the dominated convergence theorem to obtain

$$
\lim_{\mu\to 0}\int_{\mathbb{R}^N}(-\Delta)^s_\mu U(x)\nabla U(x)\cdot X(x)\,dx=\int_{\mathbb{R}^N}(-\Delta)^s U(x)\nabla U(x)\cdot X(x)\,dx.
$$

Combining this with (4.4), we obtain the claim. □

We cannot apply Lemma 4.2 directly to minimizers $u\in\mathcal{H}^s_0(\Omega)$ of $\lambda_{s,p}(\Omega)$ since these are not contained in $C^{1,1}_c(\Omega)$. The aim is therefore to apply Lemma 4.2 to $U_k:=u[\zeta_k\circ\delta]\in C^{1,1}_c(\Omega)$ with ζ_k given in (2.1) and to use Proposition 2.4. This leads to the following derivative formula which plays a key role in the proof of Proposition 4.1.

Lemma 4.3. *Let $u\in\mathcal{H}^s_0(\Omega)$ be a positive minimizer for $\lambda_{s,p}(\Omega)$. Then we have*

$$
\mathcal{V}_u'(0)=\frac{2\lambda_{s,p}(\Omega)}{p}\int_\Omega u^p\,div\,X\,dx+2\kappa_s\int_{\partial\Omega}(u/\delta^s)^2X\cdot\nu\,dx.
$$

Proof. To simplify notation, we put $\varphi_k := \zeta_k \circ \delta$ for $k \in \mathbb{N}$. By Lemma 2.3 and since Ω is of class $C^{1,1}$, we have $U_k := u\varphi_k \in C^{1,1}_c(\Omega) \subset \mathcal{H}^s_0(\Omega)$ for $k \in \mathbb{N}$, and $U_k \to u$ in $H^s(\mathbb{R}^N)$ by Lemma 2.2. Consequently, $\mathcal{V}'_u(0) = \lim\limits_{k\to\infty} \mathcal{V}'_{U_k}(0)$ by Corollary 3.1, so it remains to show that

$$\lim_{k\to\infty} \mathcal{V}'_{U_k}(0) = \frac{2\lambda_{s,p}(\Omega)}{p} \int_\Omega u^p \mathrm{div} X\,dx + 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X\cdot\nu\,dx. \tag{4.5}$$

Applying Lemma 4.2 to U_k, we find that

$$\mathcal{V}'_{U_k}(0) = -2\int_{\mathbb{R}^N} \nabla U_k \cdot X(-\Delta)^s U_k dx \qquad \text{for } k \in \mathbb{N}.$$

By the standard product rule for the fractional Laplacian, we have $(-\Delta)^s U_k = u(-\Delta)^s\varphi_k + \varphi_k(-\Delta)^s u - I(u,\varphi_k)$ with $I(u,\varphi_k)$ given by (2.11). We thus obtain

$$\begin{aligned}\mathcal{V}'_{U_k}(0) &= -2\int_{\mathbb{R}^N} \nabla U_k \cdot X\varphi_k(-\Delta)^s u\,dx - 2\int_{\mathbb{R}^N} [\nabla U_k\cdot X] u(-\Delta)^s\varphi_k\,dx \\ &\quad + 2\int_{\mathbb{R}^N} \nabla U_k \cdot X I(u,\varphi_k)\,dx \\ &= -2\lambda_{s,p}(\Omega)\int_\Omega \nabla U_k\cdot X\varphi_k u^{p-1}\,dx - 2\int_{\mathbb{R}^N} \nabla U_k\cdot X\Big(u(-\Delta)^s\varphi_k - I(u,\varphi_k)\Big)\,dx,\end{aligned} \tag{4.6}$$

where we used that $(-\Delta)^s u = \lambda_{s,p}(\Omega) u^{p-1}$ in Ω. Consequently, Proposition 2.4 yields that

$$\lim_{k\to\infty} \mathcal{V}'_{U_k}(0) = -2\lambda_{s,p}(\Omega) \lim_{k\to\infty} \int_\Omega \nabla U_k\cdot X\varphi_k u^{p-1}\,dx + 2\kappa_s \int_{\partial\Omega} \psi^2 X\cdot\nu\,dx. \tag{4.7}$$

Moreover, integration by parts, we obtain, for $k \in \mathbb{N}$,

$$\begin{aligned}\int_\Omega \nabla U_k\cdot X\varphi_k u^{p-1}\,dx &= \frac{1}{p}\int_\Omega \nabla u^p\cdot X\varphi_k^2\,dx + \int_\Omega \nabla\varphi_k\cdot X\varphi_k u^p\,dx \\ &= -\frac{1}{p}\int_\Omega u^p \mathrm{div} X\varphi_k^2\,dx - \frac{2}{p}\int_\Omega u^p\varphi_k X\cdot\nabla\varphi_k\,dx + \int_\Omega \nabla\varphi_k\cdot X\varphi_k u^p\,dx.\end{aligned} \tag{4.8}$$

Since $u^p \in C^s_0(\overline{\Omega})$ by Lemma 2.3, it is easy to see from the definition of φ_k that the last two terms in (4.8) tend to zero as $k\to\infty$, whereas

$$\lim_{k\to\infty}\int_\Omega u^p \mathrm{div} X\varphi_k^2\,dx = \int_\Omega u^p \mathrm{div} X\,dx.$$

Hence

$$\lim_{k\to\infty}\int_\Omega \nabla U_k\cdot X\varphi_k u^{p-1}\,dx = -\frac{1}{p}\int_\Omega u^p \mathrm{div} X\,dx.$$

Plugging this into (4.7), we obtain (4.5), as required. □

Our next lemma provides an upper estimate for $\partial^+_\varepsilon\big|_{\varepsilon=0}\lambda^\varepsilon_{s,p}$.

Lemma 4.4. *Let $u \in \mathcal{H}^s_0(\Omega)$ be a positive minimizer for $\lambda^0_{s,p} = \lambda_{s,p}(\Omega)$. Then*

$$\limsup_{\varepsilon\to 0^+} \frac{\lambda^\varepsilon_{s,p} - \lambda^0_{s,p}}{\varepsilon} \le 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X\cdot\nu\,dx. \tag{4.9}$$

Proof. For $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we define

$$j(\varepsilon) := \frac{\mathcal{V}_u(\varepsilon)}{\tau(\varepsilon)} \quad \text{for } k\in\mathbb{N} \text{ with } \quad \tau(\varepsilon) := \left(\int_\Omega |u|^p \mathrm{Jac}_{\Phi_\varepsilon}(x)\,dx\right)^{2/p}.$$

By (3.8), we then have $\lambda^{\varepsilon}_{s,p} \le j(\varepsilon)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover,

$$\tau(0) = \|u\|^{2/p}_{L^p(\Omega)} = 1, \quad \mathcal{V}_u(0) = [u]^2_{H^s(\mathbb{R}^N)} = \lambda_{s,p}(\Omega) \quad \text{and} \quad j(0) = \frac{\mathcal{V}_u(0)}{\tau(0)} = \lambda^0_{s,p},$$

which implies that

$$\partial^+_\varepsilon\Big|_{\varepsilon=0} \lambda^\varepsilon_{s,p} \le j'(0) = 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx,$$

by Lemma 4.3, as claimed. □

Next, we shall prove a lower estimate for $\partial^+_\varepsilon\Big|_{\varepsilon=0} \lambda^\varepsilon_{s,p}$.

Lemma 4.5. *We have*

$$\liminf_{\varepsilon \searrow 0^+} \frac{\lambda^\varepsilon_{s,p} - \lambda^0_{s,p}}{\varepsilon} \ge \inf \left\{ 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X \cdot \nu \, dx \,:\, u \in \mathcal{H} \right\}.$$

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive numbers converging to zero such that

$$\lim_{n\to\infty} \frac{\lambda^{\varepsilon_n}_{s,p} - \lambda^0_{s,p}}{\varepsilon} = \liminf_{\varepsilon \searrow 0^+} \frac{\lambda^\varepsilon_{s,p} - \lambda^0_{s,p}}{\varepsilon}. \tag{4.10}$$

For $n \in \mathbb{N}$, we let v_{ε_n} be a positive minimizer corresponding to the variational characterization of $\lambda^{\varepsilon_n}_{s,p}$ given in (3.8), i.e. we have

$$\lambda^{\varepsilon_n}_{s,p} = \mathcal{V}_{v_{\varepsilon_n}}(\varepsilon_n) \qquad \text{and} \qquad \int_\Omega v^p_{\varepsilon_n} \mathrm{Jac}_{\Phi_{\varepsilon_n}} dx = 1. \tag{4.11}$$

Since v_{ε_n} remains bounded in $\mathcal{H}^s_0(\Omega)$ by (3.12), we may pass to a sub-sequence such that $v_{\varepsilon_n} \rightharpoonup u$ in $\mathcal{H}^s_0(\Omega)$ for some $u \in \mathcal{H}^s_0(\Omega)$. Moreover,$v_{\varepsilon_n} \to u$ in $L^p(\Omega)$ as $n \to \infty$ since the embedding $\mathcal{H}^s_0(\Omega) \to L^p(\Omega)$ is compact. In the following, to keep the notation simple, we write ε in place of ε_n. By (3.11) and (4.11), we have

$$\mathcal{V}_{v_\varepsilon}(0) = \mathcal{V}_{v_\varepsilon}(\varepsilon) - \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + O(\varepsilon^2)[v_\varepsilon]^2_{H^s(\mathbb{R}^N)} = \lambda^\varepsilon_{s,p} - \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + O(\varepsilon^2) \tag{4.12}$$

and therefore

$$\mathcal{V}_u(0) = [u]^2_{H^s(\mathbb{R}^N)} \le \liminf_{\varepsilon\to 0} [v_\varepsilon]^2_{H^s(\mathbb{R}^N)} = \liminf_{\varepsilon\to 0} \mathcal{V}_{v_\varepsilon}(0) \le \limsup_{\varepsilon\to 0} \lambda^\varepsilon_{s,p} \le \lambda^0_{s,p}, \tag{4.13}$$

where the last inequality follows from Lemma 4.4. In view of (3.2) and the strong convergence $v_\varepsilon \to u$ in $L^p(\Omega)$, we see that

$$1 = \int_\Omega v^p_\varepsilon \mathrm{Jac}_{\Phi_\varepsilon} dx = \int_\Omega v^p_\varepsilon (1 + \varepsilon \mathrm{div} X) dx + O(\varepsilon^2) = \int_\Omega u^p \, dx + o(1) \tag{4.14}$$

as $\varepsilon \to 0$, and hence $\|u\|_{L^p(\Omega)} = 1$. Combining this with (4.13), we see that $u \in \mathcal{H}$ is a minimizer for $\lambda^0_{s,p}$, and that equality must hold in all inequalities of (4.13). From this we deduce that

$$v_\varepsilon \to u \text{ strongly in } H^s(\mathbb{R}^N). \tag{4.15}$$

Now (4.12) and the variational characterization of $\lambda^0_{s,p}$ imply that

$$\lambda^0_{s,p} \left(\int_\Omega v^p_\varepsilon dx \right)^{2/p} \le \mathcal{V}_{v_\varepsilon}(0) = \lambda_{s,p}(\Omega_\varepsilon) - \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + O(\varepsilon^2) \tag{4.16}$$

whereas by (4.14) we have

$$\int_\Omega v^p_\varepsilon \, dx = 1 - \varepsilon \int_\Omega v^p_\varepsilon \mathrm{div} X dx + O(\varepsilon^2) = 1 - \varepsilon \int_\Omega u^p \mathrm{div} X dx + o(\varepsilon)$$

and therefore

$$\left(\int_\Omega v_\varepsilon^p dx\right)^{2/p} = 1 - \frac{2\varepsilon}{p}\int_\Omega u^p \mathrm{div} X dx + o(\varepsilon). \tag{4.17}$$

Plugging this into (4.16), we get the inequality

$$\lambda_{s,p}^\varepsilon \geq \Big(1 - \frac{2\varepsilon}{p}\int_\Omega u^p \mathrm{div} X dx\Big)\lambda_{s,p}^0 + \varepsilon \mathcal{V}'_{v_\varepsilon}(0) + o(\varepsilon).$$

Since, moreover, $\mathcal{V}'_{v_\varepsilon}(0) \to \mathcal{V}'_u(0)$ as $\varepsilon \to 0$ by Corollary 3.1 and (4.15), it follows that

$$\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0 \geq \varepsilon\Big(\mathcal{V}'_u(0) - \frac{2\lambda_{s,p}^0}{p}\int_\Omega u^p \mathrm{div} X dx\Big) + o(\varepsilon)$$

and therefore

$$\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0 \geq 2\varepsilon\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X\cdot\nu\, dx + o(\varepsilon)$$

by Lemma 4.3. We thus conclude that

$$\lim_{\varepsilon\to 0^+} \frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon} \geq 2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X\cdot\nu\, dx.$$

Taking the infinimum over $u \in \mathcal{H}$ in the RHS of this inequality and using (4.10), we get the result. □

Proof of Proposition 4.1 (completed). Proposition 4.1 is a consequence of Lemma 4.4 and Lemma 4.5. Indeed, let

$$A_{s,p}(\Omega) := \inf\left\{2\kappa_s \int_{\partial\Omega} (u/\delta^s)^2 X\cdot\nu\, dx \;:\; u \in \mathcal{H}\right\}.$$

Thanks to (2.8) the infinimum $A_{s,p}(\Omega)$ is attained. Finally by Lemma 4.4 and Lemma 4.5 we get

$$A_{s,p}(\Omega) \geq \partial_\varepsilon^+\Big|_{\varepsilon=0}\lambda_{s,p}^\varepsilon \geq \liminf_{\varepsilon\searrow 0}\frac{\lambda_{s,p}^\varepsilon - \lambda_{s,p}^0}{\varepsilon} \geq A_{s,p}(\Omega).$$

□

5. Proof of the main results

In this section we complete the proofs of the main results stated in the introduction.

Proof of Theorem 1.1 (completed). In view of Proposition 4.1, the proof of Theorem 1.1 is complete once we show that

$$2\kappa_s = \Gamma(1+s)^2, \tag{5.1}$$

where Γ is the usual Gamma function. In view of (2.10), the constant κ_s does not depend on N, p and Ω, we consider the case $N = p = 1$ and the family of diffeomorphisms Φ_ε on $\mathbb{R}^N$ given by $\Phi_\varepsilon(x) = (1+\varepsilon)x$, $\varepsilon \in (-1,1)$, so that $X := \partial_\varepsilon\big|_{\varepsilon=0}\Phi_\varepsilon$ is simply given by $X(x) = x$. Letting $\Omega_0 := (-1,1)$, we define $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0) = (-1-\varepsilon, 1+\varepsilon)$. Moreover, we consider $w_\varepsilon \in \mathcal{H}_0^s(\Omega_\varepsilon) \cap C_0^s([-1-\varepsilon, 1+\varepsilon])$ given by

$$w_\varepsilon(x) = \ell_s((1+\varepsilon)^2 - |x|^2)_+^s \qquad \text{with} \qquad \ell_s := \frac{2^{-2s}\Gamma(1/2)}{\Gamma(s+1/2)\Gamma(1+s)}. \tag{5.2}$$

It is well known that w_ε is the unique solution of the problem

$$(-\Delta)^s w_\varepsilon = 1 \quad \text{in } \Omega_\varepsilon, \qquad w_\varepsilon \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega_\varepsilon,$$

see e.g. [23] or [13]. Recalling (1.3), we thus deduce that $u_\varepsilon = \lambda_{s,1}(\Omega_\varepsilon) w_\varepsilon$ is the unique positive minimizer corresponding to (1.1) in the case $N = p = 1$, which implies that $\|u_\varepsilon\|_{L^1(\mathbb{R})} = 1$ and therefore

$$\lambda_{s,1}(\Omega_\varepsilon) = \|w_\varepsilon\|^{-1}_{L^1(\mathbb{R})} = (1+\varepsilon)^{-(2s+1)} \|w_0\|^{-1}_{L^1(\mathbb{R})}. \tag{5.3}$$

Moreover, by standard properties of the Gamma function,

$$\begin{aligned}\|w_0\|_{L^1(\mathbb{R})} &= \ell_s \int_{-1}^{1} (1-|x|^2)^s \, dx = 2\ell_s \int_0^1 (1-r^2)\, dr = \ell_s \int_0^1 t^{-1/2}(1-t)^s \, dt \\ &= \ell_s \frac{\Gamma(1/2)\Gamma(s+1)}{\Gamma(s+3/2)} = \ell_s \frac{\Gamma(1/2)\Gamma(s+1)}{(s+1/2)\Gamma(s+1/2)} = \frac{2^{2s}\, \ell_s^2\, \Gamma(s+1)^2}{s+1/2}.\end{aligned}$$

By differentiating (5.3), we get

$$\partial_\varepsilon\Big|_{\varepsilon=0} \lambda_{s,1}(\Omega_\varepsilon) = -\frac{2s+1}{\|w_0\|_{L^1(\mathbb{R})}}. \tag{5.4}$$

On the other hand, by Proposition 4.1 and the fact that u_0 is the unique positive minimizer for $\lambda_{s,1}$, we deduce that

$$\partial_\varepsilon^+\Big|_{\varepsilon=0} \lambda_{s,1}(\Omega_\varepsilon) = -2\kappa_s[(u_0/\delta^s)^2(1) + (u_0/\delta^s)^2(-1)] = 2^{2+2s}\kappa_s\, \ell_s^2\, \lambda_{s,1}(\Omega_0)^2 = -\frac{16\kappa_s\, \ell_s^2}{\|w_0\|^2_{L^1(\mathbb{R})}}.$$

We thus conclude that

$$2\kappa_s = \frac{(2s+1)\|w_0\|_{L^1(\mathbb{R})}}{2^{1+2s}\ell_s^2} = \Gamma(s+1)^2.$$

Thus, by Proposition 4.1, we get the result as stated in the theorem. □

Proof of Corollary 1.3. Let $h \in C^3(\partial\Omega)$, with $\int_{\partial\Omega} h\, dx = 0$. Then it is well known (see e.g. [10, Lemma 2.2]) that there exists a family of diffeomorphisms $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$, $\varepsilon \in (-1,1)$ satisfying (1.5) and having the following properties:

$$|\Phi_\varepsilon(\Omega)| = |\Omega| \text{ for } \varepsilon \in (-1,1), \text{ and } X := \partial_\varepsilon\Big|_{\varepsilon=0}\Phi_\varepsilon \text{ equals } h\nu \text{ on } \partial\Omega. \tag{5.5}$$

By assumption, there exists $\varepsilon_0 \in (0,1)$ with $\lambda_{s,p}(\Phi_\varepsilon(\Omega)) \geq \lambda_{s,p}(\Omega)$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Applying Theorem 1.1 and noting that $X \cdot \nu \equiv h$ on $\partial\Omega$ by (5.5), we get

$$\min\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 h\, dx, \quad u \in \mathcal{H}\right\} \geq 0.$$

By the same argument applied to $-h$, we get

$$\max\left\{\Gamma(1+s)^2 \int_{\partial\Omega} (u/\delta^s)^2 h\, dx, \quad u \in \mathcal{H}\right\} \leq 0. \tag{5.6}$$

We thus conclude that

$$\int_{\partial\Omega} (u/\delta^s)^2 h\, dx = 0 \qquad \text{for every } u \in \mathcal{H} \text{ and for all } h \in C^3(\partial\Omega), \text{ with } \int_{\partial\Omega} h\, dx = 0.$$

By a standard argument, this implies that u/δ^s is constant on $\partial\Omega$. Now, since u solves (1.3) and $p \in \{1\} \cup [2,\infty)$, we deduce from [13, Theorem 1.2] that Ω is a ball. □

Proof of Theorem 1.4. Consider the unit centered ball $B_1 = B_1(0)$. For $\tau \in (0,1)$ and $t \in (\tau - 1, 1-\tau)$, we define $B^t := B_\tau(te_1)$, where e_1 is the first coordinate direction. To prove Theorem 1.4, we can take advantage of the invariance under rotations of the problem and may restrict our attention to domains of the form $\Omega(t) = B_1 \setminus \overline{B^t}$. We define

$$\theta : (\tau - 1, 1-\tau) \to \mathbb{R}, \qquad \theta(t) := \lambda_{s,p}(\Omega(t)). \tag{5.7}$$

We claim that θ is differentiable and satisfies

$$\theta'(t) < 0 \qquad \text{for } t \in (0, 1-\tau). \tag{5.8}$$

For this we fix $t \in (\tau-1, 1-\tau)$ and a vector field $X : \mathbb{R}^N \to \mathbb{R}^N$ given by $X(x) = \rho(x)e_1$, where $\rho \in C_c^\infty(B_1)$ satisfies $\rho \equiv 1$ in a neighborhood of B^t. For $\varepsilon \in (-1,1)$, we then define $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ by $\Phi_\varepsilon(x) = x + \delta\varepsilon X(x)$, where $\delta > 0$ is chosen sufficiently small to guarantee that Φ_ε, $\varepsilon \in (-1,1)$ is a family of diffeomorphisms satisfying (1.5) and satisfying $\Phi_\varepsilon(B_1) = B_1$ for $\varepsilon \in (-1,1)$. Then, by construction, we have

$$\Phi_\varepsilon(\Omega(t)) = \Phi_\varepsilon\left(B_1 \setminus \overline{B^t}\right) = B_1 \setminus \overline{\Phi_\varepsilon(B^t)} = B_1 \setminus \overline{B^{t+\delta\varepsilon}} = \Omega(t + \delta\varepsilon). \tag{5.9}$$

Next we recall that, since $p \in \{1,2\}$, there exists a unique positive minimizer $u \in \mathcal{H}_0^s(\Omega(t))$ corresponding to the variational characterization (1.1) of $\lambda_{s,p}(\Omega(t))$. Hence, by Corollary 1.2, the map $\varepsilon \mapsto \lambda_{s,p}(\Phi_\varepsilon(\Omega(t)))$ is differentiable at $\varepsilon = 0$. In view of (5.9), we thus find that the map θ in (5.7) is differentiable at t, and

$$\theta'(t) = \frac{1}{\delta}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda_{s,p}(\Phi_\varepsilon(\Omega(t))) = \Gamma(1+s)^2\int_{\partial\Omega(t)}\left(\frac{u}{\delta^s}\right)^2 X\cdot\nu\,dx = \Gamma(1+s)^2\int_{\partial B(t)}\left(\frac{u}{\delta^s}\right)^2\nu_1\,dx \tag{5.10}$$

by (1.7). Here ν denotes the interior unit normal on $\partial\Omega(t)$ which coincides with the exterior unit normal to $B(t)$ on $\partial B(t)$, and we used that

$$X \equiv e_1 \quad \text{on } \partial B(t), \qquad X \equiv 0 \quad \text{on } \partial B_1 = \partial\Omega(t) \setminus \partial B(t)$$

to get the last equality in (5.10). Next, for fixed $t \in (0, 1-\tau)$, let H be the hyperplane defined by $H = \{x \in \mathbb{R}^N : x\cdot e_1 = t\}$, and let $\Theta = \{x \in \mathbb{R}^N : x \cdot e_1 > t\} \cap \Omega(t)$. We also let $r_H : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection map with respect to he hyperplane H. For $x \in \mathbb{R}^N$, we denote $\bar{x} := r_H(x)$, $\overline{u}(x) := u(\overline{x})$. Using these notations, we have

$$\begin{aligned}\theta'(t) &= \Gamma(1+s)^2\int_{\partial B^t}\left(\frac{u}{\delta^s}\right)^2\nu_1\,dx \\ &= \Gamma(1+s)^2\int_{\partial B^t\cap\Theta}\left(\left(\frac{u}{\delta^s}\right)^2(x) - \left(\frac{\overline{u}}{\delta^s}\right)^2(x)\right)\nu_1\,dx.\end{aligned} \tag{5.11}$$

Let $w = \overline{u} - u \in H^s(\mathbb{R}^N)$. Then w is a (weak) solution of the problem

$$(-\Delta)^s w = \lambda_{s,p}(\Omega(t))\overline{u}^{p-1} - \lambda_{s,p}(\Omega(t))u^{p-1} = c_p w \qquad \text{in} \quad \Theta, \tag{5.12}$$

where

$$\begin{cases} c_p := \lambda_{s,p}(\Omega(t)) & \text{for } p = 2, \\ c_p = 0 & \text{for } p = 1. \end{cases}$$

Moreover, by definition, $w \equiv \overline{u} \geq 0$ in $H \setminus \overline{\Theta}$, and $w \equiv \overline{u} > 0$ in the subset $[r_H(B_1) \cap H] \setminus \overline{\Theta}$ which has positive measure since $t > 0$. Using that w is anti-symmetric with respect to H and the fact that $\lambda_{s,p}(\Theta) > c_p$, we can apply the weak maximum principle for antisymmetric functions (see [13, Proposition 3.1] or [19, Proposition 3.5]) to deduce that $w \geq 0$ in Θ. Moreover, since $w \not\equiv 0$ in $\mathbb{R}^N$, it follows from the strong maximum principle for antisymmetric functions given in [19, Proposition 3.6] that $w > 0$ in Θ. Now by the fractional Hopf lemma (see [13, Proposition 3.3]) we conclude that

$$0 > \frac{w}{\delta^s} = \frac{\overline{u}}{\delta^s} - \frac{u}{\delta^s} \quad \text{and therefore} \quad \frac{\overline{u}}{\delta^s} < \frac{u}{\delta^s} \leq 0 \qquad \text{on } \partial B_t \cap \Theta.$$

From this and (5.11) we get (5.8), since $\nu_1 > 0$ on $\partial B_t \cap \Theta$.
To conclude, we observe that the function $t \mapsto \lambda_{s,p}(t) = \lambda_{s,p}(\Omega(t))$ is even, thanks to the invariance of the problem under rotations. Therefore the function θ attains its maximum uniquely at $t = 0$. □

6. Proof of Proposition 2.4

The aim of this section is to prove Proposition 2.4. For the readers convenience, we repeat the statement here.

Proposition 6.1. *Let $X \in C^0(\overline{\Omega}, \mathbb{R}^N)$, and let $u \in C^s_0(\overline{\Omega}) \cap C^1(\Omega)$ be a function such that $\psi := \frac{u}{\delta^s} : \Omega \to \mathbb{R}$ satisfies (2.8) and (2.9). Moreover, put $U_k := u[\zeta_k \circ \delta] \in C^{1,1}_c(\Omega)$. Then*

$$\lim_{k\to\infty} \int_\Omega \nabla U_k \cdot X\Big(u(-\Delta)^s[\zeta_k \circ \delta] - I(u, \zeta_k \circ \delta)\Big)\, dx = -\kappa_s \int_{\partial\Omega} \psi^2 X \cdot \nu \, dx, \tag{6.1}$$

where

$$\kappa_s := -\int_{\mathbb{R}} h'(r)(-\Delta)^s h(r)\, dr \qquad \text{with} \quad h(r) := r^s_+ \zeta(r), \tag{6.2}$$

and where we use the notation

$$I(u, v)(x) := \int_{\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y))K_0(x, y)\, dy \tag{6.3}$$

for $u \in C^s_c(\mathbb{R}^N)$, $v \in C^{0,1}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

The remainder of this section is devoted to the proof of this proposition. For $k \in \mathbb{N}$, we define

$$g_k := \nabla U_k \cdot X\Big(u(-\Delta)^s[\zeta_k \circ \delta] - I(u, \zeta_k \circ \delta)\Big) \quad : \quad \Omega \to \mathbb{R}. \tag{6.4}$$

For $\varepsilon > 0$, we put

$$\Omega^\varepsilon = \{x \in \mathbb{R}^N : |\delta(x)| < \varepsilon\} \qquad \text{and} \qquad \Omega^\varepsilon_+ = \{x \in \mathbb{R}^N : 0 < \delta(x) < \varepsilon\} = \{x \in \Omega : \delta(x) < \varepsilon\}.$$

For every $\varepsilon > 0$, we then have

$$\lim_{k\to\infty} \int_{\Omega\setminus\Omega^\varepsilon} g_k \, dx = 0. \tag{6.5}$$

To see this, we first note that $\zeta_k \circ \delta \to 1$ pointwise on $\mathbb{R}^N \setminus \partial\Omega$, and therefore a.e. on $\mathbb{R}^N$. Moreover, choosing a compact neighborhood $K \subset \Omega$ of $\Omega \setminus \Omega^\varepsilon$, we have

$$(-\Delta)^s[\zeta_k \circ \delta](x) = b_{N,s} \int_{\mathbb{R}^N\setminus K} \frac{1 - [\zeta_k \circ \delta](y)}{|x - y|^{N+2s}} dy \qquad \text{for } x \in \Omega \setminus \Omega^\varepsilon \text{ and } k \text{ sufficiently large,}$$

where $\frac{|1-[\zeta_k\circ\delta](y)|}{|x-y|^{N+2s}} \leq \frac{C}{1+|y|^{N+2s}}$ for $x \in \Omega \setminus \Omega^\varepsilon$, $y \in \mathbb{R}^N \setminus K$ and $C > 0$ independent of x and y. Consequently, $\|(-\Delta)^s[\zeta_k \circ \delta]\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)}$ remains bounded independently of k and $(-\Delta)^s[\zeta_k \circ \delta] \to 0$ pointwise on $\Omega \setminus \Omega^\varepsilon$ by the dominated convergence theorem. Similarly, we see that $\|I(u, \zeta_k \circ \delta)\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)}$ remains bounded independently of k and $I(u, \zeta_k \circ \delta) \to 0$ pointwise on $\Omega \setminus \Omega^\varepsilon$. Consequently, we find that

$$\|g_k\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)} \text{ is bounded independently of } k \text{ and } g_k \to 0 \text{ pointwise on } \Omega \setminus \Omega^\varepsilon.$$

Hence (6.5) follows again by the dominated convergence theorem. As a consequence,

$$\lim_{k\to\infty}\int_\Omega g_k\,dx = \lim_{k\to\infty}\int_{\Omega_+^\varepsilon} g_k(x)\,dx \qquad \text{for every } \varepsilon>0. \tag{6.6}$$

Let, as before, $\nu:\partial\Omega\to\mathbb{R}^N$ denotes the unit interior normal vector field on Ω. Since we assume that $\partial\Omega$ is of class $C^{1,1}$, the map ν is Lipschitz, which means that the derivative $d\nu: T\partial\Omega\to\mathbb{R}^N$ is a.e. well defined and bounded. Moreover, we may fix $\varepsilon>0$ from now on such that the map

$$\Psi:\partial\Omega\times(-\varepsilon,\varepsilon)\to\Omega^\varepsilon, \qquad (\sigma,r)\mapsto\Psi(\sigma,r)=\sigma+r\nu(\sigma) \tag{6.7}$$

is a bi-lipschitz map with $\Psi(\partial\Omega\times(0,\varepsilon))=\Omega_+^\varepsilon$. In particular, Ψ is a.e. differentiable. Moreover, for $0<\varepsilon'\le\varepsilon$, it follows from (6.6) that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega g_k\,dx &= \lim_{k\to\infty}\int_{\Omega_+^{\varepsilon'}} g_k\,dx = \lim_{k\to\infty}\int_{\partial\Omega}\int_0^{\varepsilon'}\mathrm{Jac}_\Psi(\sigma,r)g_k(\Psi(\sigma,r))\,drd\sigma\\ &= \lim_{k\to\infty}\frac{1}{k}\int_{\partial\Omega}\int_0^{k\varepsilon'} j_k(\sigma,r)g_k(\Psi(\sigma,\frac{r}{k}))\,drd\sigma,\end{aligned} \tag{6.8}$$

where we define

$$j_k(\sigma,r)=\mathrm{Jac}_\Psi(\sigma,\frac{r}{k}) \qquad \text{for a.e. } \sigma\in\partial\Omega,\ 0\le r<k\varepsilon.$$

We note that

$$\begin{aligned}&\|j_k\|_{L^\infty(\partial\Omega\times[0,k\varepsilon))}\le\|\mathrm{Jac}_\Psi\|_{L^\infty(\Omega_\varepsilon)}<\infty \quad \text{for all } k, \text{ and}\\ &\lim_{k\to\infty} j_k(\sigma,r)=\mathrm{Jac}_\Psi(\sigma,0)=1 \quad \text{for a.e. } \sigma\in\partial\Omega,\ r>0.\end{aligned} \tag{6.9}$$

Moreover, we write $g_k=g_k^0(g_k^1-g_k^2)$ with the functions

$$g_k^0=\nabla U_k\cdot X, \qquad g_k^1=u(-\Delta)^s[\zeta_k\circ\delta] \qquad \text{and} \qquad g_k^2=I(u,\zeta_k\circ\delta), \tag{6.10}$$

which are all defined on $\overline{\Omega}$. We provide estimates for the functions g_k^0,g_k^1,g_k^2 separately in the following lemmas.

Lemma 6.2. *Let $\alpha\in(0,1)$ be given by Lemma 2.3. Then we have*

$$k^{s-1}|g_k^0(\sigma,\frac{r}{k})|\le C(r^{s-1}+r^{s-1+\alpha}) \qquad \text{for } k\in\mathbb{N},\ 0\le r<k\varepsilon \tag{6.11}$$

with a constant $C>0$, and

$$\lim_{k\to\infty}k^{s-1}g_k^0(\sigma,\frac{r}{k})=h'(r)\psi(\sigma)[X(\sigma)\cdot\nu(\sigma)] \quad \text{for } \sigma\in\partial\Omega,\ r>0 \tag{6.12}$$

with the function $r\mapsto h(r)=r^s\zeta(r)$ given in (6.2).

Proof. Since $u=\psi\delta^s$, we have

$$\nabla u = s\delta^{s-1}\psi\nabla\delta+\delta^s\nabla\psi = s\delta^{s-1}\psi\nabla\delta+O(\delta^{s-1+\alpha}) \qquad \text{in } \Omega$$

by Lemma 2.3, and therefore

$$\nabla U_k=\Big(s\zeta\circ(k\delta)+k\delta\zeta'\circ(k\delta)\Big)\psi\delta^{s-1}\nabla\delta+O(\delta^{s-1+\alpha}) \qquad \text{in } \Omega.$$

Consequently,

$$\big[(\nabla u_k)\circ\Psi\big](\sigma,r)=\Big(s\zeta(kr)+kr\zeta'(kr)\Big)\psi(\sigma+r\nu(\sigma))r^{s-1}\nabla\delta(\sigma+r\nu(\sigma))+O(r^{s-1+\alpha})$$

for $\sigma \in \partial\Omega$, $0 \le r < \varepsilon$ with $O(r^{s-1+\alpha})$ independent of k, and therefore

$$g_k^0(\Psi(\sigma, \frac{r}{k})) = \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma + \frac{r}{k}\nu(\sigma))\nabla\delta(\sigma + \frac{r}{k}\nu(\sigma)) \cdot X(\sigma + \frac{r}{k}\nu(\sigma))k^{1-s}r^{s-1} + k^{1-s-\alpha}O(r^{s-1+\alpha}) \qquad \text{for } \sigma \in \partial\Omega,\ 0 \le r < k\varepsilon.$$

Since $\alpha > 0$, we deduce that

$$k^{s-1}g_k^0(\Psi(\sigma, \frac{r}{k})) \to \Big(s\zeta(r) + r\zeta'(r)\Big)\psi(\sigma)\nabla\delta(\sigma) \cdot X(\sigma)r^{s-1} = h'(r)\psi(\sigma)X(\sigma) \cdot \nu(\sigma) \qquad \text{as } k \to \infty$$

for $\sigma \in \partial\Omega$, $r > 0$, while

$$k^{s-1}|g_k^0(\sigma, \frac{r}{k})| \le C(r^{s-1} + r^{s-1+\alpha}) \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon$$

with a constant $C > 0$ independent of k and r, as claimed. □

Next we consider the functions g_k^1 defined in (6.10), and we first state the following estimate.

Lemma 6.3. *There exists $\varepsilon' > 0$ with the property that*

$$|k^{-2s}(-\Delta)^s[\zeta_k \circ \delta](\Psi(\sigma, \frac{r}{k}))| \le \frac{C}{1 + r^{1+2s}} \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon' \tag{6.13}$$

with a constant $C > 0$. Moreover,

$$\lim_{k\to\infty} k^{-2s}(-\Delta)^s[\zeta_k \circ \delta](\Psi(\sigma, \frac{r}{k})) = (-\Delta)^s\xi(r) \quad \text{for } \sigma \in \partial\Omega,\ r > 0. \tag{6.14}$$

Before giving the somewhat lengthy proof of this lemma, we infer the following corollary related to the functions g_k^1.

Corollary 6.4. *There exists $\varepsilon' > 0$ with the property that*

$$|k^{-s}g_k^1(\Psi(\sigma, \frac{r}{k}))| \le \frac{Cr^s}{1 + r^{1+2s}} \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon' \tag{6.15}$$

with a constant $C > 0$. Moreover,

$$\lim_{k\to\infty} k^{-s}g_k^1(\Psi(\sigma, \frac{r}{k})) = \psi(\sigma)r^s(-\Delta)^s\xi(r) \quad \text{for } \sigma \in \partial\Omega,\ r > 0. \tag{6.16}$$

Proof. Since $u = \psi\delta^s$ we have $u(\Psi(\sigma, \frac{r}{k})) = k^{-s}\psi(\sigma + \frac{r}{k}\nu(\sigma))r^s$ for $k \in \mathbb{N}$, $0 \le r < k\varepsilon$, and

$$\lim_{k\to\infty} k^s u(\Psi(\sigma, \frac{r}{k})) = \psi(\sigma)r^s \quad \text{for } \sigma \in \partial\Omega,\ r > 0.$$

Since moreover $\|\psi\|_{L^\infty(\Omega_\varepsilon)} < \infty$, the claim now follows from Lemma 6.3 by noting that $g_k^1 = u(-\Delta)^s[\zeta_k \circ \delta]$. □

Proof of Lemma 6.3. We start with some preliminary considerations. We define

$$a_{N,s} := \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |z|^2)^{\frac{N+2s}{2}}} dz \tag{6.17}$$

and we recall that

$$b_{N,s}a_{N,s} = b_{1,s}, \tag{6.18}$$

where $b_{N,s}$ is given in (1.2), see e.g. [11]. Since $\partial\Omega$ is of class $C^{1,1}$ by assumption, there exists an open ball $B \subset \mathbb{R}^{N-1}$ centered at the origin and, for any fixed $\sigma \in \partial\Omega$, a parametrization

$$f_\sigma : B \to \partial\Omega$$

of class $C^{1,1}$ such that $f_\sigma(0) = \sigma$ and $df_\sigma(0) : \mathbb{R}^{N-1} \to \mathbb{R}^N$ is a linear isometry. For $z \in B$ we then have

$$f(0) - f(z) = -df_\sigma(0)z + O(|z|^2)$$

and therefore

$$|f(0) - f(z)|^2 = |df_\sigma(0)z|^2 + O(|z|^3) = |z|^2 + O(|z|^3), \tag{6.19}$$

$$(f(0) - f(z)) \cdot \nu(\sigma) = -df_\sigma(0)z \cdot \nu(\sigma) + O(|z|^2) = O(|z|^2), \tag{6.20}$$

$$\tag{6.21}$$

where we used in (6.20) that $df_\sigma(0)z$ belongs to the tangent space $T_\sigma\partial\Omega = \{\nu(\sigma)\}^\perp$. Here and in the following, the term $\mathcal{O}(\tau)$ stands for a function depending on τ and possibly other quantities but satisfying $|\mathcal{O}(\tau)| \le C\tau$ with a constant $C > 0$. By (6.19), we can make B smaller if necessary to guarantee that

$$|f(0) - f(z)|^2 \ge \frac{3}{4}|z|^2. \tag{6.22}$$

Recalling the definition of the map Ψ in (6.7) and writing $\nu_\sigma(z) := \nu(f_\sigma(z))$ for $z \in B$, we now define

$$\Psi_\sigma : (-\varepsilon, \varepsilon) \times B \to \Omega^\varepsilon, \qquad \Psi_\sigma(r, z) = \Psi(f_\sigma(z), r) = f_\sigma(z) + r\nu_\sigma(z). \tag{6.23}$$

Then Ψ_σ is a bi-lipschitz map which maps $(-\varepsilon, \varepsilon) \times B$ onto a neighborhood of σ. Consequently, there exists $\varepsilon' \in (0, \frac{\varepsilon}{2})$ with the property that

$$|\sigma - y| \ge 3\varepsilon' \qquad \text{for all } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B). \tag{6.24}$$

Moreover, ε' can be chosen independently of $\sigma \in \partial\Omega$. For fixed $\sigma \in \partial\Omega$, $r \in [0, \varepsilon')$, we can now write

$$(-\Delta)^s[\zeta_k \circ \delta](\Psi(\sigma, r))| = b_{N,s}\Big(A_k(\sigma, r) + B_k(\sigma, r)\Big) \tag{6.25}$$

with

$$A_k(\sigma, r) := \int_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{\zeta_k(r) - \zeta_k(\delta(y))}{|\Psi(\sigma, r) - y|^{N+2s}}\, dy$$

and

$$B_k(\sigma, r) = \int_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{\zeta_k(r) - \zeta_k(\delta(y))}{|\Psi(\sigma, r) - y|^{N+2s}}\, dy.$$

By (6.24) and since $r < \varepsilon'$, we then have

$$|\Psi(\sigma, r) - y| = |\sigma - y + r\nu(\sigma)| \ge |\sigma - y| - r \ge \frac{|\sigma - y|}{3} + \varepsilon' \quad \text{for } y \in \mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon, \varepsilon) \times B)$$

and therefore

$$\begin{aligned} |B_k(\sigma, r)| &\le \int_{\mathbb{R}^N \setminus \Psi_\sigma((-\varepsilon,\varepsilon)\times B)} \frac{|\rho(kr) - \rho(k\delta(y))|}{|\Psi(\sigma, r) - y|^{N+2s}}\, dy \\ &\le 3^{N+2s}|\rho(kr)| \int_{\mathbb{R}^N} \Big(|\sigma - y| + 3\varepsilon'\Big)^{-N-2s} dy + \big(\varepsilon'\big)^{-N-2s} \int_{\mathbb{R}^N} |\rho(k\delta(y))|\, dy \\ &\le C\Big(|\rho(kr)| + |\Omega_{\frac{2}{k}}|\Big) \le C\Big(|\rho(kr)| + k^{-1}\Big). \end{aligned}$$

Here and in the following, the letter C stands for various positive constants. Consequently,

$$\lim_{k\to\infty} k^{-2s}|B_k(\sigma, \frac{r}{k})| = 0 \qquad \text{for every } \sigma \in \Omega,\ r \ge 0, \tag{6.26}$$

and, since ρ has compact support in $\mathbb{R}$,

$$k^{-2s}|B_k(\sigma,\frac{r}{k})| \le Ck^{-2s}\Big(|\rho(r)|+k^{-1}\Big) \\ \le \frac{C}{1+r^{1+2s}}+k^{-1-2s} \le \frac{C}{1+r^{1+2s}} \qquad \text{for } k\in\mathbb{N},\ 0\le r<k\varepsilon',\ \sigma\in\partial\Omega. \tag{6.27}$$

Hence it remains to estimate $A_k(\sigma,r)$. By definition of Ψ_σ we have

$$\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)=f(0)-f(z)-t\nu_\sigma(0)+(r+t)(\nu_\sigma(0)-\nu_\sigma(z))$$

for $z\in B$, $r\in(0,\varepsilon')$ and $t\in(-\varepsilon-r,\varepsilon-r)$. Therefore using that $(\nu_\sigma(0)-\nu_\sigma(z))\cdot\nu_\sigma(0)=\frac{1}{2}|\nu_\sigma(0)-\nu_\sigma(z)|^2$ and (6.20), we get

$$\begin{aligned}
&|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)|^2=t^2+|f(0)-f(z)|^2+(r+t)^2|\nu_\sigma(0)-\nu_\sigma(z)|^2\\
&-2t(f(0)-f(z))\cdot\nu_\sigma(0)-t(r+t)|\nu_\sigma(0)-\nu_\sigma(z)|^2+2(r+t)(f(0)-f(z))\cdot(\nu_\sigma(0)-\nu_\sigma(z))\\
&=t^2+|f(0)-f(z)|^2+r(r+t)|\nu_\sigma(0)-\nu_\sigma(z)|^2\\
&\qquad -2t(f(0)-f(z))\cdot\nu_\sigma(0)+2(r+t)(f(0)-f(z))\cdot(\nu_\sigma(0)-\nu_\sigma(z))\\
&=t^2+|f(0)-f(z)|^2+q_\sigma(r,z)+tp_\sigma(r,z)=t^2+|f(0)-f(z)|^2+(t+r)O(|z|^2)
\end{aligned} \tag{6.28}$$

for $z\in B$, $r\in(0,\varepsilon')$ and $t\in(-\varepsilon-r,\varepsilon-r)$, where

$$q_\sigma(r,z):=r^2|\nu_\sigma(0)-\nu_\sigma(z)|^2+2r(f(0)-f(z))\cdot(\nu_\sigma(0)-\nu_\sigma(z))$$

and

$$p_\sigma(r,z)=r|\nu_\sigma(0)-\nu_\sigma(z)|^2+2(f(0)-f(z))\cdot(\nu_\sigma(0)-\nu_\sigma(z)).$$

We note that there exists a constant $C_0>0$, depending only on Ω, with the property that

$$|q_\sigma(r,z)|\le rC_0|z|^2,\quad |p_\sigma(r,z)|\le C_0|z|^2 \qquad \text{for } z\in B,\ r\in(0,\varepsilon') \text{ and } t\in(-\varepsilon-r,\varepsilon-r),$$

By (6.22), this implies that

$$|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)|^2\ge t^2+\Big(\frac{3}{4}-(r+t)C_0\Big)|z|^2\ge t^2+\Big(\frac{3}{4}-(\varepsilon'+t)C_0\Big)|z|^2$$

for $z\in B$, $r\in(0,\varepsilon')$ and $t\in(-\varepsilon-r,\varepsilon-r)$ with a constant $C_0>0$ depending only on Ω. Making $\varepsilon'>0$ smaller if necessary, we thus have

$$|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)|^2\ge\frac{t^2+(1-tC_0)|z|^2}{2} \quad \text{for } z\in B,\ r\in(0,\varepsilon') \text{ and } t\in(-\varepsilon-r,\varepsilon-r)$$

and therefore

$$|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)|^{-N-2s}\le 2^{N+2s}\Big(t^2+(1-tC_0)|z|^2\Big)^{-\frac{N+2s}{2}} \tag{6.29}$$

for $z\in B$, $r\in(0,\varepsilon')$ and $t\in(-\varepsilon-r,\varepsilon-r)$, $|t|<\frac{1}{C_0}$. Moreover, combining (6.19) and (6.28) gives

$$\begin{aligned}
|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)|^{-N-2s}&=\big(t^2+|z|^2+q_\sigma(r,z)+(|z|+t)O(|z|^2)\big)^{-\frac{N+2s}{2}}\\
&=\big(t^2+|z|^2+(|z|+r+t)O(|z|^2)\big)^{-\frac{N+2s}{2}}
\end{aligned} \tag{6.30}$$

for $z\in B$, $r\in(0,\varepsilon')$ and $t\in(-\varepsilon-r,\varepsilon-r)$. Moreover, we note that, as a bi-lipschitz map, Ψ_σ is a.e. differentiable, and $d\Psi_\sigma$ is given by

$$d\Psi_\sigma(r,z)(r',z')=[df_\sigma(z)+rd\nu_\sigma(z)]z'+r'\nu_\sigma(z)=df_\sigma(0)z'+r'\nu_\sigma(0)+O\big(r+|z|\big)(|z'|+|r'|)$$

for $(r,z)\in(0,\varepsilon')\times B$, $(r',z')\in\mathbb{R}\times\mathbb{R}^{N-1}$. Since $df_\sigma(0)$ is an isometry and $df_\sigma(0)\nu_\sigma(0)=0$, we thus infer that

$$\mathrm{Jac}_{\Psi_\sigma}(r+t,z)=1+O(r+t+|z|). \tag{6.31}$$

We now define the kernel $\mathcal{K}$ on $(0,\varepsilon')\times\mathbb{R}\times B$ by

$$\mathcal{K}(r,t,z)=\begin{cases}\mathrm{Jac}_{\Psi_\sigma}(r+t,z)|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,z)|^{-N-2s}, & t\in(-\varepsilon-r,\varepsilon-r),\\ 0, & t\notin(-\varepsilon-r,\varepsilon-r).\end{cases} \tag{6.32}$$

We then have that

$$A_k(\sigma,r)=\int_{-\varepsilon}^{\varepsilon}\int_B \mathrm{Jac}_{\Psi_\sigma}(z,\tilde r)\,\frac{\zeta_k(r)-\zeta_k(\tilde r)}{|\Psi_\sigma(r,0)-\Psi_\sigma(\tilde r,z)|^{N+2s}}\,dzd\tilde r \tag{6.33}$$
$$=\int_{\mathbb{R}}\int_B\big(\zeta_k(r)-\zeta_k(r+t)\big)\mathcal{K}(r,t,z)\,dzdt$$
$$=\frac14\int_{\mathbb{R}}\int_B(2\zeta_k(r)-\zeta_k(r+t)-\zeta_k(r-t))[\mathcal{K}(r,t,z)+\mathcal{K}(r,-t,z)]dtdz$$
$$+\frac14\int_{\mathbb{R}}\int_B(\zeta_k(r+t)-\zeta_k(r-t))[\mathcal{K}(r,t,z)-\mathcal{K}(r,-t,z)]dzdt$$
$$=\frac14\int_{\mathbb{R}}t^{N-1}\int_{\frac{1}{|t|}B}\big(2\zeta_k(r)-\zeta_k(r+t)-\zeta_k(r-t)\big)\big(\mathcal{K}(r,t,|t|z)+\mathcal{K}(r,-t,|t|z)\big)dzdt$$
$$+\frac14\int_{\mathbb{R}}\int_B(\zeta_k(r+t)-\zeta_k(r-t))[\mathcal{K}(r,t,z)-\mathcal{K}(r,-t,z)]dzdt,$$
$$=k^{2s}\Big(J_k^1(\sigma,r)+J_k^2(\sigma,r)\Big), \tag{6.34}$$

where, by a further change of variable,

$$J_k^1(\sigma,r):=\frac{1}{4k^{2s}}\int_{\mathbb{R}}t^{N-1}\int_{\frac{1}{|t|}B}\big(2\zeta_k(r)-\zeta_k(r+t)-\zeta_k(r-t)\big)\big(\mathcal{K}(r,t,|t|z)+\mathcal{K}(r,r-t,|t|z)\big)dtdz$$
$$=\frac{1}{4k^{N+2s}}\int_{\mathbb{R}}t^{N-1}\int_{\frac{k}{|t|}B}\big(2\zeta(kr)-\zeta(kr+t)-\zeta(kr-t)\big)\big(\mathcal{K}(r,\frac{t}{k},\frac{|t|}{k}z)+\mathcal{K}(r,-\frac{t}{k},\frac{|t|}{k}z)\big)dtdz$$

and

$$J_k^2(\sigma,r):=\frac{1}{4k^{2s}}\int_{\mathbb{R}}\int_B(\zeta_k(r+t)-\zeta_k(r-t))[\mathcal{K}(r,t,z)-\mathcal{K}(r,-t,z)]dtdz.$$

By (6.30) we have

$$|\Psi_\sigma(r,0)-\Psi_\sigma(r+t,|t|z)|^{-N-2s}=t^{-N-2s}\left(1+|z|^2+\mathcal{O}(|t||z|+r+|t|)|z|^2\right)^{-\frac{N+2s}{2}}.$$

for $r\in(0,\varepsilon')$, $t\in(-\varepsilon-r,\varepsilon-r)\setminus\{0\}$ and $z\in\frac{1}{t}B$ and therefore, by (6.31),

$$|t|^{N+2s}\mathcal{K}(r,t,|t|z)=(1+O(r+|t|+|t||z|))\left(1+|z|^2+\mathcal{O}(|t||z|+r+|t|)|z|^2)\right)^{-\frac{N+2s}{2}}.$$

Hence

$$\lim_{k\to\infty}|\frac{t}{k}|^{N+2s}\mathcal{K}(\frac{r}{k},\frac{t}{k},\frac{|t|}{k}z)=(1+|z|^2)^{-\frac{N+2s}{2}}\qquad\text{for every } r\ge 0,\ t\in\mathbb{R} \tag{6.35}$$

while by (6.29) and (6.31) there exists $C>0$ such that, for k sufficiently large,

$$|\frac{t}{k}|^{N+2s}|\mathcal{K}(\frac{r}{k},\frac{t}{k},\frac{|t|}{k}z)|\le C(1+|z|^2)^{-\frac{N+2s}{2}}. \tag{6.36}$$

for $r \in (-k\varepsilon', k\varepsilon')$, $t \in \mathbb{R} \setminus \{0\}$ and $z \in \frac{k}{|t|}B$. Using this, we get

$$|J_k^1(\sigma, \frac{r}{k})| \le C \int_{\mathbb{R}} \frac{|2\zeta(r) - \zeta(r+t) - \zeta(r-t)|}{|t|^{1+2s}}\, dt \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}} dz$$
$$= C a_{N,s} \int_{\mathbb{R}} \frac{|2\rho(r) - \rho(r+t) - \rho(r-t)|}{|t|^{1+2s}}\, dt \le \frac{C a_{N,s}}{1 + r^{1+2s}} \tag{6.37}$$

for k sufficiently large, $r \in (-k\varepsilon', k\varepsilon')$, $t \in \mathbb{R} \setminus \{0\}$ and $z \in \frac{|k|}{t}B$. Here we used $\rho = 1 - \zeta \in C_c^\infty(\mathbb{R})$ in the last step. Moreover, by (6.35) and the dominated convergence theorem, we find that

$$\lim_{k\to\infty} J_k^1(\sigma, \frac{r}{k})$$
$$= \frac{1}{4} \int_{\mathbb{R}} \frac{2\zeta(r) - \zeta(r+t) - \zeta(r-t)}{|t|^{1+2s}} \Big(\lim_{k\to\infty} |\frac{t}{k}|^{N+2s} \int_{\frac{k}{|t|}B} \big(\mathcal{K}(\frac{r}{k}, \frac{t}{k}, \frac{|t|}{k} z) + \mathcal{K}(\frac{r}{k}, -\frac{t}{k}, \frac{|t|}{k} z) \big) dz \Big) dt$$
$$= \frac{1}{b_{1,s}} [(-\Delta)^s \zeta(r)] \int_{\mathbb{R}^{N-1}} (1+|z|^2)^{-\frac{N+2s}{2}} dz = \frac{a_{N,s}}{b_{1,s}} [(-\Delta)^s \zeta(r)] = \frac{(-\Delta)^s \zeta(r)}{b_{N,s}}, \tag{6.38}$$

where we used (6.18) in the last equality.

Next, to deal with $J_k^2(\sigma, r)$, we have to estimate the kernel difference $|\mathcal{K}(r,t,z) - \mathcal{K}(r,-t,z)|$. For this we note that by (6.30) we have

$$|\Psi_\sigma(r,0) - \Psi_\sigma(r+t,z)|^{-N-2s} = \Big(t^2 + |z|^2 + q_\sigma(r,z) \Big)^{-\frac{N+2s}{2}} \Big(1 + O(|t| + |z|) \Big)$$

and therefore

$$\Big| |\Psi_\sigma(r,0) - \Psi_\sigma(r+t,z)|^{-N-2s} - |\Psi_\sigma(r,0) - \Psi_\sigma(r-t,z)|^{-N-2s} \Big| = \frac{O(|t|+|z|)}{\big(t^2 + |z|^2 + q_\sigma(r,z)\big)^{\frac{N+2s}{2}}}$$
$$\le C\big(|t| + |z|\big)^{1-N-2s} \qquad \text{for } z \in B,\ r \in (0,\varepsilon') \text{ and } t \in (-\varepsilon + r, \varepsilon - r).$$

Moreover,

$$\big| \mathrm{Jac}_{\Psi_\sigma}(r+t,z) - \mathrm{Jac}_{\Psi_\sigma}(r-t,z) \big| \le C\big(|t| + |z|\big)$$

for $z \in B$, $r \in (0,\varepsilon')$ and $t \in (-\varepsilon + r, \varepsilon - r)$ by (6.31). From this we deduce that

$$|\mathcal{K}(r,t,z) - \mathcal{K}(r,-t,z)| \le C\big(|t| + |z|\big)^{1-N-2s} \quad \text{for } z \in B,\ r \in (0,\varepsilon') \text{ and } t \in (-\varepsilon + r, \varepsilon - r).$$

Moreover, we have

$$|\mathcal{K}(r,t,z) - \mathcal{K}(r,-t,z)| = 0 \qquad \text{for } t \in \mathbb{R} \setminus (-\varepsilon - r, \varepsilon + r),$$

while for $t \in (-\varepsilon - r, -\varepsilon + r) \cup (\varepsilon - r, \varepsilon + r)$ we have $|t| \ge \varepsilon - \varepsilon' \ge \frac{\varepsilon}{2}$ and therefore

$$|\mathcal{K}(r,t,z)| \le C(\varepsilon + |z|)^{-N-2s}.$$

Consequently,

$$
\begin{aligned}
|J_k^2(\sigma,r)| &\le \frac{1}{4k^{2s}}\int_{-\varepsilon-r}^{\varepsilon+r}|\zeta_k(r+t)-\zeta_k(r-t)|\int_B\big(\mathcal{K}(r,t,z)-\mathcal{K}(r,-t,z)\big)dzdt\\
&\le Ck^{-2s}\int_{-\varepsilon+r}^{\varepsilon-r}|\zeta_k(r+t)-\zeta_k(r-t)|\int_B\big(|t|+|z|\big)^{1-N-2s}dzdt\\
&+Ck^{-2s}\int_{-\varepsilon-r}^{\varepsilon+r}|\zeta_k(r+t)-\zeta_k(r-t)|\int_B(\varepsilon+|z|)^{-N-2s}dzdt\\
&\le Ck^{-2s}\Big(\int_{-\varepsilon+r}^{\varepsilon-r}\frac{|\zeta_k(r+t)-\zeta_k(r-t)|}{|t|^{2s}}dt+\varepsilon^{-1-2s}\int_{-\varepsilon-r}^{\varepsilon+r}|\zeta_k(r+t)-\zeta_k(r-t)|dt\Big)\\
&\le C\Big(\frac{1}{k}\int_{\mathbb{R}}\frac{|\rho(kr+t)-\rho(kr-t)|}{|t|^{2s}}dt+(\varepsilon k)^{-1-2s}\int_{\mathbb{R}}|\rho(kr+t)-\rho_k(kr-t)|dt\Big)\\
&\le C\Big(\frac{1}{k}\int_{-kr-4}^{-kr+4}|t|^{1-2s}dt+(\varepsilon k)^{-1-2s}\Big)\le C\Big(\frac{(kr+1)^{1-2s}}{k}+(\varepsilon k)^{-1-2s}\Big)
\end{aligned}
$$

and therefore

$$
|J_k^2(\sigma,\frac{r}{k})|\le C\Big(\frac{(1+r)^{1-2s}}{k}+(\varepsilon k)^{-1-2s}\Big)\qquad\text{for } k\in\mathbb{N},\ 0\le r<\varepsilon'.
$$

From this we deduce that

$$
|J_k^2(\sigma,\frac{r}{k})|\le\frac{C}{1+r^{1+2s}}\qquad\text{for } k\in\mathbb{N},\ 0\le r<k\varepsilon' \tag{6.39}
$$

and

$$
\lim_{k\to\infty}|J_k^2(\sigma,\frac{r}{k})|=0\qquad\text{for all } r\ge 0. \tag{6.40}
$$

In view of (6.25), the bound (6.13) follows by combining (6.27), (6.37) and (6.39), while the pointwise limit equality (6.14) follows from (6.26), 6.38 and (6.40). □

Lemma 6.5. *There exists $\varepsilon'>0$ with the property that the function $g_k^2=I(u,\zeta_k\circ\delta)$ satisfies*

$$
|k^{-s}g_k^2(\Psi(\sigma,\frac{r}{k}))|\le\frac{C}{1+r^{1+s}}\qquad\textit{for } k\in\mathbb{N},\ 0\le r<k\varepsilon' \tag{6.41}
$$

with a constant $C>0$. Moreover,

$$
\lim_{k\to\infty}k^{-s}g_k^2(\Psi(\sigma,\frac{r}{k}))=\psi(\sigma)\tilde{I}(r) \tag{6.42}
$$

with

$$
\tilde{I}(r)=b_{1,s}\int_{\mathbb{R}}\frac{(r_+^s-\widetilde{r}_+^s)(\zeta(r)-\zeta(\widetilde{r}))}{|r-\widetilde{r}|^{1+2s}}\,d\widetilde{r}.
$$

Proof. We keep using the notation of the proof of Lemma 6.3. We then have

$$
g_k^2(\Psi(\sigma,r))=b_{N,s}\Big(\tilde{A}_k(\sigma,r)+\tilde{B}_k(\sigma,r)\Big) \tag{6.43}
$$

with

$$
\tilde{A}_k(\sigma,r):=\int_{\Psi_\sigma((-\varepsilon,\varepsilon)\times B)}\frac{(u(\Psi(\sigma,r))-u(y))(\zeta_k(r)-\zeta_k(\delta(y)))}{|\Psi(\sigma,r)-y|^{N+2s}}\,dy
$$

and

$$
\tilde{B}_k(\sigma,r)=\int_{\mathbb{R}^N\setminus\Psi_\sigma((-\varepsilon,\varepsilon)\times B)}\frac{(u(\Psi(\sigma,r))-u(y))(\zeta_k(r)-\zeta_k(\delta(y)))}{|\Psi(\sigma,r)-y|^{N+2s}}\,dy.
$$

As noted in the proof of Lemma 6.3, we have

$$|\Psi(\sigma,r)-y| \geq \frac{|\sigma-y|}{3}+\varepsilon' \quad \text{for } y\in\mathbb{R}^N\setminus\Psi_\sigma((-\varepsilon,\varepsilon)\times B).$$

Therefore, since $u\in L^\infty(\mathbb{R}^N)$, we may estimate as in the proof of Lemma 6.3 to get

$$\begin{aligned}|\tilde{B}_k(\sigma,r)| &\leq 2\|u\|_{L^\infty}\int_{\mathbb{R}^N\setminus\Psi_\sigma((-\varepsilon,\varepsilon)\times B)}\frac{|\rho(kr)-\rho(k\delta(y))|}{|\Psi(\sigma,r)-y|^{N+2s}}\\ &\leq C\Big(|\rho(kr)|+k^{-1}\Big).\end{aligned}$$

Here and in the following, the letter C stands for various positive constants. Consequently,

$$\lim_{k\to\infty}k^{-s}|\tilde{B}_k(\sigma,\frac{r}{k})|=0 \qquad \text{for every } \sigma\in\Omega,\ r\geq 0, \tag{6.44}$$

and, since ρ has compact support in $\mathbb{R}$,

$$\begin{aligned}k^{-s}|\tilde{B}_k(\sigma,\frac{r}{k})| &\leq Ck^{-s}\Big(|\rho(r)|+k^{-1}\Big)\leq C\Big(\frac{1}{1+r^{1+s}}+k^{-1-s}\Big)\\ &\leq \frac{C}{1+r^{1+s}} \qquad \text{for } k\in\mathbb{N},\ 0\leq r<k\varepsilon',\ \sigma\in\partial\Omega.\end{aligned} \tag{6.45}$$

Hence it remains to estimate $\tilde{A}_k(\sigma,r)$. For this we note that

$$\begin{aligned}\tilde{A}_k(\sigma,\frac{r}{k}) &= \int_{-\varepsilon}^{\varepsilon}\int_B \mathrm{Jac}_{\Psi_\sigma}(z,\tilde{r})\,\frac{(u(\Psi(\frac{r}{k},0))-u(\Psi_\sigma(\tilde{r},z)))(\zeta_k(\frac{r}{k})-\zeta_k(\tilde{r}))}{|\Psi_\sigma(\frac{r}{k},0)-\Psi_\sigma(\tilde{r},z)|^{N+2s}}\,dzd\tilde{r}\\ &= \int_{\mathbb{R}}\int_B\Big((u(\Psi_\sigma(\frac{r}{k},0))-u(\Psi_\sigma(\frac{r}{k}+t,z)))(\zeta(r)-\zeta(r+kt))\Big)\mathcal{K}(\frac{r}{k},t,z)\,dzdt\end{aligned}$$

with $\mathcal{K}$ given as in the proof of Lemma 6.3, and therefore, by a change of variables,

$$\tilde{A}_k(\sigma,\frac{r}{k})=k^{2s}\int_{\mathbb{R}}\frac{\zeta(r)-\zeta(r+t)}{|t|^{1+2s}}\int_{\frac{k}{|t|}B}w_k(r,t,z)\,dzdt \tag{6.46}$$

with

$$w_k(r,t,z):=(\frac{|t|}{k})^{N+2s}\Big(u(\Psi_\sigma(\frac{r}{k},0))-u(\Psi_\sigma(\frac{r+t}{k},\frac{|t|z}{k}))\Big)\mathcal{K}(\frac{r}{k},\frac{t}{k},\frac{|t|}{k}z).$$

For k sufficiently large,

$$(\frac{|t|}{k})^{N+2s}|\mathcal{K}(\frac{r}{k},\frac{t}{k},\frac{|t|}{k}z)|\leq C(1+|z|^2)^{-\frac{N+2s}{2}} \tag{6.47}$$

by (6.36) and

$$\lim_{k\to\infty}(\frac{|t|}{k})^{N+2s}\mathcal{K}(\frac{r}{k},\frac{t}{k},\frac{|t|}{k}z)=(1+|z|^2)^{-\frac{N+2s}{2}} \qquad \text{for every } r\geq 0,\ t\in\mathbb{R} \tag{6.48}$$

by (6.35). Moreover, since $u\in C^s(\mathbb{R}^N)$,

$$\begin{aligned}&|u(\Psi_\sigma(\frac{r}{k},0))-u(\Psi_\sigma(\frac{r+t}{k},\frac{|t|}{k}z)|\leq C\Big(\min\Big\{(\frac{|t|}{k})^s,1\Big\}+\min\Big\{(\frac{|tz|}{k})^s,1\Big\}\Big)\\ &\leq C(\frac{|t|}{k})^s+(\frac{|tz|}{k})^s\leq Ck^{-s}|t|^s(1+|z|^s)\end{aligned}$$

and consequently

$$|w_k(r,t,z)|\leq Ck^{-s}|t|^s(1+|z|)^{-N-s}, \tag{6.49}$$

where the function $z \mapsto (1+|z|)^{-N-s}$ is integrable over $\mathbb{R}^{N-1}$. We conclude that

$$|\tilde{A}_k(\sigma, \frac{r}{k})| \le Ck^{-s} \int_{\mathbb{R}} \frac{|\zeta(r) - \zeta(r+t)|}{|t|^{1+s}} ds \le Ck^{-s} \int_{\mathbb{R}} \frac{|\rho(r) - \rho(r+t)|}{|t|^{1+s}} ds \le Ck^{-s} \frac{1}{1+r^s}.$$

Combining this with (6.43) and (6.45), we get (6.41). Moreover, since $u \in C^s(\mathbb{R}^N)$ and $\psi = \frac{u}{\delta^s} \in C^0(\overline{\Omega})$, we have that

$$k^s \left[u(\Psi_\sigma(\frac{r}{k}, 0)) - u(\Psi_\sigma(\frac{r+t}{k}, \frac{|t|}{k} z)) \right] \to \psi(\sigma)(r^s - (r+t)^s) \qquad \text{as } k \to \infty \tag{6.50}$$

which by (6.48) implies that

$$k^s w_k(r,t,z) \to \psi(\sigma)(r^s - (r+t)^s)(1+|z|^2)^{-\frac{N+2s}{2}}. \tag{6.51}$$

Hence, by (6.46), (6.49), (6.51) and the dominated convergence theorem, we find that

$$\begin{aligned} k^{-s} \tilde{A}_k(\sigma, \frac{r}{k}) &\to \psi(\sigma) \int_{\mathbb{R}} \frac{(r^s - (r+t)^s)(\zeta(r) - \zeta(r+t))}{|t|^{1+2s}} dt \int_{\mathbb{R}^{N-1}} (1+|z|^2)^{-\frac{N+2s}{2}} dz \\ &= \frac{a_{N,s}}{b_{1,s}} \psi(\sigma) \tilde{I}(r) \qquad \text{as } k \to \infty. \end{aligned}$$

Combining this with (6.18), (6.43) and (6.44), we obtain (6.42). □

We are now ready to complete the

Proof of Proposition 6.1. Combining (6.11), (6.15) and (6.41), we see that there exists $\varepsilon' > 0$ such that the functions g_k defined in (6.4) satisfy

$$\frac{1}{k} g_k(\Psi(\sigma, \frac{r}{k})) \le C \frac{r^{s-1} + r^{s-1+\alpha}}{1 + r^{1+s}} \qquad \text{for } k \in \mathbb{N},\ 0 \le r < k\varepsilon' \tag{6.52}$$

with a constant $C > 0$ independent of k and r. Since $s, \alpha \in (0,1)$, the RHS of this inequality is integrable over $[0, \infty)$. Moreover,

$$\frac{1}{k} g_k(\Psi(\sigma, \frac{r}{k})) \to [X(\sigma) \cdot \nu(\sigma)] \psi^2(\sigma) h'(r) \big(r^s (-\Delta)^s \zeta(r) - \tilde{I}(r) \big) \tag{6.53}$$

for every $r > 0$, $\sigma \in \partial\Omega$ as $k \to \infty$. Here we note that, by a standard computation,

$$(-\Delta)^s h(r) = (-\Delta)^s [r_+^s \zeta(r)] = \zeta(r)(-\Delta)^s r_+^s + r_+^s (-\Delta)^s \zeta(r) - \tilde{I}(r) = r_+^s (-\Delta)^s \zeta(r) - \tilde{I}(r) \tag{6.54}$$

for $r > 0$ since r_+^s is an s-harmonic function on $(0, \infty)$ see e.g [1].

Hence, by (6.8), (6.8), (6.52), (6.53), (6.54) and the dominated convergence theorem, we conclude that

$$\begin{aligned} \lim_{k \to \infty} \int_\Omega g_k dx &= \int_0^\infty h'(r)(-\Delta)^s h(r) dr \int_{\partial\Omega} [X(\sigma) \cdot \nu(\sigma)] \psi^2(\sigma) d\sigma \\ &= \int_{\mathbb{R}} h'(r)(-\Delta)^s h(r) dr \int_{\partial\Omega} [X(\sigma) \cdot \nu(\sigma)] \psi^2(\sigma) d\sigma, \end{aligned}$$

as claimed in (6.1). □

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